

How To Find A Levi Decomposition of a Lie Algebra

Synopsis

- If g is a finite-dimensional Lie algebra, then the <u>radical</u> of g is the largest solvable subalgebra x contained in g. A fundamental theorem in the structure theory of Lie algebras asserts that there exists a complementary subalgebra S such that g is a semi-direct sum g = S ⊕ x. This decomposition is called a Levi decomposition. The subalgebra S is semi-simple.
- In this worksheet we show how to compute the Levi decomposition of a Lie algebra using the command <u>LeviDecomposition</u>.

Example

Load in the required packages.

> with(DifferentialGeometry): with(LieAlgebras):

We shall calculate the Levi decomposition for a 7-dimensional Lie algebra \mathfrak{g} . The structure equations of \mathfrak{g} are

> StructureEquations := [[x1, x2] = 2*x2, [x1, x3] = 2*x3, [x1, x6] = -2*x6, [x1, x7] = -2*x7, [x2, x4] = 2*x3, [x2, x6] = x1, [x2, x7] = x4, [x3, x5] = x3, [x3, x6] = -x4, [x4, x5] = x4, [x4, x6] = -2*x7, [x5, x7] = -x7];StructureEquations := [[x1, x2] = 2x2, [x1, x3] = 2x3, [x1, x6] = -2x6, [x1, x7] = -2x7, (1) [x2, x4] = 2x3, [x2, x6] = x1, [x2, x7] = x4, [x3, x5] = x3, [x3, x6] = -x4, [x4, x5] = x4, [x4, x5] = x4, [x4, x6] = -2x7, [x5, x7] = -x7]

Initialize this Lie algebra with the LieAlgebraData and DGsetup commands.

```
> LD := LieAlgebraData(StructureEquations, [x1, x2, x3, x4, x5, x6,
x7], alg1):
> DGsetup(LD);
Lie algebra: alg1 (2)
```

Calculate the Levi decomposition. In the output, the first list of vectors defines the radical and the second list of vectors defines the semi-simple subalgebra.

```
alg1 > Levi := LeviDecomposition();

Levi := [[e3, e4, e5, e7], [e1, e2, e6]]
(3)
```

We can use the <u>Query</u> command to verify that the span of the first set of vectors is a solvable ideal and that the span of the second list is a semi-simple algebra.

Remark. The structure of the Lie algebra becomes more transparent if we make a change of basis adapted to the Levi decomposition.

alg1 > NewBasis := [e1, e2, e6, e5, e7, e4, e3];

$$NewBasis := [e1, e2, e6, e5, e7, e4, e3]$$
(7)

Calculate the structure equations of the Lie algebra in the new basis.

```
alg1 > NewLD := LieAlgebraData(NewBasis, alg2):
```

Initialize the Lie algebra in the new basis. The new basis elements are labeled by X (with dual 1-forms labeled by omega).

```
alg1 > DGsetup(NewLD, [X], [omega]);
Lie algebra: alg2 (8)
```

```
alg2 > MultiplicationTable("LieTable");
                      Xl
                           X2
                                 X3
                                      X4
                                          X5
                                               X6
                                                    X7
                           ____
                                      ____
                                          ____
                                                    ____
              Xl
                      0
                           2 X2
                                -2X3 = 0
                                         -2 X5
                                                0
                                                   2 X7
              X2
                     -2 X2
                          0
                                 Xl
                                          X6
                                               2 X7
                                                    0
                                      0
              X3 |
                     2X3 - X1
                                 0
                                               2 X5
                                      0
                                          0
                                                   X6
                                                                        (9)
              X4 |
                     0
                          0
                                  0
                                    0
                                          -X5
                                               -X6 -X7
                     2 X5 -X6 0
              X5 |
                                      X5
                                           0
                                                0
                                                     0
                          -2 X7 -2 X5 X6
                      0
                                         0
                                                0
              X6
                                                     0
              X7
                     -2X7
                            0
                                 -X6 X7
                                           0
                                                0
                                                     0
```

We see that the semi-simple part of the Levi decomposition, spanned by $\{XI, X2, X3\}$, is sl(2) with Cartan subalgebra defined by XI. We see that the radical, $\{X4, X5, X6, X7\}$, decomposes into a 1-dimensional invariant subalgebra and a 3-dimensional irreducible subalgebra with highest weight vector X7. The vectors $\{X5, X6, X7\}$ span an abelian subalgebra and define the nilradical. These statements can all be verified with commands in

the LieAlgebra package.

Commands Illustrated

• DGsetup, LieAlgebraData, LeviDecomposition, Query

Related Commands

• Decompose, CartanSubalgebra, Nilradical, Series

References

- N. Jacobson , Lie Algebras, Dover, 1979
- V. S. Varadarajan, Lie Groups, Lie Algebras and their Representations Graduate Texts in Mathematics 102, Springer
- <u>http://en.wikipedia.org/wiki/Levi_decomposition</u>

Release Notes

• The illustrated commands are available in Maple 11 and subsequent releases.

Author

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