Appendix A. Taylor’s Theorem and Taylor Series

Taylor’s theorem and Taylor’s series constitute one of the more important tools used by mathematicians, physicists and engineers. They provide a means of approximating a function in terms of polynomials. To begin, we present Taylor’s theorem, which is an identity satisfied by any function \( f(x) \) that has continuous derivatives of, say, order \((n+1)\) on some interval \( a \leq x \leq b \). Taylor’s theorem asserts that

\[
f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \ldots + \frac{1}{n!} f^{(n)}(a)(x - a)^n + R_{n+1},
\]

(A.1)

where \( R_{n+1} \) – the remainder – can be expressed as

\[
R_{n+1} = \frac{1}{(n+1)!} (x - a)^{n+1} f^{(n+1)}(\xi),
\]

(A.2)

for some \( \xi \) with \( a \leq \xi \leq b \). Here we are using the notation

\[
f^{(k)}(c) = \frac{d^k f}{dx^k} \bigg|_{x=c}.
\]

(A.3)

The number \( \xi \) is not arbitrary; it is determined (though not uniquely) via the mean value theorem of calculus. For our purposes we just need to know that it lies between \( a \) and \( b \). The equation (A.1) is an identity; it involves no approximations.

The idea is that for many functions the value of \( n \) can be chosen so that the remainder is sufficiently small compared to the polynomial terms that we can omit it. In this case we get Taylor’s approximation:

\[
f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \ldots + \frac{1}{n!} f^{(n)}(a)(x - a)^n.
\]

(A.4)

Typically, the approximation is reasonable provided \( x \) is close enough to \( a \) and none of the derivatives of \( f \) get too large in the region of interest. As you can see, if \( (x - a) \) is small, \( i.e., \ x - a < < 1 \), successive powers of \( (x - a) \) become smaller and smaller so that one need only keep a few terms in the polynomial expansion to get a good approximation.

If you can prove that

\[
\lim_{n \to \infty} R_n = 0,
\]

(A.5)

then it makes sense to consider expressing \( f(x) \) as a power series:

\[
f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \ldots + \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x - a)^n,
\]

(A.6)

which is known in this context as the Taylor series for \( f \). (Note that here we use the definitions \( 0! = 1 \) and \( f^{(0)}(x) = f(x) \).) Normally the Taylor series of a function will

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converge in some neighborhood of $x = a$ and diverge outside of this neighborhood.* In any case, for a sufficiently “well-behaved” function, one can get a good approximation to it using (A.4) by keeping $n$ relatively small. How small $n$ needs to be depends, in large part, on how big $(x - a)$ is. Often times one can get away with just choosing $n = 1$ or perhaps $n = 2$ for $x$ sufficiently close to $a$.

As a simple example, consider the sine function $f(x) = \sin(x)$. Let us approximate the sine function in the vicinity of $x = 0$, so that we are taking $a = 0$ in the above formulas. The zeroth-order approximation amounts to using $n = 0$ in (A.4). We get

$$\sin(x) \approx \sin(0) = 0.$$ (A.7)

This is obviously not a terribly good approximation. But you can check (using your calculator in radian mode) that if $x$ is nearly zero, so is $\sin(x)$. A better approximation, the first-order approximation, arises when $n = 1$ in (A.4). We get (exercise)

$$\sin(x) \approx \sin(0) + \cos(0)x = x.$$ (A.8)

Again, you can check this approximation on your calculator. If $x$ is kept sufficiently small (in radians), this approximation does a pretty good job. As $x$ gets larger the approximation gets less accurate. For example, at $x = 0.1$ the error in the approximation is about 0.2%. At $x = 0.75$, the error is about 10%. The second-order approximation is identical to the first-order approximation, as you can check explicitly (exercise). The third-order approximation (exercise),

$$\sin(x) \approx x - \frac{1}{6}x^3$$ (A.9)

is considerably better than the first-order approximation. It gives good results out to, say, $x = 1.7$, where the error is about 11%. Incidentally, the remainder term for the sine function satisfies (A.5) (exercise), and we can represent the analytic function $\sin(x)$ by its (everywhere convergent) Taylor series.

* Functions that can be represented by a convergent Taylor series are called analytic.