An Introduction to Differential Geometry with Maple
PCMI - July 5, 2013

Einstein Metrics On Homogeneous Spaces

Synopsis
Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. In this worksheet we present an example of a $G$ invariant Einstein metric on the homogeneous space $G/H$. These are purely algebraic calculations which can be done with the Lie algebra $\mathfrak{g}$, without the need to introduce local coordinates on $G/H$. In this setting the Einstein equations are purely algebraic equations.

The centralizer $K$ of $H$ in $G$ plays a special role. Indeed, the action of $K$ on $G$ descends to an action on $G/H$ which preserves the space of $G$-invariant metrics $G/H$. It is essential to use these diffeomorphisms to simplify the general form of the $G$ invariant metrics and thereby simplify the algebraic equations defining the $G$ invariant Einstein metrics.

Jensen showed there are 2 symplectically invariant Einstein metrics on the spheres $S^{4k+1}$. Shortly thereafter Ziller showed that these are the only possibilities. We check these results for the special case $k = 1$.

The Symplectic Algebra $\mathfrak{g} = \mathfrak{sp}(4)$, the Subalgebra $\mathfrak{h} = \mathfrak{sp}(2, \mathbb{R})$, and the Reductive Decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$

```
> with(DifferentialGeometry): with(LieAlgebras): with(Tensor): with(GroupActions):

Step 1. We begin by initializing the 10-dimensional Lie algebra $\mathfrak{sp}(4)$ -- the algebra of $4 \times 4$ symplectic matrices. This is done with the commands `SimpleLieAlgebraData` and `DGsetup`.
```
> LD1 := SimpleLieAlgebraData("sp(4, 0)", sp4);

> DGsetup(LD1);

\textit{Lie algebra: sp4}

(1.2)

**Step 2.** We now identify the sub-algebra \( sp(2) \subset sp(4) \). The DifferentialGeometry software provides a general mechanism for creating such Lie algebras. Let \( \rho : \mathfrak{g} \rightarrow \text{aut}(V) \) be the standard representation of \( \mathfrak{g} \). From this representation one can define representations of \( \mathfrak{g} \) on the various tensor spaces of \( V \). Given a list of tensors on \( V \), the command \texttt{MatrixSubalgebra} will find the subalgebra of \( \mathfrak{g} \) which leaves invariant the given tensors.

Since \( sp(2, R) \subset sp(4, R) \) is the set of matrices which leaves invariant the 2nd and 4th vectors in the standard representation of \( sp(4, R) \), we can proceed as follows. Define a 4-dimensional space to serve as the representation space for the standard representation.

\[ \text{sp4} > \text{DGsetup}([x1, x2, x3, x4], V); \]

(frame name: \( V \))

(1.3)

Create the standard representation.

\[ \text{V} > \rho := \text{StandardRepresentation}(\text{sp4, representation space = V}); \]

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(1.4)
Find the sub-algebra which leaves the vectors $D_{x^2}$ and $D_{x^4}$ invariant. This is the sub-algebra $sp(2)$.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
e_8, \\
0 & 0 & 0 & 0 \\
e_9, \\
0 & 0 & 0 & 0 \\
e_{10}, \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Check that $h = sp(2) = so(3)$.

\[
\begin{bmatrix}
e_1, e_2 \\= 2 e_3, \\
e_1, e_3 \\= K e_2, \\
e_2, e_2, e_3 \\= 2 e_1
\end{bmatrix}
\]

**Step 3.** Choose a complementary subspace $m$ to the sub-algebra $h$.

\[
\begin{bmatrix}
e_1, e_3, e_6, e_9, e_4, e_7, e_{10} \\
m := [e_1, e_3, e_6, e_9, e_4, e_7, e_{10}]
\end{bmatrix}
\]

We note (using the command `Query`) that the decomposition $g = m + h$ is reductive but not symmetric.

\[
\begin{bmatrix}
true
\end{bmatrix}
\]

\[
\begin{bmatrix}
false
\end{bmatrix}
\]

**The Centralizer $k$ of $h$ and the Reductive Decomposition $m = l + k$**

The next step is to determine the centralizer of $h$ in $g$.

\[
k := [e_{10}, e_7, e_4]
\]
Find a complement of $k$ in $m$.

\[
\text{sp4n} > \text{ell1} := \text{ComplementaryBasis}(k, m);
\]
\[
\quad \text{ell1} := [e1, e3, e6, e9]
\]

Create a new basis for $\text{sp}(4)$ adapted to the decomposition $\mathfrak{g} = \mathfrak{s} + \mathfrak{f} + \mathfrak{h}$.

\[
\text{sp4n} > \text{newBasis} := [\text{op(ell1)}, \text{op(k)}, \text{op(h)}];
\]
\[
\quad \text{newBasis} := [e1, e3, e6, e9, e10, e7, e4, e2, e5, e8]
\]

Re-initialize $\text{sp}(4)$ in this new basis.

\[
\text{sp4n} > \text{LD2} := \text{LieAlgebraData(newBasis, sp4n)};
\]
\[
\quad \text{sp4n} > \text{DGsetup}([\text{LD2}, [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3], [\omega]);
\]

\[
\text{Lie algebra: sp4n}
\]
\[
\text{sp4n} > H := [h1, h2, h3];
\]
\[
\quad H := [h1, h2, h3]
\]
\[
\text{sp4n} > M := [l1, l2, l3, l4, k1, k2, k3];
\]
\[
\quad M := [l1, l2, l3, l4, k1, k2, k3]
\]
\[
\text{sp4n} > K := [k1, k2, k3];
\]
\[
\quad K := [k1, k2, k3]
\]

The multiplication table shows the properties of this decomposition.

\[
\text{sp4n} > \text{interface(rtablesize = 15)};
\]
\[
\text{sp4n} > \text{MultiplicationTable("LieTable")};
\]
\[
[ [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
  [l1, l2, l3, l4, k1, k2, k3, h1, h2, h3],
]
\]
The sp(2) Invariant Metrics on m

We use the command `InvariantGeometricObjectFields` to define the general invariant symmetric quadratic form. There are 7 invariant symmetric quadratic forms.

First we generate all the rank 2 symmetric tensors on M.

```plaintext
S7 > Omega := [omega1, omega2, omega3, omega4, omega5, omega6, omega7];
Ω := [ω1, ω2, ω3, ω4, ω5, ω6, ω7] (3.1)
S7 > Q := GenerateSymmetricTensors(Omega, 2);
Ω[1] := [ω1 ω1, 1/2 ω1 ω2 + 1/2 ω2 ω1, 1/2 ω1 ω3 + 1/2 ω3 ω1, 1/2 ω1 ω4 + 1/2 ω4 ω1, 1/2 ω1 ω5 + 1/2 ω5 ω1, 1/2 ω1 ω6 + 1/2 ω6 ω1, 1/2 ω1 ω7 + 1/2 ω7 ω1, 1/2 ω2 ω2 + 1/2 ω2 ω3 + 1/2 ω3 ω2 + 1/2 ω2 ω4 + 1/2 ω4 ω2 + 1/2 ω2 ω5 + 1/2 ω5 ω2 + 1/2 ω2 ω6 + 1/2 ω6 ω2, 1/2 ω2 ω7 + 1/2 ω7 ω2, 1/2 ω3 ω3 + 1/2 ω3 ω4 + 1/2 ω4 ω3 + 1/2 ω3 ω5 + 1/2 ω5 ω3 + 1/2 ω3 ω6 + 1/2 ω6 ω3 + 1/2 ω3 ω7 + 1/2 ω7 ω3, 1/2 ω4 ω4 + 1/2 ω4 ω5 + 1/2 ω5 ω4 + 1/2 ω4 ω6 + 1/2 ω6 ω4 + 1/2 ω4 ω7 + 1/2 ω7 ω4, 1/2 ω5 ω5 + 1/2 ω5 ω6 + 1/2 ω6 ω5 + 1/2 ω5 ω7 + 1/2 ω7 ω5 + 1/2 ω6 ω6 + 1/2 ω6 ω7 + 1/2 ω7 ω6 + 1/2 ω7 ω7]

We find that the space of h-invariant quadratic forms is 7-dimensional.

```plaintext
S7 > InvQ := InvariantGeometricObjectFields(H, Q, coefficientvariables = [], output = "list");
InvQ := [ω1 ω1 + ω2 ω2 + ω3 ω3 + ω4 ω4, ω5 ω5, 1/2 ω5 ω6 + 1/2 ω6 ω5 + 1/2 ω5 ω7 + 1/2 ω7 ω5, 1/2 ω6 ω6 + 1/2 ω6 ω7 + 1/2 ω7 ω6, 1/2 ω7 ω7] (3.3)
```
It is helpful to write the general invariant metric in matrix form.

\[ Q_0 := a_1 \omega_1 \otimes \omega_1 + a_1 \omega_2 \otimes \omega_2 + a_1 \omega_3 \otimes \omega_3 + a_1 \omega_4 \otimes \omega_4 + a_2 \omega_5 \otimes \omega_5 \]
\[ + \frac{a_3}{2} \omega_5 \otimes \omega_6 + \frac{a_4}{2} \omega_5 \otimes \omega_7 + \frac{a_3}{2} \omega_6 \otimes \omega_5 + a_5 \omega_6 \otimes \omega_6 + \frac{a_6}{2} \omega_6 \otimes \omega_7 \]
\[ + \frac{a_4}{2} \omega_7 \otimes \omega_5 + \frac{a_6}{2} \omega_7 \otimes \omega_6 + a_7 \omega_7 \otimes \omega_7 \]

\[ A := \begin{bmatrix}
    a_1 & 0 & 0 & 0 & 0 & 0 \\
    0 & a_1 & 0 & 0 & 0 & 0 \\
    0 & 0 & a_1 & 0 & 0 & 0 \\
    0 & 0 & 0 & a_2 & \frac{a_3}{2} & \frac{a_4}{2} \\
    0 & 0 & 0 & \frac{a_3}{2} & a_5 & \frac{a_6}{2} \\
    0 & 0 & 0 & \frac{a_4}{2} & \frac{a_6}{2} & a_7 
\end{bmatrix} \]  

Since \( K \) acts on \( \{ \omega_5, \omega_6, \omega_7 \} \) by infinitesimal rotations, the action of the group \( \exp(K) \) on the matrix \( A \) is by conjugation by rotation matrices on the lower 3 x 3 block. Since every symmetric matrix can be diagonalized by a rotation matrix, we can use a diffeomorphism \( \exp(K) \) to transform the off-diagonal metric components to zero.

For the purposes of finding homogeneous Einstein metrics we can also normalize \( a_1 \) to 1.

\[ Q_{\text{reduced}} := \text{Tools:-DGsimplify}(\text{subs}([\{a_1 = 1, a_3 = 0, a_4 = 0, a_6 = 0\}, Q_0])); \]
\[ Q_{\text{reduced}} := \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3 + \omega_4 \otimes \omega_4 + a_2 \omega_5 \otimes \omega_5 \]
\[ + a_5 \omega_6 \otimes \omega_6 + a_7 \omega_7 \otimes \omega_7 \]
The Einstein Metrics

Use FrameData and to create the computation environment for homogeneous spaces.

\[
\text{sp4n} > \text{FD} := \text{FrameData}([], M, H, S7);
\]
\[
\text{FD} := [d \Theta 1 = \Theta 4 \wedge \Theta 10 - \Theta 2 \wedge \Theta 7 + \Theta 2 \wedge \Theta 8 - \Theta 3 \wedge \Theta 6 + \Theta 3 \wedge \Theta 9]
\]  
\[
- \Theta 4 \wedge \Theta 5, d \Theta 2 = \Theta 1 \wedge \Theta 7 - \Theta 1 \wedge \Theta 8 - \Theta 3 \wedge \Theta 10 - \Theta 3 \wedge \Theta 5 + \Theta 4 \wedge \Theta 6
\]
\[
+ \Theta 4 \wedge \Theta 9, \text{d} \Theta 3 = \Theta 1 \wedge \Theta 6 - \Theta 1 \wedge \Theta 9 + \Theta 2 \wedge \Theta 10 + \Theta 2 \wedge \Theta 5 - \Theta 4 \wedge \Theta 7
\]
\[
- \Theta 3 \wedge \Theta 8, d \Theta 4 = -\Theta 1 \wedge \Theta 10 + \Theta 1 \wedge \Theta 5 - \Theta 2 \wedge \Theta 6 - \Theta 2 \wedge \Theta 9 + \Theta 3 \wedge \Theta 7
\]
\[
+ 2 \Theta 2 \wedge \Theta 4 - 2 \Theta 5 \wedge \Theta 7, d \Theta 7 = -2 \Theta 1 \wedge \Theta 2 - 2 \Theta 3 \wedge \Theta 4 + 2 \Theta 5 \wedge \Theta 6,
\]
\[
d \Theta 8 = 2 \Theta 1 \wedge \Theta 2 - 2 \Theta 9 \wedge \Theta 10 - 2 \Theta 3 \wedge \Theta 4, d \Theta 9 = 2 \Theta 1 \wedge \Theta 3 + 2 \Theta 8 \wedge \Theta 10
\]
\[
+ 2 \Theta 2 \wedge \Theta 4, d \Theta 10 = 2 \Theta 1 \wedge \Theta 4 - 2 \Theta 2 \wedge \Theta 3 - 2 \Theta 8 \wedge \Theta 9]
\]  
\[
\text{sp4n} > \text{DGsetup(FD, [E], [sigma]);}
\]
\[
\text{frame name: S7}
\]  
Re-define the invariant metric on the Lie algebra as a metric on S7.

\[
\text{sp4n} > g := \text{subs} (\text{sp4n} = \text{S7}, \text{Qreduced});
\]
\[
g := \sigma 1 \otimes \sigma 1 + \sigma 2 \otimes \sigma 2 + \sigma 3 \otimes \sigma 3 + \sigma 4 \otimes \sigma 4 + a 2 \sigma 5 \otimes \sigma 5 + a 5 \sigma 6 \otimes \sigma 6
\]
\[
+ a 7 \sigma 7 \otimes \sigma 7
\]  
Solve the Einstein equations with DGsolve.

\[
\text{G} > h := \text{InverseMetric}(g);
\]
\[
h := E 1 \otimes E 1 + E 2 \otimes E 2 + E 3 \otimes E 3 + E 4 \otimes E 4 + \frac{1}{a 2} \ E 5 \otimes E 5 + \frac{1}{a 5} \ E 6 \otimes E 6
\]
\[
+ \frac{1}{a 7} \ E 7 \otimes E 7
\]  
\[
\text{S7} > \_\text{EnvExplicit} := \text{true};
\]
\[
\_\text{EnvExplicit} := \text{true}
\]  
\[
\text{G} > \text{EinsteinMetrics} := \text{DGsolve} (\text{EinsteinTensor}(g) - \text{Lambda}^* h, g, \{a 2, a 5, a 7, \text{Lambda}\}, \text{auxiliaryequations} = \{a 2>-\text{infinity}\});
\]
\[
\text{EinsteinMetrics} := \{\sigma 1 \otimes \sigma 1 + \sigma 2 \otimes \sigma 2 + \sigma 3 \otimes \sigma 3 + \sigma 4 \otimes \sigma 4 + \sigma 5 \otimes \sigma 5 + \sigma 6 \otimes \sigma 6
\]
\[
+ \sigma 7 \otimes \sigma 7, \sigma 1 \otimes \sigma 1 + \sigma 2 \otimes \sigma 2 + \sigma 3 \otimes \sigma 3 + \sigma 4 \otimes \sigma 4 + \frac{1}{5} \sigma 5 \otimes \sigma 5\}
We find there are 2 metrics.

\[
\begin{align*}
S7 & > g_1 := \text{EinsteinMetrics[1]}; \\
& \quad = \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3 + \sigma_4 \otimes \sigma_4 + \sigma_5 \otimes \sigma_5 + \sigma_6 \otimes \sigma_6 + \sigma_7 \otimes \sigma_7 \\
S7 & > g_2 := \text{EinsteinMetrics[2]}; \\
& \quad = \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3 + \sigma_4 \otimes \sigma_4 + \frac{1}{5} \sigma_5 \otimes \sigma_5 + \frac{1}{5} \sigma_6 \otimes \sigma_6 \\
& \quad \quad + \frac{1}{5} \sigma_7 \otimes \sigma_7
\end{align*}
\]

\textbf{Properties of the Einstein Metrics}

The first metric has covariantly constant curvature and 28 dimensional isometry algebra. It is the standard metric on $S^7$.

\[
\begin{align*}
S7 & > \text{C1 := Christoffel(g1)}; \\
S7 & > \text{R1 := CurvatureTensor(g1)}; \\
S7 & > \text{CovariantDerivative(R1, C1)}; \\
& \quad = 0 E1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \\
S7 & > \text{M := d1, isol1, isotropy1 := IsometryAlgebra(g1, alg, [])}; \\
S7 & > \text{d1}; \\
& \quad = 28 \\
S7 & > \text{Hol1 := InfinitesimalHolonomy(g1, [])}; \\
S7 & > \text{nops(Hol1)}; \\
& \quad = 21
\end{align*}
\]

\[
\begin{align*}
S7 & > \text{infolevel[IsometryAlgebra] := 3}; \\
& \quad = \text{infolevel}_{\text{IsometryAlgebra}} := 3 \\
S7 & > \text{d2, iso2, isotropy2 := IsometryAlgebra(g2, alg, [])}; \\
& \quad = \text{Computing the 1st derivative of curvature.} \\
& \quad = \text{The upper bound on the dimension of the isometry algebra is 13.} \\
& \quad = \text{Computing the 2 derivative of curvature} \\
& \quad = \text{Curvature calculation complete} \\
& \quad = \text{The upper bound on the dimension of the isometry algebra is 13.} \\
& \quad = \text{Calculating Lie algebra data.} \\
& \quad = \text{Calculating isotropy subalgebra data}
\end{align*}
\]

\[
\begin{align*}
S7 & > \text{nops(isotropy2)};
\end{align*}
\]