The Octonions and the Exceptional Lie Algebra $\mathfrak{g}_2$

Synopsis

- The octonions $\mathbb{O}$ are an 8-dimensional non-commutative, non-associative normed real algebra. The derivations of $\mathbb{O}$ are the linear maps $\phi : \mathbb{O} \to \mathbb{O}$ such that $\phi(x \cdot y) = \phi(x) \cdot y + x \cdot \phi(y)$. The set of all derivations form a real Lie algebra. It is remarkable fact, first proved by E. Cartan in 1908, that the the derivation algebra of $\mathbb{O}$ is the compact form of the exceptional Lie algebra $\mathfrak{g}_2$.
- In this worksheet we shall verify this result of Cartan and also show that the derivation algebra of the split octonions is the split real form of the exceptional Lie algebra $\mathfrak{g}_2$.

```
> with(DifferentialGeometry): with(LieAlgebras):
```

The Compact Form of $\mathfrak{g}_2$ as the Derivation Algebra of the Octonions

We use the AlgebraLibraryData command to obtain the multiplication rules for the octonions. We shall refer to this algebra as *oct*

```
> AD := AlgebraLibraryData("Octonions", oct):
```

We use the DGsetup command to initialize this algebra.

```
> DGsetup(AD,'[e0, e1, e2, e3, e4, e5, e6, e7]', '[omega]');
```

algebra name: oct **(1.1)**

The multiplication table for this algebra is

```
oct > MultiplicationTable();
```

(1.2)

$$\begin{bmatrix} & | & e0 & e1 & e2 & e3 & e4 & e5 & e6 & e7 \\ & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} \\ e0 & | & e0 & e1 & e2 & e3 & e4 & e5 & e6 & e7 \\ e1 & | & e1 & -e0 & e3 & -e2 & e5 & -e4 & -e7 & e6 \\ e2 & | & e2 & -e3 & -e0 & e1 & e6 & e7 & -e4 & -e5 \\ e3 & | & e3 & e2 & -e1 & -e0 & e7 & -e6 & e5 & -e4 \\ e4 & | & e4 & -e5 & -e6 & -e7 & -e0 & e1 & e2 & e3 \\ e5 & | & e5 & e4 & -e7 & e6 & -e1 & -e0 & -e3 & e2 \\ e6 & | & e6 & e7 & e4 & -e5 & -e2 & e3 & -e0 & -e1 \\ e7 & | & e7 & -e6 & e5 & e4 & -e3 & -e2 & e1 & -e0 \end{bmatrix} \tag{1.2}$$

It matches that obtained from the Cayley-Dickson construction.

The algebra of derivations of the octonions is a matrix algebra. A basis of matrices for the derivations is computed here.

```
oct > aut := Derivations():
oct > aut[3];
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.3}$$

The derivation algebra is 14-dimensional.

```
oct > nops(aut);
```

$$14 \tag{1.4}$$

We use the command LieAlgebraData to find the structure equations for the Lie algebra and initialize it.

```
oct > LDder := LieAlgebraData(aut, alg);
```

$$LDder := [[e1, e2] = -e7, [e1, e3] = -e8, [e1, e4] = -e5 - e9, [e1, e5] = e4 - e10, [e1, e6] = -e11, [e1, e7] = e2, [e1, e8] = e3, [e1, e9] = e4 - e10, [e1, e10] = e5 + e9, [e1, e11] = e6, [e1, e12] = -e13, [e1, e13] = e12, [e2, e3] = -e5, [e2, e4] = -2\, e6, [e2, e5] \tag{1.5}$$

$= e3, [e2, e6] = 2\, e4, [e2, e7] = -e1, [e2, e8] = e4, [e2, e9] = -e3 - e11, [e2, e10] = -e6, [e2, e11] = e5 + e9, [e2, e12] = -e14, [e2, e14] = e12, [e3, e4] = -e12, [e3, e5] = -2\, e2 - 2\, e13, [e3, e6] = -e14, [e3, e7] = e10, [e3, e8] = -e1, [e3, e9] = e2 + e13, [e3, e10] = -e7, [e3, e12] = e4, [e3, e13] = e5, [e3, e14] = e6, [e4, e5] = -e14, [e4, e6] = -2\, e2, [e4, e7] = e11, [e4, e8] = -e2, [e4, e9] = -e1 + e14, [e4, e11] = -e7, [e4, e12] = -e3, [e4, e13] = -e6, [e4, e14] = e5, [e5, e6] = -e12, [e5, e7] = e6 - e8, [e5, e8] = e7 - e12, [e5, e10] = -e1 + e14, [e5, e11] = -e2 - e13, [e5, e12] = e6, [e5, e13] = -e3, [e5, e14] = -e4, [e6, e7] = -e5 - e9, [e6, e9] = e7 - e12, [e6, e10] = e2, [e6, e11] = -e1, [e6, e12] = -e5, [e6, e13] = e4, [e6, e14] = -e3, [e7, e8] = e9, [e7, e9] = -e8, [e7, e10] = -2\, e11, [e7, e11] = 2\, e10, [e7, e13] = -e14, [e7, e14] = e13, [e8, e9] = 2\, e7 - 2\, e12, [e8, e10] = -e13, [e8, e11] = -e14, [e8, e12] = e9, [e8, e13] = e10, [e8, e14] = e11, [e9, e10] = -e14, [e9, e11] = e13, [e9, e12] = -e8, [e9, e13] = -e11, [e9, e14] = e10, [e10, e11] = -2\, e7, [e10, e12] = e11, [e10, e13] = -e8, [e10, e14] = -e9, [e11, e12] = -e10, [e11, e13] = e9, [e11, e14] = -e8, [e12, e13] = -2\, e14, [e12, e14] = 2\, e13, [e13, e14] = -2\, e12]$

```
oct > DGsetup(LDder);
```

$$\text{Lie algebra: } alg \tag{1.6}$$

Now we show that this 14-dimensional Lie algebra is the compact real form for $\mathfrak{g}_2$. First we note (see Query) that it is semi-simple.

```
alg > Query("Semisimple");
```

$$true \tag{1.7}$$

We now classify this algebra using the structure theory of semi-simple Lie algebras. This involves three steps. First, find a Cartan subalgebra. Second, find the corresponding root space decomposition. Third, identify the simple roots and calculate the Cartan matrix.

We find a Cartan subalgebra. It has dimension 2 so that the derivation algebra alg has rank 2.

```
alg > CSA := CartanSubalgebra(alg);
```

$$CSA := [e1, e4 + e10] \tag{1.8}$$

We see that all the roots in the root space decomposition are pure imaginary numbers. This shows that the Lie algebra is of compact type.

```
alg > RSD := RootSpaceDecomposition(CSA);
```

$$RSD := table([[-\mathrm{I}, -3\,\mathrm{I}] = e2 + \mathrm{I}\, e6 - \mathrm{I}\, e7 + e11, [0, 2\,\mathrm{I}] = e1 + \mathrm{I}\, e5 - \mathrm{I}\, e9 - 2\, e14, [\mathrm{I}, \mathrm{I}] = e2 + 2\, e3 - \mathrm{I}\, e6 + \mathrm{I}\, e7 + 2\,\mathrm{I}\, e8 + e11 - 2\,\mathrm{I}\, e12 + 2\, e13, [0, -2\,\mathrm{I}] = e1 - \mathrm{I}\, e5 + \mathrm{I}\, e9 - 2\, e14, [2\,\mathrm{I}, 0] = e4 + \mathrm{I}\, e5 + \mathrm{I}\, e9 - e10, [\mathrm{I}, -3\,\mathrm{I}] = e2 + \mathrm{I}\, e6 + \mathrm{I}\, e7 - e11, [\mathrm{I}, -\mathrm{I}] = e2 - 2\, e3 + \mathrm{I}\, e6 + \mathrm{I}\, e7 - 2\,\mathrm{I}\, e8 - e11 - 2\,\mathrm{I}\, e12 + 2\, e13, [\mathrm{I}, 3\,\mathrm{I}] = e2 - \mathrm{I}\, e6 + \mathrm{I}\, e7 + e11, [-\mathrm{I}, -\mathrm{I}] = e2 + 2\, e3 + \mathrm{I}\, e6 - \mathrm{I}\, e7 - 2\,\mathrm{I}\, e8 + e11 + 2\,\mathrm{I}\, e12 + 2\, e13, [\tag{1.9}$$

$$-2\,\mathrm{I}, 0] = e4 - \mathrm{I}\,e5 - \mathrm{I}\,e9 - e10, [-\mathrm{I}, \mathrm{I}] = e2 - 2\,e3 - \mathrm{I}\,e6 - \mathrm{I}\,e7 + 2\,\mathrm{I}\,e8 - e11 + 2\,\mathrm{I}\,e12 + 2\,e13, [-\mathrm{I}, 3\,\mathrm{I}] = e2 - \mathrm{I}\,e6 - \mathrm{I}\,e7 - e11])$$

Finally, we find the simple roots and the Cartan matrix to conclude that the derivation algebra of the octonions is the compact real form of $\mathfrak{g}_2$.

```
alg > SimRts := SimpleRoots(RSD, [<I,0>, <0, I>]);
```

$$SimRts := \left[\begin{bmatrix} 0 \\ 2\,\mathrm{I} \end{bmatrix}, \begin{bmatrix} \mathrm{I} \\ -3\,\mathrm{I} \end{bmatrix}\right] \tag{1.10}$$

```
alg > CM := CartanMatrix(SimRts, RSD);
```

$$CM := \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \tag{1.11}$$

```
alg > CartanMatrixToStandardForm(CM);
```

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{"G"} \tag{1.12}$$

As an alternative demonstration that alg is of compact type, we find the Killing form and use the QuadraticFormSignature command from the Tensor package to provide that it is negative-definite.

```
alg > Q := KillingForm();
```

$$Q := -8\,\theta6\,\theta8 - 16\,\theta7\,\theta7 - 8\,\theta7\,\theta12 - 8\,\theta8\,\theta6 - 16\,\theta1\,\theta1 - 8\,\theta1\,\theta14 - 16\,\theta2\,\theta2 + 8\,\theta2\,\theta13 - 16\,\theta3\,\theta3 + 8\,\theta3\,\theta11 - 16\,\theta4\,\theta4 - 8\,\theta4\,\theta10 - 16\,\theta5\,\theta5 + 8\,\theta5\,\theta9 - 16\,\theta6\,\theta6 - 16\,\theta8\,\theta8 + 8\,\theta9\,\theta5 - 16\,\theta9\,\theta9 - 8\,\theta10\,\theta4 - 16\,\theta10\,\theta10 + 8\,\theta11\,\theta3 - 16\,\theta11\,\theta11 - 8\,\theta12\,\theta7 - 16\,\theta12\,\theta12 + 8\,\theta13\,\theta2 - 16\,\theta13\,\theta13 - 8\,\theta14\,\theta1 - 16\,\theta14\,\theta14 \tag{1.13}$$

```
alg > PosNegNull := Tensor:-QuadraticFormSignature(Q);
```

$$PosNegNull := [[\ ], [e1, e1 - 2\,e14, e2, e2 + 2\,e13, e3, e3 + 2\,e11, e4, e4 - 2\,e10, e5, e5 + 2\,e9, e6, e6 - 2\,e8, e7, e7 - 2\,e12], [\ ]] \tag{1.14}$$

```
alg > map(nops, PosNegNull);
```

$$[0, 14, 0] \tag{1.15}$$

The Split Real Form of $\mathfrak{g}_2$ as the Derivation Algebra of the Split Octonions

We repeat the calculations of the previous section using the split octonions in place of the octonions. We use the AlgebraData command to obtain the multiplication rules for the split octonions. We shall refer to this algebra as octS.

```
> LD := AlgebraLibraryData("Octonions", octS, type = "Split");
```

$$LD := [e1^2 = e1, e1.e2 = e2, e1.e3 = e3, e1.e4 = e4, e1.e5 = e5, e1.e6 = e6, e1.e7 = e7, e1.e8 = e8, e2.e1 = e2, e2^2 = -e1, e2.e3 = e4, e2.e4 = -e3, e2.e5 = -e6, e2.e6 = e5, e2.e7 = \tag{2.1}$$

$-e8, e2.e8 = e7, e3.e1 = e3, e3.e2 = -e4, e3^2 = -e1, e3.e4 = e2, e3.e5 = -e7, e3.e6 = e8,$
$e3.e7 = e5, e3.e8 = -e6, e4.e1 = e4, e4.e2 = e3, e4.e3 = -e2, e4^2 = -e1, e4.e5 = -e8, e4$
$.e6 = -e7, e4.e7 = e6, e4.e8 = e5, e5.e1 = e5, e5.e2 = e6, e5.e3 = e7, e5.e4 = e8, e5^2 = e1,$
$e5.e6 = e2, e5.e7 = e3, e5.e8 = e4, e6.e1 = e6, e6.e2 = -e5, e6.e3 = -e8, e6.e4 = e7, e6.e5$
$= -e2, e6^2 = e1, e6.e7 = e4, e6.e8 = -e3, e7.e1 = e7, e7.e2 = e8, e7.e3 = -e5, e7.e4 =$
$-e6, e7.e5 = -e3, e7.e6 = -e4, e7^2 = e1, e7.e8 = e2, e8.e1 = e8, e8.e2 = -e7, e8.e3 = e6,$
$e8.e4 = -e5, e8.e5 = -e4, e8.e6 = e3, e8.e7 = -e2, e8^2 = e1]$

We use the DGsetup command to initialize this algebra.

```
> DGsetup(LD,'[s0, s1, s2, s3, s4, s5, s6, s7]', '[omega]');
```

algebra name: octS **(2.2)**

Here are the multiplication tables for the octonions and the split octonions -- for the split octonions the square of each of the last 4 basis vectors is 1 not -1.

```
octS > MultiplicationTable(oct), MultiplicationTable(octS);
```

$$\left[\begin{array}{c|cccccccc} & e0 & e1 & e2 & e3 & e4 & e5 & e6 & e7 \\ \hline e0 & e0 & e1 & e2 & e3 & e4 & e5 & e6 & e7 \\ e1 & e1 & -e0 & e3 & -e2 & e5 & -e4 & -e7 & e6 \\ e2 & e2 & -e3 & -e0 & e1 & e6 & e7 & -e4 & -e5 \\ e3 & e3 & e2 & -e1 & -e0 & e7 & -e6 & e5 & -e4 \\ e4 & e4 & -e5 & -e6 & -e7 & -e0 & e1 & e2 & e3 \\ e5 & e5 & e4 & -e7 & e6 & -e1 & -e0 & -e3 & e2 \\ e6 & e6 & e7 & e4 & -e5 & -e2 & e3 & -e0 & -e1 \\ e7 & e7 & -e6 & e5 & e4 & -e3 & -e2 & e1 & -e0 \end{array}\right], \quad \textbf{(2.3)}$$

$$\begin{array}{cccccccccc} & | & s0 & s1 & s2 & s3 & s4 & s5 & s6 & s7 \\ & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} \\ s0 & | & s0 & s1 & s2 & s3 & s4 & s5 & s6 & s7 \\ s1 & | & s1 & -s0 & s3 & -s2 & -s5 & s4 & -s7 & s6 \\ s2 & | & s2 & -s3 & -s0 & s1 & -s6 & s7 & s4 & -s5 \\ s3 & | & s3 & s2 & -s1 & -s0 & -s7 & -s6 & s5 & s4 \\ s4 & | & s4 & s5 & s6 & s7 & s0 & s1 & s2 & s3 \\ s5 & | & s5 & -s4 & -s7 & s6 & -s1 & s0 & s3 & -s2 \\ s6 & | & s6 & s7 & -s4 & -s5 & -s2 & -s3 & s0 & s1 \\ s7 & | & s7 & -s6 & s5 & -s4 & -s3 & s2 & -s1 & s0 \end{array}$$

The algebra of derivations is a matrix algebra of 8×8 matrices. We do not display all the matrices.

```
octS > autS := Derivations(octS):
octS > autS[3];
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.4}$$

The derivation algebra is 14-dimensional.

```
octS > nops(autS);
```

$$14 \tag{2.5}$$

We find the structure equations for the Lie algebra and initialize.

```
octS > LDS := LieAlgebraData(autS, algS);
```

$$LDS := [[e1, e2] = -e7, [e1, e3] = -e8, [e1, e4] = -e5 - e9, [e1, e5] = e4 - e10, [e1, e6] = -e11, [e1, e7] = e2, [e1, e8] = e3, [e1, e9] = e4 - e10, [e1, e10] = e5 + e9, [e1, e11] = e6, [e1, e12] = -e13, [e1, e13] = e12, [e2, e3] = e5, [e2, e4] = -2\, e6, [e2, e5] = -e3, [e2, e6] = 2\, e4, [e2, e7] = -e1, [e2, e8] = -e4, [e2, e9] = e3 - e11, [e2, e10] = -e6, [e2, e11] = e5 + e9, [e2, e12] = -e14, [e2, e14] = e12, [e3, e4] = e12, [e3, e5] = -2\, e2 + 2\, e13, [e3, e6] = e14, [e3, e7] = -e10, [e3, e8] = e1, [e3, e9] = e2 - e13, \tag{2.6}$$

$[e3, e10] = -e7, [e3, e12] = e4, [e3, e13] = e5, [e3, e14] = e6, [e4, e5] = -e14, [e4, e6] = 2\, e2, [e4, e7] = e11, [e4, e8] = -e2, [e4, e9] = e1 + e14, [e4, e11] = e7, [e4, e12] = -e3, [e4, e13] = e6, [e4, e14] = -e5, [e5, e6] = -e12, [e5, e7] = e6 + e8, [e5, e8] = e7 + e12, [e5, e10] = e1 + e14, [e5, e11] = e2 - e13, [e5, e12] = -e6, [e5, e13] = -e3, [e5, e14] = e4, [e6, e7] = -e5 - e9, [e6, e9] = -e7 - e12, [e6, e10] = -e2, [e6, e11] = e1, [e6, e12] = e5, [e6, e13] = -e4, [e6, e14] = -e3, [e7, e8] = -e9, [e7, e9] = e8, [e7, e10] = -2\, e11, [e7, e11] = 2\, e10, [e7, e13] = -e14, [e7, e14] = e13, [e8, e9] = 2\, e7 + 2\, e12, [e8, e10] = e13, [e8, e11] = e14, [e8, e12] = e9, [e8, e13] = e10, [e8, e14] = e11, [e9, e10] = -e14, [e9, e11] = e13, [e9, e12] = -e8, [e9, e13] = e11, [e9, e14] = -e10, [e10, e11] = 2\, e7, [e10, e12] = -e11, [e10, e13] = -e8, [e10, e14] = e9, [e11, e12] = e10, [e11, e13] = -e9, [e11, e14] = -e8, [e12, e13] = 2\, e14, [e12, e14] = -2\, e13, [e13, e14] = 2\, e12]$

```
octS > DGsetup(LDS);
```

$$\text{Lie algebra: } algS \tag{2.7}$$

Now we show that this 14 dimensional Lie algebra is the split real form for $\mathfrak{g}_2$. First we note that it is semi-simple.

```
algS > Query("Semisimple");
```

$$true \tag{2.8}$$

Now we find the Killing form and use the QuadraticFormSignature command from the Tensor package to provide that it is split signature $[8, 6]$.

```
algS > Q := KillingForm();
```

$$Q := 8\,\theta 14\,\theta 1 - 16\,\theta 14\,\theta 14 + 8\,\theta 12\,\theta 7 - 16\,\theta 1\,\theta 1 + 8\,\theta 1\,\theta 14 - 16\,\theta 2\,\theta 2 - 8\,\theta 2\,\theta 13 + 16\,\theta 3\,\theta 3 + 8\,\theta 3\,\theta 11 + 16\,\theta 4\,\theta 4 + 8\,\theta 4\,\theta 10 + 16\,\theta 5\,\theta 5 - 8\,\theta 5\,\theta 9 + 16\,\theta 6\,\theta 6 - 8\,\theta 6\,\theta 8 - 16\,\theta 7\,\theta 7 + 8\,\theta 7\,\theta 12 - 8\,\theta 8\,\theta 6 + 16\,\theta 8\,\theta 8 - 8\,\theta 9\,\theta 5 + 16\,\theta 9\,\theta 9 + 8\,\theta 10\,\theta 4 + 16\,\theta 10\,\theta 10 + 8\,\theta 11\,\theta 3 + 16\,\theta 11\,\theta 11 - 16\,\theta 12\,\theta 12 - 16\,\theta 13\,\theta 13 - 8\,\theta 13\,\theta 2 \tag{2.9}$$

```
algS > PosNegNull := Tensor:-QuadraticFormSignature(Q);
```

$$PosNegNull := [[e3, e3 - 2\, e11, e4, e4 - 2\, e10, e5, e5 + 2\, e9, e6, e6 + 2\, e8], [e1, e1 + 2\, e14, e2, e2 - 2\, e13, e7, e7 + 2\, e12], [\ ]] \tag{2.10}$$

```
algS > map(nops, PosNegNull);
```

$$[8, 6, 0] \tag{2.11}$$

We find a Cartan subalgebra. It has dimension 2 so that our derivation algebra has rank 2. Since we want a Cartan subalgebra which gives a real root space decomposition, we first use the matrices *autS,* which define a respresentation of our algebra, to calculate a Cartan decomposition.

```
algS > T, P := CartanDecomposition(autS, algS);
```

(2.12)

$$T, P := [e1, e2, e7, e12, e13, e14], [e3, e4, e5, e6, e8, e9, e10, e11] \quad (2.12)$$

A 2-dimensional Abelian subalgebra of P will give us a Cartan subalgebra. We pick an element of P and calculate its centralizer.

```
algS > Centralizer([e3]);
```

$$\left[e2 + e13, e11, \frac{1}{2} e5 + e9, e3\right] \quad (2.13)$$

We see that $e_{11} \in P$ so here is our Cartan subalgebra.

```
algS > CSA := CartanSubalgebra([e3, e11]);
```

$$CSA := [e3, e11] \quad (2.14)$$

We see that all the roots in the root space decomposition are real. This affirms that the Lie algebra is a split real form.

```
algS > RSD := RootSpaceDecomposition(CSA);
```

$$RSD := table\left(\left[[-1, -2] = e7 + e10, [-1, 0] = e4 - \frac{1}{2} e7 - \frac{1}{2} e10 - e12, [1, 1] = e1 - e6 + e8 - e14, [1, 0] = e4 + \frac{1}{2} e7 - \frac{1}{2} e10 + e12, [0, -1] = e2 + e5 + 2\, e9 + e13, [1, -1] = e1 + e6 + e8 + e14, [-2, -1] = e2 + e5 - e13, [2, 1] = e2 - e5 - e13, [1, 2] = e7 - e10, [0, 1] = e2 - e5 - 2\, e9 + e13, [-1, 1] = e1 - e6 - e8 + e14, [-1, -1] = e1 + e6 - e8 - e14]\right) \quad (2.15)$$

Finally, we find the simple roots and the Cartan matrix to conclude that the derivation algebra of the split octonions is the split real form of $\mathfrak{g}_2$.

```
algS > SimRts := SimpleRoots(RSD, [<1, 0>, <0, 1>]);
```

$$SimRts := \left[\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] \quad (2.16)$$

```
algS > CM := CartanMatrix(SimRts, RSD);
```

$$CM := \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \quad (2.17)$$

```
algS > CartanMatrixToStandardForm(CM);
```

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{"G"} \quad (2.18)$$

Commands Illustrated

AlgebraData, CartanMatrix, CartanMatrixToStandardForm, CartanSubalgebra, Derivations, KillingForm , LieAlgebraData, MultiplicationTable, QuadraticFormSignature, RootSpaceDecomposition, SimpleRoots

References

1. John Baez, *The octonions,* Bull. Amer. Math. Soc. **39**(2002), 145-205
2. Elie Cartan, *Nombres complexes,* in Encyclopedie des sciences mathematiques, 1, ed. J. Molk, 1908, 329--468
3. W. Fulton and J. Harris, *Representation Theory,* Graduate Texts in Mathematics 129, Springer, 1991
4. http://en.wikipedia.org/wiki/Octonion, http://en.wikipedia.org/wiki/Split-octonion

Release Notes

• The illustrated commands are available in Maple 17 and subsequent releases.

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