09 The Wave Equation in 3 Dimensions

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We now turn to the 3-dimensional version of the wave equation, which can be used to describe a variety of wavelike phenomena, e.g., sound waves and electromagnetic waves. One could derive this version of the wave equation much as we did the one-dimensional version by generalizing our line of coupled oscillators to a 3-dimensional array of oscillators. For many purposes, e.g., modeling propagation of sound, this provides a useful discrete model of a three-dimensional solid. We won’t be able to go into that here. The point is, though, that if we take the continuum limit as before we end up with the 3-dimensional wave equation for the displacement \( q(r, t) \) of the oscillator-medium at the point labeled \( r = (x, y, z) \) at time \( t \):

\[
\frac{1}{v^2} \frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}. \tag{9.1}
\]

The 3-dimensional wave equation is a linear, homogeneous partial differential equation with constant coefficients. Note that if we assume the wave displacement does not depend upon two of the three independent variables, e.g., \( q = q(x, t) \) we end up with the one-dimensional wave equation (exercise).

9.1 Gradient, Divergence and Laplacian

The right-hand side of (9.1) represents a very important differential operator, known as the Laplacian, so let us take a moment to discuss it. The Laplacian itself can be viewed as the composition of two other operators known as the gradient and the divergence, which we will also briefly discuss.

To begin, let \( f(r) = f(x, y, z) \) be a function of three variables. The gradient of \( f \) is a vector field, i.e., a vector at each position \( r \), denoted \( \nabla f \), defined by

\[
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}, \tag{9.2}
\]

where \( \hat{i}, \hat{j}, \hat{k} \) are the orthogonal unit vector fields in the \( x, y, z \) directions, respectively.* What is the meaning of this vector field? Clearly at any given point, the \( x, y, z \) components of \( \nabla f \) give the rate of change of \( f \) at that point in the \( x, y, z \) directions, respectively; the other variables being held fixed. More generally, at a given point, say, \( r_0 \), a unit vector \( \hat{n} \) defines the rate of change of \( f \) in the direction of \( \hat{n} \) via the directional derivative along \( \hat{n} \), defined by \( n \cdot \nabla f(r_0) \). The direction of the gradient of \( f \) at any given point is the direction

\[
\nabla f \equiv \text{grad} f. \tag{9.3}
\]

* Some texts denote the gradient operation by “grad”: \( \nabla f \equiv \text{grad} f \).
in which the function \( f \) has the greatest rate of change at that point, while the magnitude of the gradient (at the given point) is that rate of change. (Prove these last two facts as a nice exercise.) An important geometric feature of the gradient of a function is that \( \nabla f \) is always perpendicular to the surfaces \( f = \text{constant} \). This you will study in a homework problem. The gradient is an example of a linear differential operator.

The Laplacian is a combination of the gradient with another linear differential operator, called the *divergence*, which makes a function out of a vector field. Let

\[
\mathbf{V} = V^x \mathbf{i} + V^y \mathbf{j} + V^z \mathbf{k}
\]

be a vector field (each of the three components \( (V^x, V^y, V^z) \) is a function of \( \mathbf{r} \)). The divergence, denoted \( \nabla \cdot \mathbf{V} \), is the function*

\[
\nabla \cdot \mathbf{V} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} + \frac{\partial V^z}{\partial z}.
\]

You may have encountered the divergence in a discussion of the Gauss law of electromagnetism. Often, Gauss law is formulated in terms of the electric flux through a closed surface. But if you take a spherical surface and shrink the surface to infinitesimal size at a point, the divergence of the vector field at that point is what you get in the limit. More precisely, consider a small closed (“Gaussian”) surface enclosing a point. One can compute the flux of the vector field \( \mathbf{V} \) through this surface (just as one computes electric flux). Now consider shrinking this surface to the chosen point. The limit as one shrinks the surface to the point of the flux divided by the enclosed volume is precisely the divergence of the vector field at that point. So, you can think of the divergence as a sort of flux per unit volume.

It is now easy to see that the right-hand side of the wave equation (9.1) is the divergence of the gradient of the function \( q(\mathbf{r}, t) \) with \( t \) held fixed. The resulting differential operator, shown on the right-hand side of (9.1), is denoted by \( \triangle f \) or \( \nabla^2 f \) and is the Laplacian:

\[
\nabla^2 f := \nabla \cdot (\nabla f) = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}.
\]

The Laplacian is a linear differential operator that takes a (twice differentiable) function and produces another function.

The wave equation can thus be compactly written in terms of the Laplacian as

\[
\frac{1}{v^2} \frac{\partial^2 q}{\partial t^2} = \nabla^2 q.
\]

* In some texts one denotes \( \nabla \cdot \) by “div”:

\[
\nabla \cdot \mathbf{V} \equiv \text{div} \mathbf{V}.
\]
9.2 Solutions to the Three-Dimensional Wave Equation

Solutions of the 3-dimensional wave equation (9.7) are not any harder to come by than those of the 1-dimensional wave equation. Indeed, if we look for solutions that are independent of \( y \) and \( z \), we recover the solutions obtained for the 1-dimensional equation. So, for example, the (complex) wave solution

\[
q(r, t) = Ae^{ik(x-\nu t)},
\]

satisfies the 3-dimensional wave equation (exercise) for any real number \( k \). This is a (complex) wave traveling in the \( x \) direction with wavelength \( \lambda = 2\pi/|k| \) and speed \( \nu \). Of course, we can equally well write down solutions corresponding to traveling waves in the \( y \) and \( z \) directions (exercise). More generally, a (complex) wave with wavelength \( \lambda = 2\pi/|k| \) propagating in the direction \( \mathbf{k} \) with speed \( \nu \) is given by

\[
q(r, t) = Ae^{i(k \cdot r - \omega t)}, \quad \omega = |k|\nu.
\]

To check this you can just plug (9.9) into (9.7) and check that the wave equation is satisfied. Alternatively, you can use the following (slightly tricky) argument, which you should ponder as an exercise. The dot product \( \mathbf{k} \cdot \mathbf{r} \) is a scalar that is defined geometrically, that is, independently of the orientation of the \((x, y, z)\) axes. Likewise, the Laplacian is the same no matter the orientation of these axes. If you choose your \( x \) axis along the direction of \( \mathbf{k} \), it will be obvious that (9.9) solves (9.7) because of our previous comments.

Solutions to the wave equation of the form (9.9) are called plane waves because at any given time they look the same as one moves along any plane \( \mathbf{k} \cdot \mathbf{r} = \text{constant} \). We’ll have a little more to say on this in §10.

Note the relation between frequency and wavelength associated with the plane wave (9.9):

\[
\omega = |k|\nu = \frac{2\pi}{\lambda} \nu.
\]

This is the dispersion relation for plane waves in three dimensions. We can view this restriction as fixing the frequency \( \omega \) in terms of the magnitude \( |k| \) but leaving the wave vector \( \mathbf{k} \) itself free. In effect, the dispersion relation (8.67), \( \omega = |k|\nu \), still holds in the three-dimensional setting, provided we interpret \( |k| \) as the magnitude of the wave vector. This is because each plane wave appearing in the Fourier decomposition of the general solution is, mathematically, indistinguishable from a complex sinusoidal solution of the one-dimensional wave equation.

Unfortunately, it is not quite as easy to write a simple expression for the general solution to the 3-dimensional wave equation as it was in the 1-dimensional case.* In particular,

* There is a formula analogous to (7.26), but it is a little too complicated to be worth going into here.
our trick of changing variables to \( x \pm vt \) will not help here. However, Fourier analysis does generalize to any number of dimensions. The idea is that, given a function \( f(x, y, z) \), we can take its Fourier transform one variable at a time. Let us briefly see how this works. For simplicity, we will only consider the case of waves on all of 3-dimensional space, i.e., we will use the continuous version of the Fourier transform.

The Fourier transform \( h(k) = h(k_x, k_y, k_z) \) of \( f(r) = f(x, y, z) \) is defined by

\[
    h(k) = (2\pi)^{-3/2} \int_{all \ space} e^{-ik \cdot r} f(r) \, d^3x,
\]

where \( k \cdot r = k_x x + k_y y + k_z z \), and the \( 2\pi \) factor is a conventional—and convenient—normalization. Every (e.g., square-integrable) function \( f(r) \) can be expressed as

\[
    f(r) = (2\pi)^{-3/2} \int_{all \ k \ space} e^{ik \cdot r} h(k) \, d^3k
\]

for some (e.g., square-integrable) function \( h(k) \). The integration region in each of these formulas is denoted by “all space”, which means that each of \( (x, y, z) \) in (9.11) and each of \( (k_x, k_y, k_z) \) in (9.12) run from \(-\infty\) to \(\infty\).

Note that \( e^{ik \cdot r} \) can be viewed as a (complex) plane wave profile at a fixed time; the wave is traveling in the direction \( k \) with wavelength \( \lambda = 2\pi/k \), where

\[
    k = |k| = \sqrt{(k_x)^2 + (k_y)^2 + (k_z)^2}
\]

is the magnitude (length) of the vector \( k \). Thus, the essence of Fourier analysis is that every function can be expressed as a superposition of waves with (i) varying amplitudes (specified by \( h(k) \)), (ii) varying directions (specified by \( k/k \)), and (iii) varying wavelengths (specified by \( k \)). The precise contributions from the ingredients (i)–(iii) depend upon the particular function being Fourier analyzed. We now use this basic fact from Fourier analysis to get a handle on the general solution of the wave equation; this will generalize our one-dimensional result.

Suppose \( q(r, t) \) is a solution to the wave equation. Let us define

\[
    p(k, t) = \frac{1}{(2\pi)^{3/2}} \int_{all \ space} e^{-ik \cdot r} q(r, t) \, d^3x.
\]

This is just the Fourier transform of \( q \) at each time \( t \). It is easy to see that the wave equation implies

\[
    \frac{\partial^2 p}{\partial t^2} + k^2 v^2 p = 0.
\]

As also occurred in the one-dimensional case, for each value of \( k \) this is just the harmonic oscillator equation. The wave vector, in effect, labels degrees of freedom that oscillate
with a frequency $\omega = |k|v$. This is not an accident, of course, given the relation between the wave equation and a collection of harmonic oscillators which we discussed earlier. The solution to (9.15) is then of the form

$$p(k, t) = A(k) \cos(\omega t) + B(k) \sin(\omega t). \quad (9.16)$$

To fix the coefficients $A$ and $B$ we need to consider initial conditions. Suppose that at, say, $t = 0$ the wave has displacement and velocity profiles given by

$$q(r, 0) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all } k \text{ space}} e^{i\mathbf{k} \cdot r} a(k) \, d^3k \quad (9.17)$$

and

$$\frac{\partial q(r, 0)}{\partial t} = \frac{1}{(2\pi)^{3/2}} \int_{\text{all } k \text{ space}} e^{i\mathbf{k} \cdot r} b(k) \, d^3k. \quad (9.18)$$

Then it is easy to see that*

$$A(k) = a(k) \quad B(k) = \frac{1}{\omega} b(k). \quad (9.19)$$

Evidently,

$$q(r, t) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all } k \text{ space}} e^{i\mathbf{k} \cdot r} \left\{ a(k) \cos(\omega t) + b(k) \frac{\sin(\omega t)}{\omega} \right\} \, d^3k \quad (9.20)$$

satisfies the wave equation where

$$a(k) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} e^{-i\mathbf{k} \cdot r} q(r, 0) \, d^3x, \quad (9.21)$$

and

$$b(k) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} e^{-i\mathbf{k} \cdot r} \frac{\partial q(r, 0)}{\partial t} \, d^3x. \quad (9.22)$$

The general solution (9.20) to the 3-d wave equation can be viewed as a superposition of the elementary (complex) plane wave solutions that we studied earlier. To see this, just note that

$$\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}), \quad \sin(\omega t) = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}), \quad (9.23)$$

and then rearrange the terms in (9.20) to get the solution into the form

$$q(r, t) = (2\pi)^{-3/2} \int_{\text{all } k-\text{space}} \left[ c(k)e^{i(k \cdot r - kvt)} + c^*(k)e^{-i(k \cdot r - kvt)} \right] \, d^3k, \quad (9.24)$$

* Note that the coefficient $B(k)$ is not defined as $k \to 0$ (exercise), but the product $B(k) \sin(\omega t)$ is well-defined at $k = 0$. 

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where \( c(\mathbf{k}) \) are determined by the initial data. Physically, you can think of this integral formula as representing a (continuous) superposition of plane waves over their possible physical attributes. To see this, consider a plane wave of the form

\[
q(\mathbf{r}, t) = Re \left[ ce^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right],
\]

where \( c \) is a complex number. You can check that this is a wave traveling in the direction of \( \mathbf{k} \), with wavelength \( \frac{2\pi}{k} \), and with amplitude \( |c| \). The argument of \( c \), namely \( \left( \frac{c}{|c|} \right) \) adds a constant to the phase of the wave (exercise). The integral in (9.20) is then a superposition of waves in which one varies the amplitudes (\( |c| \)), relative phases (\( c/|c| \)), wavelengths (\( 2\pi/k \)), and directions of propagation (\( \mathbf{k}/k \)) from one wave to the next.

Exactly as we did for the space of solutions to the one-dimensional wave equation, we can view the space of solutions of the three-dimensional wave equation as a vector space (exercise). From this point of view, the plane waves form a basis for the vector space of solutions. Equivalently, every solution to the wave equation can be obtained by superimposing real plane wave solutions of the form

\[
q(\mathbf{r}, t) = A \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)
\]

by varying the amplitude \( A \), the wave vector \( \mathbf{k} \) and the phase \( \phi \) (exercise).