THE DIFFERENTIALGEOMETRY SOFTWARE PROJECT

Rainich-type Conditions for Null Electrovacuum Spacetimes II

Synopsis

- In this second of two worksheets I continue describing local Rainich-type conditions which are necessary and sufficient for the metric to define a null electrovacuum. In other words, these conditions, which I will call the *null electrovacuum conditions,* guarantee the existence of a null electromagnetic field such that the metric and electromagnetic field satisfy the Einstein-Maxwell equations. When it exists, the electromagnetic field is easily constructed from the metric. The results illustrated here are based upon [1].

- In this worksheet I consider the null electrovacuum conditions which apply when a certain null geodesic congruence determined by the metric is twisting. I shall illustrate the these conditions using a couple of pure radiation spacetimes taken from the literature [3, 4].

- A companion worksheet (Rainich-type Conditions for Null Electrovacuum Spacetimes I) treats the twist-free case, which is considerably simpler.

Theory

- Let (M, g) be a spacetime – a 4-dimensional manifold M endowed with a Lorentz-signature metric g. The Rainich conditions are geometric conditions on g such that there exists an electromagnetic field F with (g, F) satisfying the Einstein-Maxwell equations,

$$G_{ab} = F_a{}^c F_{bc} - \frac{1}{4} g_{ab} F_{mn} F^{mn} \quad , \qquad \nabla_{[a} F_{bc]} = 0, \ \nabla_a F^{ab} = 0.$$

The classical Rainich conditions involve the metric, the Ricci tensor R_{ij}, the covariant derivative ∇, and the volume form ϵ_{ijhk} of the metric, and are given by

$$R^i_h R^h_j - \frac{1}{4} \delta^i_j R^{hk} R_{hk} = 0, \qquad R^i_i = 0, \qquad R_{ij} t^i t^j > 0, \qquad \nabla_{[j} \alpha_{i]} = 0, \quad \text{where} \quad \alpha_i = \frac{\epsilon_{ijhk} R^j_m \nabla^h R^{mk}}{R^{ij} R_{ij}}$$

Here t^i is any timelike vector field. When these conditions are satisfied there is a straightforward procedure for constructing the electromagnetic field (see RainichElectromagneticField), which is determined by the metric up to a duality rotation:

$$F_{ab} \rightarrow \cos(\theta)\, F_{ab} - \sin(\theta) \star F_{ab}, \qquad \theta \in \mathbb{R},$$

where $\star$ denotes the Hodge dual on 2-forms determined by the metric. If a metric satisfies the Rainich conditions we say that it determines an electrovacuum spacetime.

• A spacetime admits a non-null electromagnetic source if and only if it satisfies the Rainich conditions.

• The Rainich conditions are not defined for *null electrovacua*, *i.e.*, solutions of the Einstein-Maxwell equations with a null electromagnetic field,

$$F_{ab}F^{ab} = 0, \; F_{ab} \star F^{ab} = 0$$

because such fields have a null energy-momentum tensor and hence a null Ricci tensor:

$$R_{ab}R^{ac} = 0 \; .$$

The Rainich conditions do not provide local, geometric criteria for null electrovacua.

• A geometric description of null electrovacua is as follows [1]. A metric with a null Ricci tensor – a pure radiation spacetime – determines a null vector field k^a via

$$G_{ab} = R_{ab} = \frac{1}{4} k_a k_b. \tag{1}$$

The contracted Bianchi identity implies

$$k^b \nabla_b k_a = \left(\nabla_b k^b\right) k_a,$$

which implies that the congruence generated by this vector field k^a is a geodesic congruence. The vector field k^a determines a family of 2-forms,

$$f_{ab} = k_{[a} s_{b]}, \tag{2}$$

where s^b is any spacelike unit vector orthogonal to k^a. The energy-momentum tensor of f_{ab} satisfies the Einstein equations in the sense that its energy-momentum tensor equals the Ricci tensor (which is the same as the Einstein tensor in the null case). Therefore, if there is a solution F_{ab} to the Einstein-Maxwell equations, at each point of M it must be related to f_{ab} by a duality rotation. Thus there will exist a function $\varphi : M \rightarrow \mathbf{R}$ such that the electromagnetic field takes the form

$$F_{ab} = \cos(\varphi)\, f_{ab} - \sin(\varphi) \star f_{ab} \; . \tag{3}$$

The Maxwell equations for F_{ab} impose a number of conditions on k^a and φ. In particular, the vector field k^a defines a shear-free, null, geodesic congruence. In terms of a null tetrad whose first leg is k^a, and using the associated Newman-Penrose formalism [2], these conditions on k^a take the form

$$\sigma = 0 = \kappa, \quad \frac{1}{2}(\rho + \bar{\rho}) = \epsilon + \bar{\epsilon}\,. \tag{4}$$

To analyze the conditions imposed by the Maxwell equations on φ, one must distinguish two cases: the congruence tangent to k^a is (i) twisting or (ii) is twist-free. The twist, denoted ω, is defined by

$$\omega = -\mathrm{Im}(\rho).$$

The solution spaces for φ – and hence for the electromagnetic field – are significantly different in these two cases. This worksheet considers the twisting case only (see the companion worksheet for the twist-free case).

- In the twisting case the function φ determining the duality rotation must satisfy

$$\frac{1}{i}\delta\varphi + \tau - 2\beta = 0, \quad \frac{1}{i}\bar{\delta}\varphi - \bar{\tau} + 2\bar{\beta} = 0, \quad \frac{1}{i}D\varphi - \varepsilon + \bar{\varepsilon} = 0, \quad \omega\,\Delta\varphi + i\,\mathrm{Im}(\mu)\,(\rho - 2\varepsilon) + \mathrm{Re}[\,(\delta + \bar{\beta} - \alpha)(\tau - 2\beta)\,] = 0\,. \tag{5}$$

Here the letters α, τ, β, ε, δ, ρ, μ, D, Δ denote the standard Newman-Penrose quantities [2], which are determined once k^a has been incorporated into the first leg of a null tetrad. As shown in [1], there are two non-trivial integrability conditions for these equations. We express them as a single complex-valued condition:

$$\mathcal{G} = 0,$$

where

$$\mathcal{G} = \omega\,\delta(\mathrm{Re}(\delta(\bar{\tau} - 2\bar{\beta}))) - [\delta\omega + \omega(\tau - \bar{\alpha} - \beta)]\Big[\mathrm{Re}\{(\bar{\delta} + \bar{\beta} - \alpha)(\tau - 2\beta)\} + i\,\mathrm{Im}(\mu)(\rho - 2\varepsilon) + \frac{\omega}{2}\Big\{\bar{\beta}\delta(2\bar{\alpha} + \tau - 4\beta) + \beta\,\delta(2\alpha + \bar{\tau} - 4\bar{\beta}) + 2i\delta\big[\mathrm{Im}(\mu)(\rho - 2\varepsilon)\big] + \tau\,\delta(\bar{\beta} - \alpha) + \bar{\tau}\,\delta(\beta - \bar{\alpha}) - \alpha\,\delta(\tau - 2\beta) - \bar{\alpha}\delta(\bar{\tau} - 2\bar{\beta})\Big\} - i\,\omega^2\,\Delta(\tau - 2\beta) + \omega^2\big[\bar{\nu}(\omega + 2\,\mathrm{Im}(\varepsilon)) - (\tau - 2\beta)(2\,\mathrm{Im}(\gamma) + i\,\mu) + i\,\bar{\lambda}(\bar{\tau} - 2\bar{\beta})\big]. \tag{6}$$

- Conditions (4) – (6) are suitably invariant under the set of local Lorentz transformations which fix k^a; they represent invariant conditions which are defined independently of the choice of null tetrad adapted to k^a.

• Conditions (4) and (6) on the null congruence determined by the Ricci tensor (1) provide geometric conditions on the spacetime geometry which are necessary and sufficient for the existence of a (null) electromagnetic source [1]. Thus we obtain Rainich-type conditions for null electrovacuum spacetimes.

• Any two solutions to (5) differ by a solution to $\delta\varphi = \bar{\delta}\varphi = D\varphi = \Delta\varphi = 0$, i.e, a constant.

A metric satisfying (4) and (6) therefore determines the electromagnetic field up to a duality rotation.

Procedures for computing the Maxwell equations and their integrability conditions.

The following procedure computes the Maxwell equations (3) for the function φ and isolates the 4 coordinate derivatives. The input is a table of spin coefficients and a table of directional derivatives, both computed from a null tetrad adapted to the principal null vector k using the commands NPSpinCoefficients and NPDirectionalDerivatives.

```
NullMaxwellEquations := proc(NPS, NPD)
```

The following procedure computes the integrability conditions $\mathcal{J}$ in (4). The input is a table of spin coefficients and a table of directional derivatives, both computed from a null tetrad adapted to the principal null vector k.

```
NullElectrovacuumConditions := proc(NPS, NPD)
```

If the command NullElectrovacuumConditions is executed with no arguments, the formula for the integrability conditions is displayed. The quantity $\delta 1$ is the conjugate operator to δ. The symbols conj(x), im(x), re(x) denote the complex conjugate, imaginary part, and real part of x, respectively.

```
> NullElectrovacuumConditions();
```

$$\begin{aligned}
&\omega\,\delta(re(\delta 1(\tau-2\beta))) + \omega^2\,conj(\nu)\,(\omega + 2\,im(\varepsilon)) - I\,\omega^2\,\Delta(\tau - 2\beta) - (\delta(\omega) \\
&\quad + \omega\,(\tau - conj(\alpha) - \beta))\,(re(\delta 1(\tau - 2\beta) + (conj(\beta) - \alpha)\,(\tau - 2\beta)) \\
&\quad + I\,im(\mu)\,(\rho - 2\varepsilon)) + \frac{1}{2}(\omega\,(conj(\beta)\,\delta(2\,conj(\alpha) + \tau - 4\beta) + \beta\,\delta(2\alpha \\
&\quad + conj(\tau) - 4\,conj(\beta)) + 2\,I\,\delta(im(\mu)\,(\rho - 2\varepsilon)) + \tau\,\delta(conj(\beta) - \alpha) \\
&\quad + conj(\tau)\,\delta(\beta - conj(\alpha)))) + \omega^2\,((\tau - 2\beta)\,(-2\,im(\gamma) - I\,\mu) \\
&\quad + I\,conj(\lambda)\,conj(\tau - 2\beta)) - \frac{\omega\,(\alpha\,\delta(\tau - 2\beta) + conj(\alpha)\,\delta(conj(\tau - 2\beta)))}{2}
\end{aligned} \tag{2.1}$$

These two bits of code must be executed (e.g., by clicking on the code edit region) before running the following examples.

Example 1: Nurowski-Tafel electrovacuum

Nurowski and Tafel have constructed a class of algebraically special solutions of the Einstein-Maxwell equations [4] with null electromagnetic field. In particular, they have found the only known solutions which have twisting rays and a purely radiative electromagnetic field. These solutions are built from a freely specifiable holomorphic function b(ξ) and two parameters, α and c2. We consider a special case of these solutions, specialized to Petrov type **III** (c2 = 0) and with b(ξ) = b/ξ, where b is a real constant. To avoid confusion with the Newman-Penrose spin coefficients, we relabel α as the parameter a. We verify that the integrability conditions (6) are satisfied and construct the electromagnetic field from the metric using (5) and (3).

Set-up.

We begin by defining the spacetime manifold and various auxiliary quantities needed to define the metric.

```
> with(DifferentialGeometry): with(Tensor): with(Tools):
> DGsetup([u, r, xi, xi1], M, complexconjugatepairs=[[xi, xi1]])
  ;
```

$$\textit{frame name: M} \quad (3.1.1)$$

```
M > bet := evalDG((r+I*Sigma)/b(xi)/(1+xi*xi1) * dxi);
```

$$bet := \frac{r + \mathrm{I}\,\Sigma}{b(\xi)\,(\xi\,\xi 1 + 1)}\, d\xi \quad (3.1.2)$$

```
M > bet1 := simplify(evalDG((r - I* Sigma)/b1(xi1)/(1+ xi*xi1)
    * dxi1), symbolic);
```

$$bet1 := -\frac{\mathrm{I}\,\Sigma - r}{b1(\xi 1)\,(\xi\,\xi 1 + 1)}\, dxi1 \quad (3.1.3)$$

```
M > theta3 := evalDG(du + L* dxi + L1 * dxi1);
```

$$\theta 3 := du + L\, d\xi + L1\, dxi1 \quad (3.1.4)$$

```
> Sigma := _alpha*(1 - xi*xi1)/(1+xi*xi1);
```

$$\Sigma := \frac{\_\alpha\,(-\xi\,\xi 1 + 1)}{\xi\,\xi 1 + 1} \quad (3.1.5)$$

```
M > L := I * _alpha*xi1/(b(xi)*b1(xi1)*(1 + xi * xi1)^2) + I *
    _alpha* int(xi1*diff(ln(b1(xi1)), xi1)/(b(xi)*b1(xi1)*(1+
    xi*xi1)^2), xi1);
```

$$L := \frac{\mathrm{I}\,\_\alpha\,\xi 1}{b(\xi)\,b1(\xi 1)\,(\xi\,\xi 1 + 1)^2} + \mathrm{I}\,\_\alpha \left(\int \frac{\xi 1 \left(\frac{d}{d\xi 1} b1(\xi 1) \right)}{b1(\xi 1)^2\, b(\xi)\,(\xi\,\xi 1 + 1)^2}\, d\xi 1 \right) \quad (3.1.6)$$

```
M > L1 := -I * _alpha*xi/(b1(xi1)*b(xi)*(1 + xi * xi1)^2) - I
```

```
* _alpha* int(xi*diff(ln(b(xi)), xi)/(b1(xi1)*b(xi)*(1+
xi*xi1)^2), xi);
```

$$L1 := -\frac{I\_\alpha\,\xi}{b1(\xi1)\,b(\xi)\,(\xi\,\xi1+1)^2} - I\_\alpha\left(\int \frac{\xi\left(\frac{d}{d\xi}\,b(\xi)\right)}{b(\xi)^2\,b1(\xi1)\,(\xi\,\xi1+1)^2}\,d\xi\right) \quad (3.1.7)$$

Here is a general form of the metric considered in [4].

```
M > g0 := evalDG(2*(r^2 + Sigma^2)/(b(xi)*b1(xi1)*(1+ xi*xi1)
^2) * dxi &s dxi1 - 2*theta3 &s (dr + 2*_alpha*I/(1 + xi*
xi1)^2*(xi*dxi1 - xi1 * dxi) + (b(xi)*b1(xi1) - c2*r/(r^2
+ Sigma^2))*theta3)):
```

Here we specialize to the case $b(\xi) = \frac{b}{\xi}$. We also fix c2=0 which means the spacetime is Petrov type III. The metric g is quite complicated, so we do not display it.

```
M > ch := [b(xi) = b/xi, b1(xi1) = b/xi1, c2=0];
```

$$ch := \left[b(\xi) = \frac{b}{\xi},\, b1(\xi1) = \frac{b}{\xi1},\, c2 = 0\right] \quad (3.1.8)$$

```
M > g := factor(eval(g0, ch)):
```

Adapted tetrad, spin coefficients and the integrability conditions.

Our next task is to identify the tangent vector k to the preferred congruence and construct a tetrad adapted to it. To this end, we begin by constructing a convenient null tetrad which is used in [4].

```
M > omega0 := evalDG((dr + 2*_alpha*I/(1 + xi*xi1)^2*(xi*dxi1
- xi1 * dxi) + (b(xi)*b1(xi1) - c2*r/(r^2 + Sigma^2))*
theta3)):
M > omega1 := eval(omega0, ch):
```

```
M > coframe := factor(eval(evalDG([omega1, theta3, bet, bet1]
), ch));
```

$$coframe := \left[\frac{b^2}{\xi\,\xi1}\,du + dr - \frac{1}{\xi^2\,\xi1\,(\xi\,\xi1+1)^2}\left(I\_\alpha\left(\ln(\xi\,\xi1+1)\,\xi^2\,\xi1^2 + \xi1^2\,\xi^2 + 2\ln(\xi\,\xi1+1)\,\xi\,\xi1 + \xi\,\xi1 + \ln(\xi\,\xi1+1) + 1\right)\right)d\xi \right. \quad (3.2.1)$$

$$+ \frac{1}{\xi\,\xi1^2\,(\xi\,\xi1+1)^2}\left(I\_\alpha\left(\ln(\xi\,\xi1+1)\,\xi^2\,\xi1^2 + \xi1^2\,\xi^2 + 2\ln(\xi\,\xi1\right.\right.$$

$+ 1) \xi \xi1 + \xi \xi1 + \ln(\xi \xi1 + 1) + 1)) \, dxi1, du$

$- \frac{1}{\xi b^2 (\xi \xi1 + 1)^2} (\mathrm{I} (\ln(\xi \xi1 + 1) \xi^2 \xi1^2 - \xi1^2 \xi^2 + 2 \ln(\xi \xi1 + 1) \xi \xi1$

$+ \xi \xi1 + \ln(\xi \xi1 + 1) + 1) \_\alpha) \, d\xi + \frac{1}{\xi1 \, b^2 (\xi \xi1 + 1)^2} (\mathrm{I} (\ln(\xi \xi1$

$+ 1) \xi^2 \xi1^2 - \xi1^2 \xi^2 + 2 \ln(\xi \xi1 + 1) \xi \xi1 + \xi \xi1 + \ln(\xi \xi1 + 1) + 1) \_\alpha)$

$dxi1, - \frac{\mathrm{I} (\_\alpha \xi \xi1 + \mathrm{I} \, r \xi \xi1 - \_\alpha + \mathrm{I} \, r) \xi}{(\xi \xi1 + 1)^2 b} d\xi,$

$- \frac{\mathrm{I} (\mathrm{I} \, r \xi \xi1 - \_\alpha \xi \xi1 + \mathrm{I} \, r + \_\alpha) \xi1}{(\xi \xi1 + 1)^2 b} dxi1 \Big]$

Here we check that this coframe does define a null tetrad.

```
N > simplify(map(expand,TensorInnerProduct(g, coframe,
    coframe)), symbolic);
```

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \textbf{(3.2.2)}$$

```
M > Fr := simplify(DualBasis(coframe), symbolic);
```

$Fr := \Big[D\_r, D\_u - \frac{b^2}{\xi \xi1} D\_r, - \frac{1}{b \xi^2 (\_\alpha \xi \xi1 + \mathrm{I} \, r \xi \xi1 - \_\alpha + \mathrm{I} \, r)} (\_\alpha (\ln(\xi \xi1$ **(3.2.3)**

$+ 1) \xi^2 \xi1^2 - \xi1^2 \xi^2 + 2 \ln(\xi \xi1 + 1) \xi \xi1 + \xi \xi1 + \ln(\xi \xi1 + 1) + 1)) \, D\_u$

$- \frac{2 \_\alpha \xi1 \, b}{(\_\alpha \xi \xi1 + \mathrm{I} \, r \xi \xi1 - \_\alpha + \mathrm{I} \, r) \xi} D\_r$

$+ \frac{\mathrm{I} (\xi \xi1 + 1)^2 b}{(\_\alpha \xi \xi1 + \mathrm{I} \, r \xi \xi1 - \_\alpha + \mathrm{I} \, r) \xi} D\_\xi,$

$\frac{1}{b \xi1^2 (\mathrm{I} \, r \xi \xi1 - \_\alpha \xi \xi1 + \mathrm{I} \, r + \_\alpha)} (\_\alpha (\ln(\xi \xi1 + 1) \xi^2 \xi1^2 - \xi1^2 \xi^2$

$+ 2 \ln(\xi \xi1 + 1) \xi \xi1 + \xi \xi1 + \ln(\xi \xi1 + 1) + 1)) \, D\_u$

$$+ \frac{2\,\_\alpha\,\xi\,b}{\left(\mathrm{I}\,r\,\xi\,\xi 1 - \_\alpha\,\xi\,\xi 1 + \mathrm{I}\,r + \_\alpha\right)\xi 1} D\_r$$

$$\left. + \frac{\mathrm{I}\,(\xi\,\xi 1 + 1)^2\,b}{\left(\mathrm{I}\,r\,\xi\,\xi 1 - \_\alpha\,\xi\,\xi 1 + \mathrm{I}\,r + \_\alpha\right)\xi 1} D\_xi1 \right]$$

We verify that the frame satisfies the reality conditions of a null tetrad.

```
M > DGIm(Fr[1]);
```

$$0\,D\_u \tag{3.2.4}$$

```
M > DGIm(Fr[2]);
```

$$0\,D\_u \tag{3.2.5}$$

```
M > DGconjugate(Fr[3]) &minus Fr[4];
```

$$0\,D\_u \tag{3.2.6}$$

Next we compute the Ricci tensor; it is of the form $R_{ab} = \frac{1}{4} k_a k_b$, where k_a is the 1-form $\theta 3$ defined above. This is also the second 1-form in the null coframe.

```
M > Ric1 := factor(RicciTensor(g)):
M > TensorInnerProduct(g, Ric1, Ric1, tensorindices=[1]);
```

$$0\,du \otimes du \tag{3.2.7}$$

```
M > R22 := op(factor(GetComponents(Ric1, [coframe[2] &t coframe[2]])));
```

$$R22 := \frac{2\,b^4\,(\xi\,\xi 1 + 1)^4}{\left(\_\alpha^2\,\xi^2\,\xi 1^2 + r^2\,\xi^2\,\xi 1^2 - 2\,\_\alpha^2\,\xi\,\xi 1 + 2\,r^2\,\xi\,\xi 1 + \_\alpha^2 + r^2\right)\xi^3\,\xi 1^3} \tag{3.2.8}$$

```
M > chi := _alpha^2*xi^2*xi1^2+r^2*xi^2*xi1^2-2*_alpha^2*xi*xi1+2*r^2*xi*xi1+_alpha^2+r^2;
```

$$\chi := \_\alpha^2\,\xi^2\,\xi 1^2 + r^2\,\xi^2\,\xi 1^2 - 2\,\_\alpha^2\,\xi\,\xi 1 + 2\,r^2\,\xi\,\xi 1 + \_\alpha^2 + r^2 \tag{3.2.9}$$

For this to be a pure radiation spacetime (a necessary condition for null electrovacuum) we must have chi > 0.

We now construct the vector field k tangent to the preferred congruence.

```
M > Phi := simplify(sqrt(R22), symbolic) assuming chi > 0, xi*xi1 >0, b>0;
```

$$\Phi := \frac{\sqrt{2}\,b^2\,(\xi\,\xi 1 + 1)^2}{\xi^{3/2}\,\xi 1^{3/2}\sqrt{\_\alpha^2\,\xi^2\,\xi 1^2 + r^2\,\xi^2\,\xi 1^2 - 2\,\_\alpha^2\,\xi\,\xi 1 + 2\,r^2\,\xi\,\xi 1 + \_\alpha^2 + r^2}} \tag{3.2.10}$$

```
M > Kdn := simplify(evalDG(2*Phi*coframe[2])) assuming chi > 0, xi*xi1 >0, b>0;
```

$$Kdn \tag{3.2.11}$$

$$:= \frac{2\sqrt{2}\, b^2\, (\xi\,\xi 1 + 1)^2}{\xi^{3/2}\, \xi 1^{3/2} \sqrt{\_\alpha^2 \xi^2 \xi 1^2 + r^2 \xi^2 \xi 1^2 - 2\,\_\alpha^2 \xi\,\xi 1 + 2 r^2 \xi\,\xi 1 + \_\alpha^2 + r^2}}$$

$$du - \Big(2\,\mathrm{I}\sqrt{2}\,\big(\ln(\xi\,\xi 1 + 1)\,\xi^2\,\xi 1^2 - \xi 1^2\,\xi^2 + 2\ln(\xi\,\xi 1 + 1)\,\xi\,\xi 1 + \xi\,\xi 1 + \ln(\xi\,\xi 1 + 1) + 1\big)\,\_\alpha\Big) \Big/ \Big(\xi^{5/2}\,\xi 1^{3/2} \sqrt{\_\alpha^2 \xi^2 \xi 1^2 + r^2 \xi^2 \xi 1^2 - 2\,\_\alpha^2 \xi\,\xi 1 + 2 r^2 \xi\,\xi 1 + \_\alpha^2 + r^2}\Big)$$

$$d\xi + \Big(2\,\mathrm{I}\sqrt{2}\,\big(\ln(\xi\,\xi 1 + 1)\,\xi^2\,\xi 1^2 - \xi 1^2\,\xi^2 + 2\ln(\xi\,\xi 1 + 1)\,\xi\,\xi 1 + \xi\,\xi 1 + \ln(\xi\,\xi 1 + 1) + 1\big)\,\_\alpha\Big) \Big/ \Big(\xi^{3/2}\,\xi 1^{5/2} \sqrt{\_\alpha^2 \xi^2 \xi 1^2 + r^2 \xi^2 \xi 1^2 - 2\,\_\alpha^2 \xi\,\xi 1 + 2 r^2 \xi\,\xi 1 + \_\alpha^2 + r^2}\Big)$$

$$dxi1$$

```
M > factor(evalDG(Ric - 1/4*Kdn &t Kdn));
```

$$0\, du \otimes du \qquad \textbf{(3.2.12)}$$

```
M > Kup := RaiseLowerIndices(InverseMetric(g), Kdn, [1]);
```

$$Kup := -\frac{2\sqrt{2}\, b^2\, (\xi\,\xi 1 + 1)^2}{\xi^{3/2}\, \xi 1^{3/2} \sqrt{\_\alpha^2 \xi^2 \xi 1^2 + r^2 \xi^2 \xi 1^2 - 2\,\_\alpha^2 \xi\,\xi 1 + 2 r^2 \xi\,\xi 1 + \_\alpha^2 + r^2}}\, D\_r \qquad \textbf{(3.2.13)}$$

We verify that Kup is a principal null direction, as follows from the Goldberg-Sachs theorem.

```
M > WT := WeylTensor(g):
M > GRQuery(convert(Kup, DGvector), g, WT,
    "PrincipalNullDirection");
```

$$true \qquad \textbf{(3.2.14)}$$

We extend Kup into an adapted null tetrad, NT, by boosting the original null frame.

```
M > w := Hook([convert(Kup, DGvector)], dr);
```

$$w := -\frac{2\sqrt{2}\, b^2\, (\xi\,\xi 1 + 1)^2}{\xi^{3/2}\, \xi 1^{3/2} \sqrt{\_\alpha^2 \xi^2 \xi 1^2 + r^2 \xi^2 \xi 1^2 - 2\,\_\alpha^2 \xi\,\xi 1 + 2 r^2 \xi\,\xi 1 + \_\alpha^2 + r^2}} \qquad \textbf{(3.2.15)}$$

```
M > NT := map(expand,NullTetradTransformation(Fr, "boost", w))
    ;
```

$$NT := \Bigg[\qquad \textbf{(3.2.16)}$$

$$-\frac{2\sqrt{2}\, b^2\, (\xi\,\xi 1 + 1)^2}{\xi^{3/2}\, \xi 1^{3/2} \sqrt{\_\alpha^2 \xi^2 \xi 1^2 + r^2 \xi^2 \xi 1^2 - 2\,\_\alpha^2 \xi\,\xi 1 + 2 r^2 \xi\,\xi 1 + \_\alpha^2 + r^2}}$$

$D\_r,$

$$-\frac{1}{4\,b^2\,(\xi\,\xi 1+1)^2}\Big(\sqrt{2}\,\xi^{3/2}\,\xi 1^{3/2}$$

$$\sqrt{\_\alpha^2\,\xi^2\,\xi 1^2+r^2\,\xi^2\,\xi 1^2-2\,\_\alpha^2\,\xi\,\xi 1+2\,r^2\,\xi\,\xi 1+\_\alpha^2+r^2}\Big)\,D\_u$$

$$+\frac{1}{4\,(\xi\,\xi 1+1)^2}\Big(\sqrt{2}\,\sqrt{\xi}\,\sqrt{\xi 1}$$

$$\sqrt{\_\alpha^2\,\xi^2\,\xi 1^2+r^2\,\xi^2\,\xi 1^2-2\,\_\alpha^2\,\xi\,\xi 1+2\,r^2\,\xi\,\xi 1+\_\alpha^2+r^2}\Big)\,D\_r,$$

$$-\frac{1}{b\,\xi^2\,(\_\alpha\,\xi\,\xi 1+\mathrm{I}\,r\,\xi\,\xi 1-\_\alpha+\mathrm{I}\,r)}\Big(\_\alpha\,\big(\ln(\xi\,\xi 1+1)\,\xi^2\,\xi 1^2-\xi 1^2\,\xi^2$$

$$+2\ln(\xi\,\xi 1+1)\,\xi\,\xi 1+\xi\,\xi 1+\ln(\xi\,\xi 1+1)+1\big)\Big)\,D\_u$$

$$-\frac{2\,\_\alpha\,\xi 1\,b}{(\_\alpha\,\xi\,\xi 1+\mathrm{I}\,r\,\xi\,\xi 1-\_\alpha+\mathrm{I}\,r)\,\xi}\,D\_r$$

$$+\frac{\mathrm{I}\,(\xi\,\xi 1+1)^2\,b}{(\_\alpha\,\xi\,\xi 1+\mathrm{I}\,r\,\xi\,\xi 1-\_\alpha+\mathrm{I}\,r)\,\xi}\,D\_\xi,$$

$$\frac{1}{b\,\xi 1^2\,(\mathrm{I}\,r\,\xi\,\xi 1-\_\alpha\,\xi\,\xi 1+\mathrm{I}\,r+\_\alpha)}\Big(\_\alpha\,\big(\ln(\xi\,\xi 1+1)\,\xi^2\,\xi 1^2-\xi 1^2\,\xi^2$$

$$+2\ln(\xi\,\xi 1+1)\,\xi\,\xi 1+\xi\,\xi 1+\ln(\xi\,\xi 1+1)+1\big)\Big)\,D\_u$$

$$+\frac{2\,\_\alpha\,\xi\,b}{(\mathrm{I}\,r\,\xi\,\xi 1-\_\alpha\,\xi\,\xi 1+\mathrm{I}\,r+\_\alpha)\,\xi 1}\,D\_r$$

$$+\frac{\mathrm{I}\,(\xi\,\xi 1+1)^2\,b}{(\mathrm{I}\,r\,\xi\,\xi 1-\_\alpha\,\xi\,\xi 1+\mathrm{I}\,r+\_\alpha)\,\xi 1}\,D\_xi1\Big]$$

We check that NT has all the required properties.

```
M > evalDG(NT[1] - convert(Kup, DGvector));
```

$$0 \qquad \textbf{(3.2.17)}$$

```
M > evalDG(EinsteinTensor(g) - 1/4*NT[1] &t NT[1]);
```

$$0\,D\_u \otimes D\_u \quad (3.2.18)$$

```
M > simplify(map(factor,TensorInnerProduct(g, NT, NT)),
    symbolic);
```

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (3.2.19)$$

```
M > DGIm(NT[1]);
```

$$0\,D\_u \quad (3.2.20)$$

```
M > DGIm(NT[2]);
```

$$0\,D\_u \quad (3.2.21)$$

```
M > simplify(DGconjugate(NT[3]) &minus NT[4]);
```

$$0\,D\_u \quad (3.2.22)$$

Compute the spin coefficients and directional derivatives.

```
M > NPS01 := map(factor,simplify(NPSpinCoefficients(NT))):
M > NPS1 := simplify(NPS01) assuming chi > 0, xi*xi1 >0, b>0:
M > NPD1 := NPDirectionalDerivatives(NT):
```

Check that the congruence defined by k is shear-free, geodesic and parametrized according to (2).

```
M > NPS1["sigma"];
```

$$0 \quad (3.2.23)$$

```
M > NPS1["kappa"];
```

$$0 \quad (3.2.24)$$

```
M > factor(simplify(DGRe(NPS1["rho"]) - 2*DGRe(NPS1["epsilon"]
    ))) assuming chi > 0, xi*xi1 >0, b>0;
```

$$0 \quad (3.2.25)$$

Here we compute the twist of the congruence, which is non-vanishing in general.

```
M > factor(simplify(DGIm(-NPS1["rho"]))) assuming chi > 0, xi*
    xi1 >0, b>0;
```

$$\left(2\left(\xi\,\xi 1+1\right)^{3} b^{2}\sqrt{2}\left(\xi\,\xi 1-1\right)\_\alpha\right) \Big/ \left(\xi^{3/2}\,\xi 1^{3/2}\sqrt{\_\alpha^{2}\,\xi^{2}\,\xi 1^{2}+r^{2}\,\xi^{2}\,\xi 1^{2}-2\,\_\alpha^{2}\,\xi\,\xi 1+2\,r^{2}\,\xi\,\xi 1+\_\alpha^{2}+r^{2}}\left(\_\alpha\,\xi\,\xi 1+\mathrm{I}\,r\,\xi\,\xi 1-\_\alpha+\mathrm{I}\,r\right)\left(\mathrm{I}\,r\,\xi\,\xi 1-\_\alpha\,\xi\,\xi 1+\mathrm{I}\,r+\_\alpha\right)\right) \quad (3.2.26)$$

Therefore this spacetime is an electrovacuum if and only if $\mathcal{J} = 0$.

```
M > NullElectrovacuumConditions(NPS1, NPD1);
```

$$0 \quad (3.2.27)$$

This spacetime is an electrovacuum.

The electromagnetic field.

We now compute the electromagnetic field by solving equations (5) for φ. These equations can be put into the following form.

```
> EQ := NullMaxwellEquations(NPS1, NPD1);
```

$$EQ := \left\{ \frac{\partial}{\partial r} \phi(u, r, \xi, \xi 1) = \frac{(\xi \xi 1 + 1)(\xi \xi 1 - 1)\_\alpha}{\_\alpha^2 \xi^2 \xi 1^2 + r^2 \xi^2 \xi 1^2 - 2\_\alpha^2 \xi \xi 1 + 2 r^2 \xi \xi 1 + \_\alpha^2 + r^2}, \frac{\partial}{\partial u} \phi(u, r, \xi, \xi 1) = 0, \frac{\partial}{\partial \xi} \phi(u, r, \xi, \xi 1) = \left(-\frac{I}{2}\left(-3\_\alpha^2 \xi^2 \xi 1^2 - 3 r^2 \xi^2 \xi 1^2 + 4 I \_\alpha r \xi \xi 1 + 6\_\alpha^2 \xi \xi 1 - 6 r^2 \xi \xi 1 - 3\_\alpha^2 - 3 r^2\right)\right) \Big/ \left((I r \xi \xi 1 - \_\alpha \xi \xi 1 + I r + \_\alpha)(\_\alpha \xi \xi 1 + I r \xi \xi 1 - \_\alpha + I r)\xi\right), \frac{\partial}{\partial \xi 1} \phi(u, r, \xi, \xi 1) = \left(-\frac{I}{2}\left(3\_\alpha^2 \xi^2 \xi 1^2 + 3 r^2 \xi^2 \xi 1^2 + 4 I \_\alpha r \xi \xi 1 - 6\_\alpha^2 \xi \xi 1 + 6 r^2 \xi \xi 1 + 3\_\alpha^2 + 3 r^2\right)\right) \Big/ \left((\_\alpha \xi \xi 1 + I r \xi \xi 1 - \_\alpha + I r)(I r \xi \xi 1 - \_\alpha \xi \xi 1 + I r + \_\alpha)\xi 1\right) \right\} \quad \textbf{(3.3.1)}$$

The solution to this system is as follows.

```
M > phisol := combine(simplify(combine(pdsolve(EQ, {phi(u, r,
    xi, xi1)}), symbolic), symbolic), symbolic);
```

$$phisol := \left\{ \phi(u, r, \xi, \xi 1) = -\frac{3 I \ln\left(\frac{\xi}{\xi 1}\right)}{2} + \arctan\left(\frac{r(\xi \xi 1 + 1)}{(\xi \xi 1 - 1)\_\alpha}\right) + \_C1 \right\} \quad \textbf{(3.3.2)}$$

To compute the electromagnetic field it is convenient to work with an anholonomic frame, defined as follows.

```
M > DGsetup([u, r, xi, xi1], M);
```

$$\textit{frame name: M} \tag{3.3.3}$$

```
M > FD := FrameData([NT[1], NT[2], 1/sqrt(2)*(NT[3] + NT[4]),
    1/sqrt(2)/I*(NT[3] - NT[4])], null):
```

The third leg of this tetrad is the unit vector field $\overset{a}{s}$ orthogonal to k^a, used in (2).

```
M > DGsetup(FD):
```

The metric in this frame is the following.

```
null > eta := evalDG(-2*Theta1 &s Theta2 + Theta3 &t Theta3
       + Theta4 &t Theta4);
```

$$\eta := -\Theta1 \otimes \Theta2 - \Theta2 \otimes \Theta1 + \Theta3 \otimes \Theta3 + \Theta4 \otimes \Theta4 \tag{3.3.4}$$

The following 2-form, defined in (2), solves the Einstein equations, but not the Maxwell equations.

```
null > f := evalDG(-1/2*Theta2 &w Theta3);
```

$$f := -\frac{1}{2}\,\Theta2 \wedge \Theta3 \tag{3.3.5}$$

```
null > Gnull := EinsteinTensor(eta):
null > Tnull := EnergyMomentumTensor("Electromagnetic",
       eta, f):
null > factor(evalDG(Gnull - Tnull));
```

$$0\,E1 \otimes E1 \tag{3.3.6}$$

We now construct from f and φ the 2-form F, according to (3), which solves the Einstein-Maxwell equations.

```
null > fd := HodgeStar(eta, f, detmetric=-1);
```

$$fd := -\frac{1}{2}\,\Theta2 \wedge \Theta4 \tag{3.3.7}$$

```
null > Cphi := eval(cos(phi(u, r, xi, xi1)), phisol);
```

$$Cphi := \cos\left(-\frac{3\,\mathrm{I}\,\ln\left(\frac{\xi}{\xi1}\right)}{2} + \arctan\left(\frac{r\,(\xi\,\xi1+1)}{(\xi\,\xi1-1)\,\_\alpha}\right) + \_C1\right) \tag{3.3.8}$$

```
null > Sphi := eval(sin(phi(u, r, xi, xi1)), phisol);
```

$$Sphi := \sin\left(-\frac{3\,\mathrm{I}\,\ln\left(\frac{\xi}{\xi1}\right)}{2} + \arctan\left(\frac{r\,(\xi\,\xi1+1)}{(\xi\,\xi1-1)\,\_\alpha}\right) + \_C1\right) \tag{3.3.9}$$

```
null > F := evalDG(Cphi*f - Sphi*fd);
```

$$F := -\frac{\cos\left(-\frac{3\,\mathrm{I}\,\ln\left(\frac{\xi}{\xi 1}\right)}{2} + \arctan\left(\frac{r\,(\xi\,\xi 1 + 1)}{(\xi\,\xi 1 - 1)\,\_\alpha}\right) + \_C1\right)}{2}\,\Theta 2 \wedge \Theta 3 + \frac{\sin\left(-\frac{3\,\mathrm{I}\,\ln\left(\frac{\xi}{\xi 1}\right)}{2} + \arctan\left(\frac{r\,(\xi\,\xi 1 + 1)}{(\xi\,\xi 1 - 1)\,\_\alpha}\right) + \_C1\right)}{2}\,\Theta 2 \wedge \Theta 4 \quad (3.3.10)$$

Because $\frac{\xi}{\xi 1}$ is a phase, its natural logarithm is pure imaginary, therefore F is real.

We verify the Maxwell equations:

```
null > MatterFieldEquations("Electromagnetic", eta, F);
```

$$0\,E1,\,0\,\Theta 1 \wedge \Theta 2 \wedge \Theta 3 \quad (3.3.11)$$

We verify the Einstein equations:

```
null > evalDG(Gnull - EnergyMomentumTensor
       ("Electromagnetic", eta, F));
```

$$0\,E1 \otimes E1 \quad (3.3.12)$$

Example 2: A pure radiation spacetime

This example is taken from a paper by Lewandowski and Nurowski [3]. They relate pure radiation spacetimes with a shear-free, null, geodesic congruence to CR geometry, and exhibit all solutions with a minimum of three conformal Killing vector fields. One class of solutions has Petrov type II and a 3-dimensional conformal symmetry group of type VI (in the Bianchi classification of three-dimensional groups), and it is this class of solutions we consider here. We shall see that these pure radiation spacetimes cannot be electrovacua (for any values of the parameters).

Set-up.

Initialize the manifold and define various quantities which are used in [3] to construct the spacetime. We put an underscore in front of some of the quantities used in [3] so they don't get confused with Newman-Penrose spin coefficients later on.

```
M > restart: with(DifferentialGeometry): with(Tensor): with
    (Tools):
```

```
NullElectrovacuumConditions := proc(NPS, NPD)
```

```
> DGsetup([u, r, x, y], M);
```

$$\text{frame name: } M \tag{4.1.1}$$

```
M > Omega1 := evalDG(1/y*dx + I/y*dy + d/(d+1)*(y^d*du - 1/y*dx));
```

$$\Omega 1 := \frac{d\, y^d}{d+1}\, du + \frac{1}{(d+1)\, y}\, dx + \frac{I}{y}\, dy \tag{4.1.2}$$

```
M > Omega := evalDG(-2/(d+1)*(y^d*du - 1/y*dx));
```

$$\Omega := -\frac{2\, y^d}{d+1}\, du + \frac{2}{(d+1)\, y}\, dx \tag{4.1.3}$$

```
M > _alpha := -I/2*(1-d); _beta := -1/4*d; _theta := -I/4*d; a := I/8; b := 1/4; C := -1/4*I; w := w0*y^(3/4); W := 2*I*a*exp(I*r) + b;
```

$$\_\alpha := -\frac{I}{2}\,(1-d)$$

$$\_\beta := -\frac{d}{4}$$

$$\_\theta := -\frac{I}{4}\, d$$

$$a := \frac{I}{8}$$

$$b := \frac{1}{4}$$

$$C := -\frac{I}{4}$$

$$w := w0\, y^{3/4}$$

$$W := -\frac{e^{I r}}{4} + \frac{1}{4} \tag{4.1.4}$$

```
M > h := - 6*a*DGconjugate(a) + _alpha * DGconjugate(_alpha) - 1/2*_beta + I*(DGconjugate(_alpha)*b - _alpha*DGconjugate(b));
```

$$h := -\frac{11}{32} + \frac{(1-d)^2}{4} + \frac{3\, d}{8} \tag{4.1.5}$$

```
M > G := G1 + I*G2;
```

$$G := G1 + I\, G2 \tag{4.1.6}$$

```
M > G1 := simplify(1/2*(h + 2 *DGRe(a*(DGconjugate(_alpha - I * b))) + 4*C*DGconjugate(C)));
```

$$G1 := -\frac{1}{64} + \frac{d^2}{8} \tag{4.1.7}$$

```
M > H := factor(simplify(2*DGRe(G*exp(2*I*r)) + 2*DGRe((2*G - (DGconjugate(_alpha) + I*DGconjugate(b))*a)*exp(I*r)) + h,
```

```
symbolic));
```

$$H := -\frac{e^{2\,I\,r}}{64} + \frac{e^{2\,I\,r}\,d^2}{8} + I\,e^{2\,I\,r}\,G2 - \frac{e^{-2\,I\,r}}{64} + \frac{e^{-2\,I\,r}\,d^2}{8} + 2\,I\,e^{I\,r}\,G2 + \frac{e^{I\,r}}{16} + \frac{e^{I\,r}\,d^2}{4} - I\,e^{-2\,I\,r}\,G2 - \frac{e^{I\,r}\,d}{16} + \frac{e^{-I\,r}}{16} + \frac{e^{-I\,r}\,d^2}{4} - 2\,I\,e^{-I\,r}\,G2 - \frac{e^{-I\,r}\,d}{16} - \frac{3}{32} + \frac{d^2}{4} - \frac{d}{8} \quad \textbf{(4.1.8)}$$

The metric g is given in terms of an null coframe (e^1, e^2, e^3, e^4) by $g = 2\,e^1 \odot e^2 - 2e^3 \odot e^4$. The null coframe is:

```
M > e1 := evalDG(w/cos(1/2*r)*Omega1);
```

$$e1 := \frac{w0\,y^{\frac{3}{4}+d}\,d}{\cos\left(\frac{r}{2}\right)(d+1)}\,du + \frac{w0}{y^{1/4}\cos\left(\frac{r}{2}\right)(d+1)}\,dx + \frac{I\,w0}{y^{1/4}\cos\left(\frac{r}{2}\right)}\,dy \quad \textbf{(4.1.9)}$$

```
M > e2 := DGconjugate(e1);
```

$$e2 := \frac{w0\,y^{\frac{3}{4}+d}\,d}{\cos\left(\frac{r}{2}\right)(d+1)}\,du + \frac{w0}{y^{1/4}\cos\left(\frac{r}{2}\right)(d+1)}\,dx - \frac{I\,w0}{y^{1/4}\cos\left(\frac{r}{2}\right)}\,dy \quad \textbf{(4.1.10)}$$

```
M > e3 := evalDG(w^2/cos(1/2*r)^2*Omega);
```

$$e3 := -\frac{2\,w0^2\,y^{\frac{3}{2}+d}}{\cos\left(\frac{r}{2}\right)^2(d+1)}\,du + \frac{2\,w0^2\sqrt{y}}{\cos\left(\frac{r}{2}\right)^2(d+1)}\,dx \quad \textbf{(4.1.11)}$$

```
M > e4 := factor(simplify(evalDG(dr + 2*DGRe(W *Omega1) + H*
    Omega), symbolic));
```

$$e4 := -\frac{1}{d+1}\Bigg(\frac{I}{32}\Big(-8\,I\,e^{-2\,I\,r}\,d^2 - 16\,I\,d^2 - 16\,I\,e^{I\,r}\,d^2 - 16\,I\,e^{-I\,r}\,d^2 - 4\,I\,e^{I\,r} + I\,e^{-2\,I\,r} - 4\,I\,e^{-I\,r} - 8\,I\,e^{2\,I\,r}\,d^2 + 64\,e^{2\,I\,r}\,G2 - 64\,e^{-2\,I\,r}\,G2 + 128\,e^{I\,r}\,G2 - 128\,e^{-I\,r}\,G2 + 24\,I\,d + 4\,I\,e^{I\,r}\,d - 16\,I\,d\cos(r) + 4\,I\,e^{-I\,r}\,d + I\,e^{2\,I\,r} + 6\,I\Big)\,y^d\Bigg)\,du + dr + \frac{1}{(d+1)\,y}\Bigg(\frac{I}{32}\Big(-8\,I\,e^{-2\,I\,r}\,d^2 - 16\,I\,d^2 - 16\,I\,e^{I\,r}\,d^2 - 16\,I\,e^{-I\,r}\,d^2 + 16\,I\cos(r) - 4\,I\,e^{I\,r} + I\,e^{-2\,I\,r} + 64\,e^{2\,I\,r}\,G2 - 64\,e^{-2\,I\,r}\,G2 + 128\,e^{I\,r}\,G2 - 128\,e^{-I\,r}\,G2 - 4\,I\,e^{-I\,r} - 8\,I\,e^{2\,I\,r}\,d^2 + 8\,I\,d + 4\,I\,e^{I\,r}\,d + 4\,I\,e^{-I\,r}\,d + I\,e^{2\,I\,r} - 10\,I\Big)\Bigg)\,dx + \frac{\sin(r)}{2\,y}\,dy \quad \textbf{(4.1.12)}$$

```
> coFrame := [e1, e2, e3, e4]:
```

Here we check the reality properties of the coframe: e^1 is the conjugate of e^2, e^3 and

e^4 are real.

```
M > evalDG(DGconjugate(e1) - e2);
```

$$0 \tag{4.1.13}$$

```
M > DGIm(e3);
```

$$0\, du \tag{4.1.14}$$

```
M > DGIm(e4);
```

$$0\, du \tag{4.1.15}$$

This is the dual basis of vector fields.

```
M > Frame := simplify(factor(convert(DualBasis(coFrame), exp)
    ), symbolic);
```

$$Frame := \left[\frac{\cos\left(\frac{r}{2}\right) y^{-\frac{3}{4}-d}}{2\, w0} D\_u + \frac{\frac{I}{2} \sin\left(\frac{r}{2}\right) \cos\left(\frac{r}{2}\right) e^{\frac{I}{2} r}}{y^{3/4}\, w0} D\_r + \frac{y^{1/4} \cos\left(\frac{r}{2}\right)}{2\, w0} D\_x - \frac{\frac{I}{2} y^{1/4} \cos\left(\frac{r}{2}\right)}{w0} D\_y, \frac{\cos\left(\frac{r}{2}\right) y^{-\frac{3}{4}-d}}{2\, w0} D\_u - \frac{\frac{I}{2} \sin\left(\frac{r}{2}\right) \cos\left(\frac{r}{2}\right) e^{-\frac{I}{2} r}}{y^{3/4}\, w0} D\_r + \frac{y^{1/4} \cos\left(\frac{r}{2}\right)}{2\, w0} D\_x + \frac{\frac{I}{2} y^{1/4} \cos\left(\frac{r}{2}\right)}{w0} D\_y, -\frac{(\cos(r) + 1)\, y^{-d-\frac{3}{2}}}{4\, w0^2} D\_u - \frac{1}{y^{3/2}\, w0^2} \left(\frac{I}{64} \left(-8\, I\, e^{-2 I r} d^2 - 16\, I\, d^2 - 16\, I\, e^{I r} d^2 - 16\, I\, e^{-I r} d^2 - 4\, I\, e^{I r} + I\, e^{-2 I r} - 4\, I\, e^{-I r} + 64\, e^{2 I r} G2 - 64\, e^{-2 I r} G2 + 128\, e^{I r} G2 - 128\, e^{-I r} G2 - 8\, I\, e^{2 I r} d^2 + 8\, I\, d + 4\, I\, e^{I r} d + 4\, I\, e^{-I r} d + I\, e^{2 I r} + 6\, I \right) \cos\left(\frac{r}{2}\right)^2 \right) D\_r + \frac{\cos\left(\frac{r}{2}\right)^2 d}{2 \sqrt{y}\, w0^2} D\_x, D\_r \right] \tag{4.1.16}$$

From this frame we can construct a null tetrad which matches the conventions used in DifferentialGeometry. We will use this as an anholonomic frame. The anholonomic vector basis is denoted NT= [E1, E2, E3, E4] and the dual basis of 1-forms is denoted [$\Theta 1, \Theta 2, \Theta 3, \Theta 4$].

```
M > NT := [Frame[4], Frame[3], Frame[1], Frame[2]]:
M > FD := simplify(factor((map(convert,FrameData(NT, null),
     exp))), symbolic):
M > DGsetup(FD);
```

$$\textit{frame name: null} \tag{4.1.17}$$

This is the spacetime metric expressed in the null anholonomic coframe.

```
M > eta := evalDG(-2*Theta1 &s Theta2 + 2*Theta3 &s Theta4);
```

$$\eta := -\Theta 1 \otimes \Theta 2 - \Theta 2 \otimes \Theta 1 + \Theta 3 \otimes \Theta 4 + \Theta 4 \otimes \Theta 3 \tag{4.1.18}$$

Here we verify the Petrov type. (This computation takes a little time.)

```
null > PetrovType([E1, E2, E3, E4], [u = 0, r = 0, x = 0, y
       = 1]);
```

$$\text{"II"} \tag{4.1.19}$$

Adapted tetrad, spin coefficients and the integrability conditions.

Now we compute the Einstein tensor to verify that this is a pure radiation spacetime with the preferred null vector field k being parallel to $E1$. (This computation takes a little time.)

```
null > Ein := EinsteinTensor(eta):
null > Ein := factor(DGsimplify(simplify(convert(Ein, exp),
       symbolic)));
```

$$Ein := \frac{d\,(4\,d+1)\,(2\,d-1)\cos\left(\frac{r}{2}\right)^6}{32\,y^3\,w0^4}\,E1 \otimes E1 \tag{4.2.1}$$

```
null > chi := op(DGinfo(Ein, "CoefficientList", [E1 &t E1])
       );
```

$$\chi := \frac{d\,(4\,d+1)\,(2\,d-1)\cos\left(\frac{r}{2}\right)^6}{32\,y^3\,w0^4} \tag{4.2.2}$$

For this to be pure radiation spacetime the function χ must be positive.

The vector field k we use to define the null congruence is given by the following.

```
null > Kup := simplify(evalDG(2*sqrt(chi)*E1), symbolic);
```

$$Kup := \frac{\sqrt{2}\cos\left(\frac{r}{2}\right)^3 \sqrt{d}\,\sqrt{4\,d+1}\,\sqrt{2\,d-1}}{4\,y^{3\,/\,2}\,w0^2}\,E1 \tag{4.2.3}$$

We can create a null tetrad adapted to k by applying a suitable boost to the original null frame.

```
null > ANT := simplify(NullTetradTransformation([E1, E2,
       E3, E4], "boost", evalDG(2*sqrt(chi))), symbolic);
```

$$ANT := \left[\frac{\sqrt{2}\cos\left(\frac{r}{2}\right)^3 \sqrt{d}\,\sqrt{4\,d+1}\,\sqrt{2\,d-1}}{4\,y^{3\,/\,2}\,w0^2}\,E1, \frac{2\sqrt{2}\,y^{3\,/\,2}\,w0^2}{\cos\left(\frac{r}{2}\right)^3 \sqrt{d}\,\sqrt{4\,d+1}\,\sqrt{2\,d-1}}\,E2, E3, E4\right] \tag{4.2.4}$$

We check that this is in fact a null tetrad and that the first leg of this tetrad satisfies $G^{ab} = \frac{1}{4} k^a k^b$.

```
null > TensorInnerProduct(eta, ANT, ANT);
```

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.2.5}$$

```
null > evalDG(Ein - 1/4*ANT[1] &t ANT[1]);
```

$$0\,E1 \otimes E1 \tag{4.2.6}$$

The spin coefficients and directional derivatives for this adapted tetrad are computed in the following.

```
null > NPS2 := simplify(map(convert, NPSpinCoefficients
       (ANT), exp), symbolic):
null > NPD2 := NPDirectionalDerivatives(ANT):
```

We verify that the congruence generated by k is geodesic, shear-free and parametrized as in (2):

```
null > NPS2["kappa"];
```

$$0 \tag{4.2.7}$$

```
null > NPS2["sigma"];
```

$$0 \tag{4.2.8}$$

```
null > DGRe(NPS2["rho"] - 2*NPS2["epsilon"]);
```

$$0 \tag{4.2.9}$$

The twist of the congruence is given by $\omega = -\mathrm{Im}(\rho)$:

```
null > simplify(- DGIm(NPS2["rho"]));
```

$$\frac{\sqrt{2}\,\cos\left(\frac{r}{2}\right)^3 \sqrt{d}\,\sqrt{4\,d+1}\,\sqrt{2\,d-1}}{8\,y^{3/2}\,w0^2} \tag{4.2.10}$$

The twist is non-vanishing since $\chi > 0$. Therefore, this spacetime is a null electrovacuum if and only if $\mathcal{J} = 0$, with $\mathcal{J}$ defined in (4).

```
null > IC := NullElectrovacuumConditions(NPS2, NPD2):
null > factor(simplify(convert(IC, exp), symbolic));
```

$$\frac{-\frac{I}{256}\,d^{3/2}\,\sqrt{2}\,\sqrt{4\,d+1}\,\sqrt{2\,d-1}\,(4\,d-1)\,\cos\left(\frac{r}{2}\right)^6}{y^{15/4}\,w0^5} \tag{4.2.11}$$

This family of spacetimes does not admit a null electrovacuum.

References

1. Torre, C. G., "The Spacetime Geometry of a Null Electromagnetic Field", arXiv:1308.2323 (2013).
2. Stephani, H. Kramer, D MacCallum, M. Hoenselaers, C., and Herlt, E., *Exact Solutions to Einstein's Field Equations*. 2nd ed. (Cambridge Monographs on Mathematical Physics, 2003) .
3. Lewandowski, J. and Nurowski, P. *Class. Quantum Grav.* **7**, 309 (1990).
4. Nurowski, P. and Tafel, J., *Class. Quantum Grav.* **9**, 2069 (1992).

Release Notes

• The illustrated commands are all available in Maple 17 and subsequent releases.

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