12 Cylindrical Coordinates

Charles G. Torre

Department of Physics, Utah State University, Charles.Torre@usu.edu
12. Cylindrical Coordinates.

We have seen how to build solutions to the wave equation by superimposing plane waves with various choices for amplitude, phase and wave vector $k$. In this way we can build up solutions which need not have the plane symmetry (exercise), or any symmetry whatsoever. Still, as you know by now, many problems in physics are fruitfully analyzed when they are modeled as having various symmetries, such as cylindrical symmetry or spherical symmetry. For example, the magnetic field of a long, straight wire carrying a steady current can be modeled as having cylindrical symmetry. Likewise, the sound waves emitted by a pointlike source are nicely approximated as spherically symmetric. Now, using the Fourier expansion in plane waves we can construct such symmetric solutions — indeed, we can construct any solution to the wave equation. But, as you also know, we have coordinate systems that are adapted to a variety of symmetries, e.g., cylindrical coordinates, spherical polar coordinates, etc. When looking for waves with some chosen symmetry it is advantageous to get at the solutions to the wave equation directly in these coordinates, without having to express them as a superposition of plane waves. Our task now is to see how to express solutions of the wave equations in a useful fashion in terms of such curvilinear coordinate systems.

12.1 The Wave Equation in Cylindrical Coordinates

Our first example will involve solutions to the wave equation in cylindrical coordinates $(\rho, \phi, z)$, which we shall now define. To begin, we point out that the purpose of coordinates is to provide a unique label for every point in some region of space. A common method of defining a new system of coordinates is to take some existing coordinate system and define the new system relative to it. To this end, fix some Cartesian (i.e., $(x, y, z)$) coordinates, which provide unique labels for all of space in a very well-known way. Cylindrical coordinates are denoted by $(\rho, \phi, z)$ and are related to Cartesian coordinates as follows. A point in space is labeled by (i) its perpendicular distance $\rho$ from the $z$-axis; (ii) the angle $\phi$ between a line obtained by projecting into the $x$-$y$ plane a line drawn perpendicularly from the $z$-axis to the point and the positive $x$-axis; and (iii) the $z$ coordinate of the point. If this verbal description leaves you a little dazed, an analytic statement of this is somewhat simpler. A point with Cartesian coordinates $(x, y, z)$ is labeled with cylindrical coordinates $(\rho, \phi, z)$, where

$$\rho = \sqrt{x^2 + y^2},$$

$$\phi = \tan^{-1} \left( \frac{y}{x} \right).$$

Note that $\phi$ is only defined modulo integral multiples of $2\pi$. For example, given a fixed value of $\rho$ and $z$, $\phi = \frac{\pi}{4}$ and $\phi = -\frac{7\pi}{4}$ define the same point in space.
You can think of cylindrical coordinates as a simple 3-dimensional extension of the usual two-dimensional polar coordinates in the sense that on each plane \( z = \text{constant} \), \( \rho \) and \( \phi \) are polar coordinates. Given cylindrical coordinates \((\rho, \phi, z)\) of a point, we can recover the Cartesian coordinates \((x, y, z)\) of that point from (exercise)

\[
\begin{align*}
x &= \rho \cos \phi, \\
y &= \rho \sin \phi
\end{align*}
\]

and the \( z \) value, of course. Note that \( \rho \geq 0 \) and, without loss of generality, \( 0 \leq \phi \leq 2\pi \).

**Warning:** cylindrical coordinates are not well behaved on the \( z \)-axis. For example, what is \( \phi \) for a point on the \( z \)-axis?

*Figure 14. Illustration of cylindrical coordinates \( \rho \), \( \phi \), and \( z \).*

As we shall explore a little later, cylindrical coordinates are particularly useful when
one is studying waves with cylindrical symmetry because the surface \( \rho = \text{constant} \) is a right cylinder of radius \( \rho \). Of course, this is why these coordinates are called cylindrical! Note that \( z \) and \( \phi \) provide coordinates for points on the cylinder \( \rho = \text{constant} \) (exercise).

One way to get solutions of the wave equation in terms of cylindrical coordinates we can simply translate our Fourier expansions in terms of this coordinate system. In particular, the plane wave solution takes the form

\[
q(\rho, \phi, z, t) = \text{Re} \left( Ae^{i(kr-\omega t)} \right) = \text{Re} \left( Ae^{i[k_x \rho \cos \phi + k_y \rho \sin \phi + k_z \rho - \omega t]} \right). \tag{12.4}
\]

Because the cylindrical coordinates are not adapted to the plane symmetry (as are the Cartesian coordinates), the plane symmetry of the plane wave is hidden in (12.4). General solutions in cylindrical coordinates can be obtained by taking superpositions of such solutions. But as mentioned above, we want to be able to get solutions which have cylindrical symmetry without having to perform complicated integrals. To do this, we return to the wave equation, express it in terms of cylindrical coordinates and try to solve it directly.

To translate the wave equation into cylindrical coordinates amounts to finding the cylindrical coordinate expression of the Laplacian. There are a number of tricks for getting this expression, and you can find the result in any text on mathematical methods for physicists or on electromagnetism. It is a nice exercise in multivariable calculus — mainly the chain rule — to get the result, so let us sketch the computation. Recall the Laplacian in Cartesian coordinates

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{12.5}
\]

Given any function \( g(x, y, z) \) we can express it as a function, denoted \( f \), of \((\rho, \phi, z)\) by simply substituting for \((x, y, z)\) in terms of \((\rho, \phi, z)\):

\[
f(\rho, \phi, z) = g(\rho \cos \phi, \rho \sin \phi, z). \tag{7.25}
\]

An example of this appears in (12.4). Our goal is to express \( \nabla^2 g \) in terms of \( f \) and its derivatives with respect to \((\rho, \phi, z)\).*

Let us begin with the term \( \frac{\partial^2}{\partial x^2} \) in \( \nabla^2 g \). The partial derivative of \( g \) with respect to \( x \) is related to the partial derivatives of \( f \) with respect to \((\rho, \phi, z)\) by

\[
\frac{\partial g}{\partial x} = \frac{\partial f}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x}. \tag{12.6}
\]

* Note that we using our picky – but clear – notation which distinguishes the form of a function in different coordinate systems.
To get an expression in cylindrical coordinates for \( \frac{\partial g}{\partial x} \) in terms of partial derivatives of \( f \) with respect to \((\rho, \phi, z)\) we take the expression (12.6) and express \((x, y, z)\) in terms of \((\rho, \phi, z)\). We get (exercise)

\[
\frac{\partial g}{\partial x} = \cos \phi \frac{\partial f}{\partial \rho} - \sin \phi \frac{\partial f}{\partial \phi}.
\]  
(12.7)

Here we used the fact that if \( u = u(x) \), then (exercise)

\[
\frac{d}{dx} \tan^{-1} u = \frac{1}{1 + u^2} \frac{du}{dx},
\]  
(12.8)

also

\[
\frac{\partial z}{\partial x} = 0,
\]  
(12.9)

and

\[
\frac{\partial \rho}{\partial x} = \cos \phi, \quad \frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{\rho}.
\]  
(12.10)

Make sure you understand the basic equations (12.8)–(12.10).

We now need to take another derivative—what a mess! Computers equipped with algebraic computing software, e.g., MathCad, Mathematica or Maple, are very good at these calculations. But it is instructive to do it ourselves. The chain rule says:

\[
\frac{\partial^2 g}{\partial x^2} = \left( \frac{\partial^2 f}{\partial \rho^2} \frac{\partial \rho}{\partial x} + \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \phi}{\partial x} \right) \frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \rho} \frac{\partial^2 \rho}{\partial x^2} + \left( \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \rho}{\partial x} + \frac{\partial^2 f}{\partial \phi^2} \frac{\partial \phi}{\partial x} \right) \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \phi} \frac{\partial^2 \phi}{\partial x^2}. \tag{12.11}
\]

Similarly,

\[
\frac{\partial^2 g}{\partial y^2} = \left( \frac{\partial^2 f}{\partial \rho^2} \frac{\partial \rho}{\partial y} + \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \phi}{\partial y} \right) \frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \rho} \frac{\partial^2 \rho}{\partial y^2} + \left( \frac{\partial^2 f}{\partial \rho \partial \phi} \frac{\partial \rho}{\partial y} + \frac{\partial^2 f}{\partial \phi^2} \frac{\partial \phi}{\partial y} \right) \frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial \phi} \frac{\partial^2 \phi}{\partial y^2}. \tag{12.12}
\]

We now combine these two expressions with \( \frac{\partial^2 f}{\partial z^2} \), and use the facts (exercises)

\[
\frac{\partial \rho}{\partial y} = \sin \phi, \quad \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho},
\]  
(12.13)

\[
\frac{\partial^2 \rho}{\partial x^2} = \frac{\sin^2 \phi}{\rho} \quad \frac{\partial^2 \rho}{\partial y^2} = \frac{2 \cos \phi \sin \phi}{\rho^2},
\]  
(12.14)

\[
\frac{\partial^2 \rho}{\partial y^2} = \frac{\cos^2 \phi}{\rho} \quad \frac{\partial^2 \phi}{\partial y^2} = \frac{2 \cos \phi \sin \phi}{\rho^2}.
\]  
(12.15)

to get the remarkably simple result

\[
\nabla^2 g \equiv \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}
\]

\[
= \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\rho \frac{\partial f}{\partial \rho}}{\rho^2} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \tag{12.16}
\]
We can thus express the wave equation for \( q(\rho, \phi, z, t) \) in cylindrical coordinates as

\[
\frac{1}{v^2} \frac{\partial^2 q}{\partial t^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial q}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 q}{\partial \phi^2} + \frac{\partial^2 q}{\partial z^2}.
\] (12.17)

Note that here and henceforth we are using the same symbol \( q \) for the wave displacement irrespective of coordinate system.

12.2 Separation of Variables in Cylindrical Coordinates

How are we to solve equation (12.17)? The only method we have exhibited for Cartesian coordinates that has a chance of generalizing to cylindrical coordinates is the method of separation of variables. Let’s try it. Set

\[
q(\rho, \phi, z, t) = R(\rho) \Phi(\phi) Z(z) T(t),
\] (12.18)

substitute this trial solution into the wave equation, and then divide the resulting equation by \( q \) to find (exercise)

\[
\frac{1}{v^2} \frac{T'''}{T} = \frac{1}{\rho} \left( \frac{\rho R'}{R} \right)' + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} + \frac{Z''}{Z}.
\] (12.19)

Again we see a separation of variables: on the left hand side we have a function of \( t \) only, while on the right hand side we have only a function of \((\rho, \phi, z)\). Thus both sides must equal a constant; let us call it \(-k^2\). We can solve the resulting equation for \( T \),

\[
T''' = -v^2 k^2 T,
\] (12.20)

via

\[
T = Ae^{\pm ivkt},
\] (12.21)

where we are using the complex form for the solution and, as usual, we must remember to take the real part of the solution at the end.

We have reduced (12.19) to an equation of the form

\[
\frac{1}{\rho} \left( \frac{\rho R'}{R} \right)' + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} = -k^2 - \frac{Z''}{Z},
\] (12.22)

which equates a function of \( \rho \) and \( \phi \) with a function of \( z \). The left and right hand sides must therefore equal a constant, call it \(-a^2\):

\[
\frac{Z''}{Z} = a^2 - k^2.
\] (12.23)

The solution to (12.23) is of the form

\[
Z = Be^{\pm i \sqrt{k^2 - a^2} z}.
\] (12.24)
We are now left with
\[
\frac{1}{\rho} \left( \frac{\rho R'}{R} \right)' + \frac{1}{\rho^2} \Phi'' = -a^2. \tag{12.25}
\]
which can be written as
\[
\rho \left( \frac{\rho R'}{R} \right)' + a^2 \rho^2 = -\frac{\Phi''}{\Phi}. \tag{12.26}
\]
The separation of variables continues! Both sides of this equation must equal a constant. Let us call it \(b^2\). We have
\[
\Phi'' = -b^2 \Phi, \tag{12.27}
\]
which has the (complex) solution
\[
\Phi = Ce^{\pm ib\phi}, \tag{12.28}
\]
where \(C\) is a complex constant.

Here we encounter a slightly novel feature arising from our use of cylindrical coordinates. If \(q(t, \rho, \phi, z)\) is to be a function associating a physically measurable quantity to each point of space at each instant of time we should demand that
\[
q(\rho, 0, z, t) = q(\rho, 2\pi, z, t). \tag{12.29}
\]
Equation (12.29) forces \(b\) in (12.28) to be an integer \(n = 0, 1, 2, 3, \ldots\) (exercise).

With this restriction in mind, we can finally try to solve for \(R\). The equation we must solve is (exercise)
\[
\rho \left( \frac{\rho R'}{R} \right)' + a^2 \rho^2 - n^2 = 0, \tag{12.30}
\]
or
\[
R'' + \frac{1}{\rho} R' + (a^2 - \frac{n^2}{\rho^2}) R = 0. \tag{12.31}
\]

Let us make a simple change of variables to put this equation into a standard form. Let us set \(x := a\rho\).* Of course, this \(x\) has nothing to do with the \(x\) of Cartesian coordinates. Note that \(x\) is a dimensionless quantity (exercise). With \(R(x) := R(x/a)\), the radial equation is (exercise)
\[
\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{n^2}{x^2}\right) R = 0. \tag{12.32}
\]
This linear, homogeneous, ordinary differential equation is known as Bessel’s equation. The solutions are called Bessel functions and Neumann functions. These solutions will, of course, depend on which integer \((n)\) is used in the equation. The Bessel and Neumann functions are examples of “special functions”. There is a menagerie of special functions which are useful in analyzing the basic equations of mathematical physics (wave equation,

* Recall that the symbol “:=” means “is defined to be”.

100
You may not have seen the Bessel or Neumann functions before, but they are really no more sophisticated than, say, the very familiar sine, cosine, or exponential functions. Indeed, we could define the sine and cosine functions as the two “independent” real solutions of

\[ f'' = -\omega^2 f, \]

where \( \omega \) is a given constant (analogous to \( n \) in Bessel’s equation (12.32)). This is a familiar pattern in mathematical physics: special functions can be defined as solutions to certain (ordinary) differential equations. Just as in the case of sine and cosine, for a given \( n \) there are actually 2 independent solutions of Bessel’s equation — the Bessel and Neumann functions. But in the case at hand, only the Bessel functions, are well-behaved at \( \rho = 0 \), and we will limit our attention to this type of solution, which is normally denoted \( J_n(x) \). The singular Neumann solutions are useful when one is interested in solving the wave equation in regions not including the axis \( \rho = 0 \).
Figure 15. (a) Bessel functions $J_0(x)$, $J_1(x)$, and $J_{10}(x)$. For clarity $J_1(x)$ and $J_{10}(x)$ have been displaced in the vertical direction by −1 and −2, respectively. (b) Neumann functions $Y_0(x)$, $Y_1(x)$, and $Y_{10}(x)$. For clarity $Y_1(x)$ and $Y_{10}(x)$ have been displaced in the vertical direction by −1 and −2, respectively.
Remark:

It is worth mentioning that this sort of situation is quite common when solving differential equations that have singularities in their coefficients — the $1/x$ in (12.32). Since the equation is, strictly speaking, only defined when $x \neq 0$, the mathematics is quite happy to supply solutions that are only defined on the domain $x \neq 0$. Moreover, differential equations are by their nature local — they really only constrain the behavior of a function in the neighborhood of any given point. Global information, such as boundary conditions, domains of validity, etc. must be imposed as additional conditions on the solutions to the differential equations.

Needless to say, the Bessel and Neumann functions have been exhaustively investigated, and one can look up their values and basic properties in any set of math tables. Alternatively, any decent computer mathematics package will know about Bessel and Neumann functions. We will not need any details about these functions, but for the purposes of the following discussion it is worth noting the following qualitative features possessed by the Bessel functions. The Bessel functions $J_n(x)$ are real functions and oscillatory in $x$ with decreasing amplitude for increasing $x$ and, as mentioned above, they take on finite values at $x = 0$ (corresponding to points on the $z$-axis). In fact, it turns out that only $J_0$ is non-zero on the at $x = 0$; $J_n(0) = 0$ for $n > 0$. It is clear that Bessel’s equation (12.32) does not distinguish between $n$ and $-n$ (exercise) so we do not expect $J_n$ and $J_{-n}$ to represent different solutions. This is true, but it is customary to define $J_n$ such that

$$J_{-n} = (-1)^n J_n.$$  

(12.34)

12.3 Solutions to the Wave Equation in Cylindrical Coordinates

Let us summarize the results of the last section. Using separation of variables we can get a complex solution to the wave equation in cylindrical coordinates via

$$q(\rho, \phi, z, t) = Ce^{\pm ikvt} J_n(a \rho)e^{\pm i\rho \phi} e^{\pm i\sqrt{k^2-a^2}z}.$$  

(12.35)

Sometimes this elementary solution is expressed as

$$q(\rho, \phi, z, t) = Ce^{\pm i\omega t} J_n(a \rho)e^{\pm i\rho \phi} e^{\pm i\kappa z},$$  

(12.36)

where

$$\omega = kv, \quad \kappa = \sqrt{\frac{\omega^2}{v^2} - a^2}.$$  

Of course, we must take the real (or imaginary) parts of (12.35) to get a real solution; for example, we have a real solution of the form (exercise)

$$q(\rho, \phi, z, t) = b \cos(kv t + n\phi + \kappa z + \alpha)J_n(a \rho),$$  

(12.37)

where all parameters $(b, k, a, \alpha)$ are real numbers.
Figure 16. Spatial part of the separation-of-variables solutions in cylindrical coordinates. The solutions shown have no $z$ dependence, i.e., $k = a$. (a) $J_0(\rho)$. (b) $\cos(2\phi)J_2(\rho)$. 
From (12.37) we see that, at a given point of space, the amplitude of the displacement will oscillate sinusoidally in time. If we take a snapshot of the displacement profile at an instant of time we see a rather complicated oscillatory behavior in \( \rho, \phi, \) and \( z. \) To get a feel for this, let us suppose that \( \kappa = 0 \) and \( n = 0. \) Then the solution (12.37) is of the form (exercise)

\[
q(\rho, \phi, z, t) = b \cos(\omega t + \alpha) J_0(a \rho).
\]

At any fixed location the displacement oscillates harmonically with frequency \( \omega. \) At any given instant of time the displacement is the same at all points on the cylinder \( \rho = \text{constant}. \) Equivalently, the solution does not depend on the coordinates \( \phi \) and \( z. \) We say that the solution exhibits \textit{cylindrical symmetry}. As \( \rho \) increases from zero — still at a fixed time — the displacement oscillates with decreasing amplitude.

A simple physical picture of such a cylindrically symmetric wave can be obtained as follows. Consider the (two-dimensional) surface of a lake. Since the cylindrically symmetric solution doesn’t depend upon \( z, \) we simply ignore \( z. \) Let \( q \) denote the displacement of the water in the vertical direction at a given time. The displacement \( q = 0 \) means that the lake is calm, that is, its surface is flat. Now drop a stone in the lake and choose the origin of our coordinates on the lake at the point where the stone is dropped. You know what happens physically: concentric circular waves are produced which spread from the point where the stone was dropped with amplitude decreasing as the distance from the disturbance increases. But this is precisely the kind of situation our cylindrically symmetric solution represents! Indeed, have a look at the cylindrically symmetric solution (12.38), which represents traveling waves propagating radially with decreasing amplitude as one moves away from the center. That the cylindrically symmetric solution to the wave equation matches up with qualitative behavior of water waves is no accident: for small displacements the restoring force on small elements of the water can be approximated as a Hooke’s law force, and all our work shows that this leads to the wave equation. You might complain that a more realistic wave on the lake is not as symmetric as our simplest solution (12.38) above. But more complicated solutions can be obtained by considering superpositions of solutions of the form (12.35).

This last point deserves some elaboration. In Cartesian coordinates we saw how one could build the general solution to the wave equation using Fourier integrals, which could be viewed as a superposition of elementary plane wave solutions. The plane wave solutions could be obtained using separation of variables in Cartesian coordinates. A similar result can be established (though we won’t do it here) when solving the wave equation in cylindrical coordinates. Essentially, any solution of the wave equation in cylindrical coordinates can be obtained by a suitable superposition of the elementary solutions (12.35) we have found via separation of variables.