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# Explicit *Synchronized Solitary Waves* For Some Models For the *Interaction of Long and Short Waves* in Dispersive Media.

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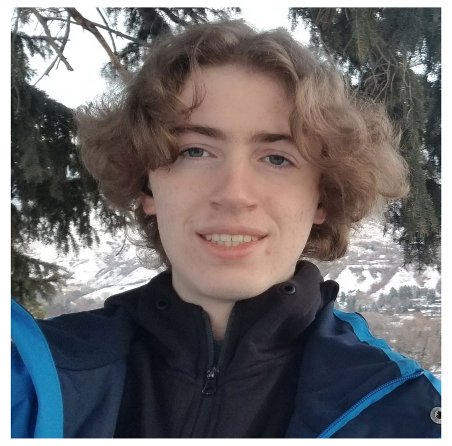
**Def**: A ***solitary wave*** has a single maximum, both tails decaying to zero, and a *constant shape through time.*

# Explicit *synchronized solitary waves* for some models for the *interaction of long and short waves* in dispersive media.

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## Introduction

In 2020, Nguyen et al. proposed the following system for study of the interaction of long and short gravity-capillary waves on surfaces of shallow water.

$$\begin{cases} u_t + \mu_0 u_x + a_0 L_1 u_{xx} + ibu_{xx} &= -(uv)_x - i\mu_1 uv, \\ v_t + v_x + vv_x + cL_2 v_{xx} &= -\frac{1}{2}|u|_x^2. \end{cases}$$

Here $L_1$ and $L_2$ can be either $\partial_x$ or $-\partial_t$. Different choices of $L_1$ and $L_2$ yield four different systems. Explicit synchronized solitary wave solutions to all four systems can be found in the paper linked by the QR code above. This poster will analyze the simplest case where $L_1=L_2=\partial_x$.

$$\begin{cases} u_t + \mu_0 u_x + a_0 u_{xxx} + ibu_{xx} &= -(uv)_x - i\mu_1 uv, \\ v_t + v_x + vv_x + cv_{xxx} &= -\frac{1}{2}|u|_x^2. \end{cases}$$

The transport part facilitates movement. The dispersive and non-linear parts can balance to form a *solitary wave* that retains its shape.

## Derivation

I searched for *synchronized solitary wave solutions* of the following form with the condition that $f$ and its derivatives decay to zero at infinity.

$$\begin{cases} u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} f(x-\sigma t), \\ v(x,t) = Af(x-\sigma t). \end{cases}$$

Plugging this ansatz into the first equation yields the following non-linear ODE for $f$.

$$\begin{cases} i[\omega + B\mu_0 - B\sigma - a_0 B^3 - bB^2] f \\ +[\mu_0 - \sigma - 3a_0 B^2 - 2bB] f' \\ +i[3a_0 B + b] f'' + [a_0] f''' \\ = -iA[B+\mu_1] f^2 - 2Aff'. \end{cases}$$

Plugging the ansatz into the second equation yields another non-linear ODE for $f$.

$$[A - A\sigma] f' + [A^2 + 1] ff' + Acf''' = 0.$$

Splitting the real and imaginary parts of the first ODE results in two separate ODEs for $f$. This means that in total, we have three ODEs for $f$. These ODEs can be simplified to the following.

$$\begin{cases} f'^2 = \frac{\sigma + 3a_0 B^2 + 2bB - \mu_0}{a_0} f^2 - \frac{2A}{3a_0} f^3, \\ f'^2 = \frac{B\sigma + a_0 B^3 + bB^2 - \omega - B\mu_0}{3a_0 B + b} f^2 - \frac{2A[B+\mu_1]}{3(3a_0 B + b)} f^3, \\ f'^2 = \frac{\sigma - 1}{c} f^2 - \frac{A^2+1}{3Ac} f^3. \end{cases}$$

These ODEs must all be satisfied together, so we require that all coefficients of $f^2$ must be equal and all coefficients $f^3$ must be equal. When this is the case, all three ODEs collapse into one.

$$f'^2 = k_1 f^2 - k_2 f^3.$$

## Solution

The solution to this ODE is

$$f(\xi) = \frac{k_1}{k_2} \operatorname{sech}^2\left(\frac{\sqrt{k_1}}{2}\xi\right).$$

This function is plotted in the bottom right corner of the title. Notice that it meets the definition of a solitary wave: It has a single maximum, its tails decay to zero at infinity, and it maintains a constant shape through time.

The values for the solution parameters are obtained by equating the coefficients in front of $f^2$ and equating the coefficients in front of $f^3$.

$$\begin{cases} \sigma = \frac{4a_0^2 + 3a_0^2 c\mu_1^2 - 2a_0 bc\mu_1 - 4a_0 c\mu_0 - b^2 c}{4a_0(a_0 - c)}, \\ \omega = b\mu_1^2 + \mu_0\mu_1 - a_0\mu_1^3 - \mu_1\sigma, \\ A^2 = \frac{a_0}{2c - a_0}, \\ B = \frac{a_0\mu_1 - b}{2a_0}. \end{cases}$$

This fully determines the solitary wave solutions.