



Higher Order Symmetries of the KdV Equation

Synopsis

Recursion operators for the higher order symmetries (or generalized symmetries) of integrable equations such as the KdV equation are usually defined in terms of the formal inverse of the total derivative operator D_x . In this worksheet we shall:

- construct the operator using the [HorizontalHomotopy](#) command in the [JetCalculus](#) package;
- construct the recursion operator $\mathcal{R}(Q) = D_{xx}(Q) + \frac{2}{3}uQ + \frac{1}{3}u_x D_x^{-1}(Q)$ for the higher order symmetries of the KdV equation (Here Q is the characteristic function for the symmetry);
- use this recursion operator to generate the first 5 higher order symmetries of the KdV equation; and
- check that these higher order symmetries all commute.

The Program InverseTotalDiff

Load in the required packages.

```
[> with(DifferentialGeometry): with(JetCalculus):
```

The DG software supports 2 different notational conventions for coordinates on jet spaces. It is convenient here to use to the [Preferences](#) command to change to JetNotation2.

```
[> Preferences("JetNotation", "JetNotation2");  
                    "JetNotation1" (1.1)
```

The following calling sequence to [DGsetup](#) creates the 1st order jet space with 1 independent variable and 1 dependent variable u . Our jet notation is

$u = u[0], u_x = u[1], u_{xx} = u[2], \dots$

```
[> DGsetup([x], [u], Jet, 1);  
                    frame name: Jet (1.2)
```

In our subsequent calculations we shall require higher order derivatives of u . The order of the jet space will automatically be increased as needed.

The command [TotalDiff](#) calculates the total derivative of a function on jet space, for example

$$\text{Jet} > f := x*u[1]*u[2]^3; \quad f := x u_1 u_2^3 \quad (1.3)$$

$$\text{Jet} > g := \text{TotalDiff}(f, x); \quad g := u_1 u_2^3 + x u_2^4 + 3 x u_1 u_2^2 u_3 \quad (1.4)$$

$$\text{Jet} > k := \text{TotalDiff}(u[2], [2]); \quad k := u_4 \quad (1.5)$$

To re-construct the function from the function $g = D_x f$, we use to make a type (1,0) differential bi-form and apply the horizontal homotopy operator from the variational bicomplex.

$$\text{Jet} > \text{HorizontalHomotopy}(g*Dx); \quad x u_1 u_2^3 \quad (1.6)$$

Here is a 2-line program to calculate the inverse of the total derivative. (Click on this box and then enter control-E (PC) or command-E(Mac) to execute this code).

$$\text{Jet} > \text{InverseTotalDiff}(g); \quad x u_1 u_2^3 \quad (1.7)$$

The Recursion Operators for the KdV Equation and Its Higher Order Symmetries

Define the KdV recursion operator using the InverseTotalDiff command from the previous section. (The sort command is used on the next line to insure that the highest order term in the generalized symmetry appears last).

$$\begin{aligned} \text{M} > \text{KdVRecursion} &:= Q \rightarrow \text{sort}(\text{expand}(\text{TotalDiff}(Q, [2]) + 2/3*u[0] \\ &\quad *Q + 1/3*u[1]*\text{InverseTotalDiff}(Q))); \\ \text{KdVRecursion} &:= Q \mapsto \text{sort} \left(\text{expand} \left(\text{JetCalculus} : - \text{TotalDiff}(Q, [2]) + \frac{2 u_0 Q}{3} \right. \right. \\ &\quad \left. \left. + \frac{u_1 \text{InverseTotalDiff}(Q)}{3} \right) \right) \end{aligned} \quad (2.1)$$

Define the first order characteristic Q0 for the KdV equation (corresponding to the translational symmetry) and generate the higher symmetries using the recursion operator.

$$\text{M} > Q0 := u[1];$$

$$Q0 := u_1 \quad (2.2)$$

The 3rd order generalized symmetry is just the KdV flow itself:

```
M > Q1 := KdVRecursion(Q0);
```

$$Q1 := u_0 u_1 + u_3 \quad (2.3)$$

The 5rd order generalized symmetry is:

```
M > Q2 := KdVRecursion(Q1);
```

$$Q2 := \frac{5}{6} u_0^2 u_1 + \frac{5}{3} u_0 u_3 + \frac{10}{3} u_1 u_2 + u_5 \quad (2.4)$$

The 7th order generalized symmetry is:

```
M > Q3 := KdVRecursion(Q2);
```

$$Q3 := \frac{35}{54} u_0^3 u_1 + \frac{35}{18} u_0^2 u_3 + \frac{70}{9} u_0 u_1 u_2 + \frac{35}{18} u_1^3 + \frac{7}{3} u_0 u_5 + 7 u_1 u_4 + \frac{35}{3} u_2 u_3 + u_7 \quad (2.5)$$

The 9th order generalized symmetry is:

```
M > Q4 := sort(KdVRecursion(Q3));
```

$$Q4 := \frac{35}{72} u_0^4 u_1 + \frac{35}{18} u_0^3 u_3 + \frac{35}{3} u_0^2 u_1 u_2 + \frac{35}{6} u_0 u_1^3 + \frac{7}{2} u_0^2 u_5 + 21 u_0 u_1 u_4 + 35 u_0 u_2 u_3 + \frac{161}{6} u_1^2 u_3 + \frac{217}{6} u_1 u_2^2 + 3 u_0 u_7 + 12 u_1 u_6 + 28 u_2 u_5 + 42 u_3 u_4 + u_9 \quad (2.6)$$

Commutators of the Higher Order Symmetries

The command [GeneralizedLieBracket](#) can be used to show that the flows generated by the higher order symmetries all commute. Here are the evolutionary vector fields defined by the higher order symmetries calculated in the previous section.

```
Jet > F := evalDG([Q0*D_u[0], Q1*D_u[0], Q2*D_u[0], Q3*D_u[0], Q4*D_u[0]]);
```

$$F := \left[u_1 D_{-u_0}, (u_0 u_1 + u_3) D_{-u_0}, \left(\frac{5}{6} u_0^2 u_1 + \frac{5}{3} u_0 u_3 + \frac{10}{3} u_1 u_2 + u_5 \right) D_{-u_0}, \left(\frac{35}{54} u_0^3 u_1 + \frac{35}{18} u_0^2 u_3 + \frac{70}{9} u_0 u_1 u_2 + \frac{35}{18} u_1^3 + \frac{7}{3} u_0 u_5 + 7 u_1 u_4 + \frac{35}{3} u_2 u_3 \right) D_{-u_0}, \left(\frac{35}{72} u_0^4 u_1 + \frac{35}{18} u_0^3 u_3 + \frac{35}{3} u_0^2 u_1 u_2 + \frac{35}{6} u_0 u_1^3 + \frac{7}{2} u_0^2 u_5 + 21 u_0 u_1 u_4 + 35 u_0 u_2 u_3 + \frac{161}{6} u_1^2 u_3 + \frac{217}{6} u_1 u_2^2 + 3 u_0 u_7 + 12 u_1 u_6 + 28 u_2 u_5 + 42 u_3 u_4 + u_9 \right) D_{-u_0} \right] \quad (3.1)$$

$$\begin{aligned}
& + u_7) D_{-u_0}, \left(\frac{35}{72} u_0^4 u_1 + \frac{35}{18} u_0^3 u_3 + \frac{35}{3} u_0^2 u_1 u_2 + \frac{35}{6} u_0 u_1^3 + \frac{7}{2} u_0^2 u_5 \right. \\
& + 21 u_0 u_1 u_4 + 35 u_0 u_2 u_3 + \frac{161}{6} u_1^2 u_3 + \frac{217}{6} u_1 u_2^2 + 3 u_0 u_7 + 12 u_1 u_6 \\
& \left. + 28 u_2 u_5 + 42 u_3 u_4 + u_9 \right) D_{-u_0} \Big]
\end{aligned}$$

The flows F[1] and F[2] commute:

$$\text{Jet} > \text{GeneralizedLieBracket}(\text{F}[1], \text{F}[2]); \quad 0 D_x \quad (3.2)$$

We can use the Matrix command to calculate all the commutators:

$$\text{Jet} > \text{interface}(\text{typesetting}=\text{standard}); \quad \text{extended} \quad (3.3)$$

$$\text{Jet} > \text{Matrix}(5, 5, (i,j) \rightarrow \text{GeneralizedLieBracket}(\text{F}[i], \text{F}[j])); \quad (3.4)$$

$$\begin{bmatrix}
0 D_x & 0 D_x & 0 D_x & 0 D_x & 0 D_x \\
0 D_x & 0 D_x & 0 D_x & 0 D_x & 0 D_x \\
0 D_x & 0 D_x & 0 D_x & 0 D_x & 0 D_x \\
0 D_x & 0 D_x & 0 D_x & 0 D_x & 0 D_x \\
0 D_x & 0 D_x & 0 D_x & 0 D_x & 0 D_x
\end{bmatrix}$$

Commands Illustrated

- [JetCalculus](#), [GeneralizedLieBracket](#), [HorizontalHomotopy](#), [TotalDiff](#)

Related Commands

- [EulerLagrange](#), [EvolutionaryVector](#), [Prolong](#), [Noether](#)

References

- P. J. Olver, *Applications of Lie Groups to Differential Equations* (2nd edition) pages 312-315.

Release Notes

- The illustrated commands are available in Maple 11 and subsequent releases.

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