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Discrete dynamics on noncommutative CW complexes

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ABSTRACT

The concept of discrete multivalued dynamical systems for noncommutative CW complexes is developed. Stable and unstable manifolds are introduced and their role in geometric and topological configurations of noncommutative CW complexes is studied. Our technique is illustrated by an example on the noncommutative CW complex decomposition of the algebra of continuous functions on two dimensional torus.

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1. INTRODUCTION

The theory of CW complexes was invented by Whitehead in 1949 [14]. The concept of CW complex structures on topological manifolds has been a great development in the category of topological spaces [8]. It is a well known fact that the topology of a manifold can be reconstructed from the commutative C*-algebra of continuous functions on it [7, 10]. In other words commutative C*-algebras play as the dual concept for topological manifolds. Away from

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commutativity, C*-algebras are still substitutes for noncommutative topological manifolds and provide building blocks of noncommutative topology theory [2, 4, 10]. In the category of noncommutative manifolds, noncommutative CW complexes were introduced in [6, 13]. A great development in the theory of noncommutative topology would be the study of noncommutative manifolds (C*-algebras) which are endowed with a noncommutative CW complex structure.

In the study of CW complexes there exist two classical approaches. One approach comes from differential topology and Morse theory [11]. The second one is the dynamics point of view and the relation between dynamical properties of a flow and the homological configuration of the CW complex. Our aim is to develop the two approaches in the framework of noncommutative topology in order to study noncommutative CW complexes:

$$\text{Dynamics} \xrightarrow{\text{Primitive Spectrum}} \text{NCCW Complexes} \xleftarrow{\text{C*-Algebra}} \text{Diff. Topology}$$

In this regard our first attempt was the development of the Morse theory approach in [12]. In the present paper we are developing the second approach. In both approaches we apply techniques from combinatorial topology [3, 5] and the primitive spectrum of C*-algebras [10] as basic tools.

The paper is organized as follows. In section 2 we review fundamental notions in the theory of noncommutative CW complexes. We explain the role of the primitive spectrum as a bridge between CW complexes and noncommutative CW complexes. Section 3 is devoted to a review from [12] on the basics of modified Morse theory on noncommutative CW complexes. Discrete multivalued dynamical systems have been introduced in [1, 9]. In section 4 we develop discrete multivalued dynamical systems on noncommutative CW complexes and provide tools to relate a dynamical picture to the topology and geometry of noncommutative CW complexes. We will see how the dynamical properties of the trajectories are related to the configuration of noncommutative CW complexes. In this section, stable and unstable manifolds are introduced and some of their properties are studied. An example will serve to illustrate our dynamical construction. Section 5 is devoted to the explanation of this example. In this section we study the noncommutative CW complex structure of $C(T^2)$: the algebra of continuous functions on the 2-dimensional torus. We associate a discrete multivalued dynamical system with it. We shall see how the configuration of this noncommutative CW complex is explained by the stable and unstable manifolds.

2. Noncommutative CW complexes

In this section we review basic definitions and results on the theory of noncommutative CW complexes from [6, 13]. we explain the technique of the primitive spectrum and its role as a link between CW complexes and noncommutative CW complexes. Details on the structure of primitive spectrum can be found in [7, 10, 12]. First we review the concept of CW complex structure for a topological space from [8].

A sequence

$$X_0 \subset X_1 \subset ... \subset X_n = X$$

is an n-dimensional CW complex structure for a compact topological space X, where X_0 is a finite discrete space consisting of 0-cells, and for $k = 1, ..., n$ each k-skeleton X_k is obtained by attaching λ_k number of k-disks to X_{k-1} via the attaching maps

$$\varphi_k : \bigcup_{\lambda_k} S^{k-1} \to X_{k-1}.$$

In other words

$$X_k = \frac{X_{k-1} \bigcup (\cup_{\lambda_k} I^k)}{x \sim \varphi_k(x)} := X_{k-1} \bigcup_{\varphi_k} (\cup_{\lambda_k} I^k) \tag{2.1}$$

where $I^k := [0,1]^k$ and $S^{k-1} := \partial I^k$. The quotient map is denoted by

$$\rho : X_{k-1} \bigcup (\cup_{\lambda_k} I^k) \to X_k.$$

For a continuous map $\phi : X \to Y$ between compact topological spaces X and Y, the C*-morphism induced on their associated C*-algebra of functions is denoted by $C(\phi) : C(Y) \to C(X)$ which is defined by $C(\phi)(g) := g \circ \phi$ for $g \in C(Y)$.

Definition 2.1. Let A_1, A_2 and C be C*-algebras. *A pull back for C via morphisms $\alpha_1 : A_1 \to C$ and $\alpha_2 : A_2 \to C$* is the C*-subalgebra of $A_1 \oplus A_2$ denoted by $PB(C, \alpha_1, \alpha_2)$ defined by

$$PB(C, \alpha_1, \alpha_2) := \{a_1 \oplus a_2 \in A_1 \oplus A_2 : \alpha_1(a_1) = \alpha_2(a_2)\}.$$

For any C*-algebra A, let

$$S^n A := C(S^n \to A), I^n A := C([0,1]^n \to A), I_0^n A := C_0((0,1)^n \to A),$$

where S^n is the n-dimensional unit sphere.

Definition 2.2. A *0-dimensional noncommutative CW complex* is any finite dimensional C*-algebra A_0. Recursively an *n-dimensional noncommutative CW complex* is any C*-algebra appearing in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & I_0^n F_n \arrow[r] \arrow[d, equal] & A_n \arrow[r, "\pi"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "\varphi_n"] & 0 \\
0 \arrow[r] & I_0^n F_n \arrow[r] & I^n F_n \arrow[r, "\delta"] & S^{n-1} F_n \arrow[r] & 0
\end{tikzcd}$$

Where the rows are extensions, A_{n-1} an $(n-1)$-dimensional noncommutative CW complex, F_n some finite (linear) dimensional C*-algebra of dimension λ_n, δ the boundary restriction map, φ_n an arbitrary morphism (called the connecting morphism), for which

$$A_n = PB(S^{n-1}F_n, \delta, \varphi_n) := \{(\alpha, \beta) \in I^n F_n \oplus A_{n-1} : \delta(\alpha) = \varphi_n(\beta)\},$$

and f_n and π are respectively projections onto the first and second coordinates.

With these notations $\{A_0, ..., A_n\}$ is called *the noncommutative CW complex decomposition of dimension n for $A = A_n$.*

For each $k = 0, 1, ..., n$, A_k is called *the k-th decomposition cell.*

Let A be a unital C*-algebra. The *primitive spectrum* of A is the space of kernels of irreducible *-representations of A. It is denoted by $Prim(A)$. The topology on this space is given by the closure operation as follows:

For any subset $U \subseteq Prim(A)$, the closure of U is defined by

$$\overline{U} := \{I \in Prim(A) : \bigcap_{J \in U} J \subset I\}$$

Obviously $U \subseteq \overline{U}$. This operation defines a topology on $Prim(A)$ (the hull-kernel topology), making it into a T_0-space [10].

Definition 2.3. A subset $U \subseteq Prim(A)$ is called *absorbing* if it satisfies the following condition:

$$I \in U, I \subseteq J \Rightarrow J \in U.$$

Remark 2.4. The closed subsets of $Prim(A)$ are exactly its absorbing subsets.

In the special case, when M is a compact topological space, and $A = C(M)$ is the commutative unital C*-algebra of complex continuous functions on M, a homeomorphism between M and $prim(A)$ is obtained in the following way. For each $x \in M$ let

$$I_x := \{f \in A : f(x) = 0\};$$

I_x is a closed maximal ideal of A. It is in fact the kernel of the evaluation map

$$\begin{aligned}(ev)_x :& A \longrightarrow \mathbb{C}\\ & f \longmapsto f(x).\end{aligned}$$

Now

$$I : M \to Prim(A)$$

defined by $I(x) := I_x$ is the desired homeomorphism.

let

$$X_0 \subset X_1 \subset ... \subset X_n = X$$

be an n-dimensional CW complex structure for the compact space X. A cell complex structure is induced on $Prim(C(X))$ by the following procedure:

Let $A_k = C(X_k)$, $k = 0, 1, ..., n$. Set $A = C(X) = C(X_n) = A_n$. Consider the homeomorphism $I : X \to Prim(C(X))$. For each k-cell C_k in the k-skeleton X_k, let

$$I_{C_k} = \bigcap_{x \in C_k} I_x = \{f \in A : f(x) = 0; x \in C_k\},$$

for $0 \leq k \leq n$. By considering the restriction of functions on X to X_k, I_{C_k} will be an ideal in A_k.

In the above notations, the closed sets

$$W_{i_0,...,i_k} := \{J \in Prim(A_k) : J \supseteq I_{C_k}\}$$

are corresponded to the ideals I_{C_k}.

In general we can have

Proposition 2.5 ([12]). *Let X be an n-dimensional CW complex containing cells of each dimension $k = 0, ..., n$. Then there exists a noncommutative CW complex decomposition of dimension n for $A = C(X)$.*

Conversely if $\{A_0, ..., A_n\}$ be a noncommutative CW complex decomposition for the C-algebra A such that $A_i s$ $(i = 0, .., n)$ are unital, Then there exists an n-dimensional CW complex structure on $Prim(A)$.*

Example 2.6. Let $X_0 = \{0, 1\}$ and $X_1 = [0, 1]$ be the zero and the one skeleton for a CW complex structure of [0,1]. Then we have $A_0 = C(X_0) \simeq \mathbb{C} \oplus \mathbb{C}$ and $A = A_1 = C(X_1)$. The 0-ideals I_0 and I_1 and their corresponding 0-chains W_0 and W_1 are as follow:

$$I_0 = \{f \in A_0 : f(0) = 0\} \simeq \mathbb{C}, I_1 = \{f \in A_0 : f(1) = 0\} \simeq \mathbb{C},$$

$$W_0 = \{J \in Prim(A_0) : J \supseteq I_0\} = \{I_0\}, W_1 = \{J \in Prim(A_0) : J \supseteq I_1\} = \{I_1\}.$$

Corresponding to the 1-chain $C_1 = [0, 1]$, the only 1-ideal is

$$I = \bigcap_{x \in C_1} I_x = \{f \in A : f(x) = 0; x \in [0, 1]\} = \{0\},$$

with the corresponding 1-chain

$$W_I = \{J \in Prim(A) : J \supseteq I\} = Prim(A) \simeq [0, 1].$$

Proposition (2.5) can be extended to an arbitrary unital C*-algebra.

Let A be an arbitrary unital C*-algebra. To each $I \in Prim(A)$, there corresponds an absorbing set

$$W_I := \{J \in Prim(A) : J \supseteq I\},$$

and an open set

$$O_I := \{J \in Prim(A) : J \subseteq I\},$$

containing I.

We have the following equivalent statements:

$$I \subseteq J \Leftrightarrow O_I \subseteq O_J \Leftrightarrow W_I \supseteq W_J$$

In [12] we have seen how $Prim(A)$ is made into a finite lattice with vertices $I_0, ..., I_n$.

Let

$$J_{i_0,...,i_k} := I_{i_0} \cap ... \cap I_{i_k},$$

where $1 \leq i_0, ..., i_k \leq n, 1 \leq k \leq n$.Set

$$W_{i_0,...,i_k} := \{J \in Prim(A) : J \supseteq J_{i_0,...,i_k}\}.$$

As we have seen in [12], these are the k-chain closed subsets of $Prim(A)$ having the following property

If $J_{i_0,...,i_k} = 0$ for some $1 \leq i_0, ..., i_k \leq n$, $1 \leq k \leq n$, then $W_{i_0,...,i_k} = Prim(A)$. Also for each pair of indices $(i_0, ..., i_t)$, $\sigma(i_0, ..., i_{t+m})$,

$$W_{i_0,...,i_t} \subseteq W_{\sigma(i_0,...,i_{t+m})}$$

where σ is a permutation on $t+m+1$ elements and $1 \leq i_0, ..., i_{t+m} \leq n$.

In the case of $Prim(C(X))$ when X has a CW complex structure, the k-chains are the closed sets

$$W_{i_0,...,i_k} = \{J \in Prim(A_k) : J \supseteq I_{C_k}\}$$

corresponding to the k-ideals I_{C_k} [12].

3. Basics of Modified Morse Theory on C*-Algebras

The first step towards understanding the geometry of noncommutative CW complexes was the idea of modified Morse theory on C*-algebras that we have done in [12]. In this section we review some of the results.

For a unital C*-algebra A let

$$\Sigma = \{W_{i_1,...,i_k}\}_{1\leq i_1,...,i_k\leq n, 1\leq k\leq n}$$

be the set of all k-chains $(k = 1, ..., n)$ in $Prim(A)$, and

$$\Gamma = \{I_{i_1,...,i_k}\}_{1\leq i_1,...,i_k\leq n, 1\leq k\leq n}$$

be the absorbing set of all k-ideals corresponding to the k-chains of Σ for $k = 1, ..., n$.

We recall the following definitions from [12].

Definition 3.1. Let $f : \Sigma \to \mathbb{R}$ be a function. The k-chain $W_k = W_{i_1,...,i_k}$ is called *a critical chain of order* k for f, if for each $(k+1)$-chain W_{k+1} containing W_k and for each $(k-1)$-chain W_{k-1} contained in W_k, we have

$$f(W_{k-1}) \leq f(W_k) \leq f(W_{k+1}).$$

The corresponding ideal I_k to W_k is called *the critical ideal of order* k.

Definition 3.2. Let f has a critical chain of order k. We say f is an *acceptable Morse function*, if it has a critical chain of order i, for all $i \leq k$.

Definition 3.3. A function $f : \Sigma \to \mathbb{R}$ is called *a modified Morse function* on the C*-algebra A, if for each k-chain W_k in Σ, there is at most one $(k+1)$-chain W_{k+1} containing W_k and at most one (k-1)-chain W_{k-1} contained in W_k, such that

$$f(W_{k+1}) \leq f(W_k) \leq f(W_{k-1}).$$

Definition 3.4. If A, B are two C*-algebras, two morphisms $\alpha, \beta : A \to B$ are *homotopic*, written $\alpha \sim \beta$,if there exists a family $\{H_t\}_{t\in[0,1]}$ of morphisms $H_t : A \to B$ such that for each $a \in A$ the map $t \mapsto H_t(a)$ is a norm continuous path in B with $H_0 = \alpha$ and $H_1 = \beta$.The C*-algebras A and B are said *to have the same homotopy type*, if there exists morphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi \sim id_B$ and $\psi \circ \varphi \sim id_A$. In this case the morphisms φ and ψ are called *homotopy equivalence*.

Definition 3.5. Let A and B be unital C*-algebras. We say A is *of pseudo-homotopy type* as B if $C(Prim(A))$ and B have the same homotopy type.

Remark 3.6. In the case of unital commutative C*-algebras, by the GNS construction [7], $C(Prim(A)) = A$, . So the notions of pseudo-homotopy type and the same homotopy type are equivalent.

Theorem 3.7. *If f is an acceptable modified Morse function on A, then $Prim(A)$ is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical chain of order p. Consequently every unital C*-algebra A with an acceptable modified Morse function f on it, is of pseudo-homotopy type as a noncommutative CW complex having a k-th decomposition cell for each critical chain of order k.*

4. Dynamical systems on noncommutative CW complexes

In this section we develop tools to relate a dynamical picture to the topology and geometry of noncommutative CW complexes. We will see how the dynamical properties of the trajectories are related to the homological configuration of noncommutative CW complexes.

Definition 4.1. Let X, Y be topological spaces, $\mathcal{P}(Y)$ be the power set of Y (the set of all subsets of Y) and $F : X \to \mathcal{P}(Y)$ be a mapping.

- The mapping F is called open hemi-continuous at $x \in X$, if for each open subset $B \subseteq Y$ such that $F(x) \subseteq B$, there exists an open set $U \subseteq X$ containing x such that
$$F(U) := \bigcup\{F(x) : x \in U\} \subseteq B.$$
- The mapping F is called closed hemi-continuous at $x \in X$, if for each closed subset $B \subseteq Y$ such that $F(x) \subseteq B$, there exists a closed set $K \subseteq X$ containing x such that
$$F(K) := \bigcup\{F(x) : x \in K\} \subseteq B.$$
- The mapping F is with compact value if for all $x \in X$, $F(x) \subseteq Y$ is a compact subset.

Definition 4.2. Let X be a topological space and $\mathcal{P}(X)$ be the power set of X. A mapping $\varphi : X \times \mathbb{Z} \to \mathcal{P}(X)$ is a discrete multivalued dynamical system on X if the following conditions satisfy:

- For each $n \in \mathbb{Z}$ the mapping $F_n : X \to \mathcal{P}(X)$ defined by $F_n(x) := \varphi(x, n)$, for all $x \in X$, is closed hemi-continuous for $n \in \mathbb{Z}^+$ and is open hemi-continuous for $n \in \mathbb{Z}^-$.
- The mapping F_1 is with compact value.
- For all $x \in X$, $\varphi(x, 0) = \{x\}$.
- For all $n, m \in \mathbb{Z}$ with $nm \geq 0$ and for all $x \in X$, $\varphi(\varphi(x, n), m) = \varphi(x, n + m)$.
- For all $x, y \in X$, $x \in \varphi(y, -1) \Leftrightarrow y \in \varphi(x, 1)$.

Remark 4.3. With the above notations if we let $(\varphi(x,1)) := F(x)$, then it follows that for all $x \in X$ and $n \geq 1$,

$$\varphi(x,n) = F^n(x),$$

where

$$F^n(x) = F(F^{n-1}(x)) := \bigcup\{F(z) : z \in F^{n-1}(x)\}$$

is defined inductively. So $F : X \to X$ is called the generator of the discrete multivalued dynamical system.

Let A be a unital C*-algebra and let $Prim(A)$ be the topological space associated with it as in the construction of the previous sections. Define two mappings

$$F, G : Prim(A) \to \mathcal{P}(Prim(A))$$
$$F(I) = W_I = \{J \in Prim(A) : J \supseteq I\}$$
$$G(I) = O_I = \{J \in Prim(A) : J \subseteq I\}$$

.

Lemma 4.4. *For all $I, J \in prim(A)$ we have $J \in O_I \Leftrightarrow I \in W_J$.*

Proof. We have

$$J \in O_I \Leftrightarrow J \subseteq I \Leftrightarrow I \in W_J.$$

□

For the mappings F, G defined above we have

Proposition 4.5. *The mapping F is closed hemi-continuous and the mapping G is open hemi-continuous.*

Proof. Let $I \in Prim(A)$, $W \subseteq Prim(A)$ be closed and $F(I) = W_I \subseteq W$. We show that there exists a closed subset $K \subseteq Prim(A)$ with $I \in K$ and $F(K) \subseteq W$. Set $K := W_I$. Then we have

$$F(W_I) = \bigcup\{F(J) : J \in W_I\} = \bigcup\{F(J) : J \supseteq I\} \subseteq W_I \subseteq W.$$

Since for $J \supseteq I$, we have $W_J \subseteq W_I$.

Now let $I \in Prim(A)$, $O \subseteq Prim(A)$ be open and $G(I) = O_I \subseteq O$. We show that there exists a open subset $U \subseteq Prim(A)$ with $I \in U$ and $G(U) \subseteq O$. Set $U := O_I$. Then we have

$$G(O_I) = \bigcup\{G(J) : J \in O_I\} = \bigcup\{G(J) : J \subseteq I\} \subseteq O_I \subseteq O.$$

Since for $J \subseteq I$, we have $O_J \subseteq O_I$. □

In the following we will see how the above mappings F, G generate a discrete multivalued dynamical system on $Prim(A)$.

Let $\varphi : Prim(A) \times \mathbb{Z}^+ \to \mathcal{P}(Prim(A))$ be defined in the following way:

For all $I \in Prim(A)$, set $\varphi(I,1) := F(I) = W_I$. For $n \in \mathbb{Z}^+$, define $\varphi(I,n)$ inductively by

$$\varphi(I,n) := F^n(I) = F(F^{n-1}(I)) = \bigcup\{F(I_{n-1}) : I_{n-1} \in F^{n-1}(I)\}$$

$$= \bigcup_{I_{n-1} \supseteq I_{n-2}} \dots \bigcup_{I_1 \supseteq I} F(I_{n-1})$$
$$= \bigcup_{I_{n-1} \supseteq \dots \supseteq I_1 \supseteq I} F(I_{n-1}).$$

The mapping φ has the following property:

Lemma 4.6. *For all $I \in Prim(A)$ and all $n, m \in \mathbb{Z}^+$, we have*

$$\varphi(I, n+m) = \varphi(\varphi(I, n), m).$$

Proof. We have $\varphi(I, n+m) = F^{n+m}(I)$. On the other hand

$$\varphi(\varphi(I, n), m) = \bigcup\{\varphi(J, m) : J \in \varphi(I, n)\}$$
$$= \bigcup\{F^m(J) : J \in F^n(I)\} = F^m(F^n(I)) = F^{m+n}(I).$$

□

In the same way let $\psi : Prim(A) \times \mathbb{Z}^- \to \mathcal{P}(Prim(A))$ be defined in the following way:

For all $I \in Prim(A)$, set $\psi(I, -1) := G(I) = O_I$. For $n \in \mathbb{Z}^+$, define $\psi(I, -n)$ inductively by

$$\psi(I, -n) := G^n(I) = G(G^{n-1}(I)) = \bigcup\{G(I_{n-1}) : I_{n-1} \in G^{n-1}(I)\}$$
$$= \bigcup_{I_{n-1} \subseteq I_{n-2}} \dots \bigcup_{I_1 \subseteq I} G(I_{n-1})$$
$$= \bigcup_{I_{n-1} \subseteq \dots \subseteq I_1 \subseteq I} G(I_{n-1}).$$

The mapping ψ has the following property:

Lemma 4.7. *For all $I \in Prim(A)$ and all $n, m \in \mathbb{Z}^+$, we have*

$$\psi(I, -n-m) = \psi(\psi(I, -n), -m).$$

Proof. We have $\psi(I, -n-m) = G^{n+m}(I)$. On the other hand

$$\psi(\psi(I, -n), -m) = \bigcup\{\psi(J, -m) : J \in \psi(I, -n)\}$$
$$= \bigcup\{G^m(J) : J \in G^n(I)\} = G^m(G^n(I)) = G^{m+n}(I)$$

. □

Proposition 4.8. *Let F, G, φ, ψ be as before. Let*

$$\Theta : Prim(A) \times \mathbb{Z} \to \mathcal{P}(Prim(A))$$

be defined by

$$\Theta(I, n) = \varphi(I, n) = F^n(I); \Theta(I, -n) = \psi(I, -n) = G^n(I); \Theta(I, 0) = \{I\}$$

for all $I \in Prim(A), n \in \mathbb{Z}^+$. Then Θ defines a discrete multivalued dynamical system on $Prim(A)$ with generators F, G.

Proof. We have to check the properties of definition (4.2) for Θ. First of all for each $I \in Prim(A)$, $\Theta(I,1)$ is compact. Moreover from proposition (4.5), the hemi-continuity property satisfies for Θ. Also

- For all $I \in Prim(A)$, we have $\Theta(I,0) = \{I\}$.
- For all $n, m \in \mathbb{Z}$ with $nm \geq 0$ it follows from lemmas (4.6) and (4.7),

$$\Theta(I, n+m) = \Theta(\Theta(I,n), m).$$

And eventually from the lemma (4.4), for all $I, J \in Prim(A)$ we have

$$J \in \Theta(I,1) = F(I) = W_I \Leftrightarrow I \in \Theta(J,-1) = G(I) = O_I.$$

$\square$

Remark 4.9. With the notations of the previous proposition, if for each $W \subseteq Prim(A)$ we define $F^{-1}(W) := \{J \in Prim(A) : F(J) \subseteq W\}$, then the proof of the above proposition shows that $G = F^{-1}$. For this reason sometimes we refer to F as the only generator of the system.

Definition 4.10. Let A be a unital C*-algebra, $\Theta : Prim(A) \times \mathbb{Z} \to \mathcal{P}(Prim(A))$ be a discrete multivalued dynamical system with generator F, $k, m \in \mathbb{Z}^+$ and $[-k,m]$ be an interval in $\mathbb{Z}$ containing $0 \in \mathbb{Z}$. Let $\{I_i\}_{-k \leq i \leq m}$ be a sequence in $Prim(A)$ such that

$$\forall -k \leq i \leq m \quad ; \quad I_{i+1} \in F(I_i).$$

Define a map $\alpha : [-k,m] \to Prim(A)$ by $\alpha(i) = I_i$, for all $-k \leq i \leq m$. obviously $\alpha(i+1) \in F(\alpha(i))$.

With these notations α is called a solution for F and the sequence $\{I_i\}_{-k \leq i \leq m}$ is called a trajectory for F passing through $\alpha(0) = I_0$.

With these notations:

Proposition 4.11. *If $\alpha : [-k,m] \to Prim(A)$ is a solution for F, then for each $i \in [-k,m]$, $\alpha(i) \in F^i(\alpha(0))$.*

Proof. We prove the statement by induction on k, m. The induction is in two parts: positive and negative parts of the interval.

For $k = 0, m = 1$, we have $\alpha(1) \in F(\alpha(0))$. Now suppose $\alpha(i) \in F^i(\alpha(0))$, for $0 \leq i \leq m$. We show that $\alpha(m+1) \in F^{m+1}(\alpha(0))$. We have

$$F^{m+1}(\alpha(0)) = F(F^m(\alpha(0))) = \bigcup\{F(J) : J \in F^m(\alpha(0))\}.$$

Set $J = \alpha(m)$. Then $F(\alpha(m)) \subseteq F^{m+1}(\alpha(0))$. On the other hand we have $\alpha(m+1) \in F(\alpha(m))$. So $\alpha(m+1) \in F^{m+1}(\alpha(0))$. So the induction on the positive part is completed.

Now we go through the second part of the induction. The proof of this part is the same as the first part with a minor difference. We just have to note that for $k = -1, m = 0$, we have $\alpha(0) \in F(\alpha(-1)) = W_{\alpha(-1)}$. Consequently $\alpha(-1) \in O_{\alpha(0)}$, which means $\alpha(-1) \in G(\alpha(0)) = F^{-1}(\alpha(0))$. Now if $\alpha(i) \in F^i(\alpha(0))$, for $-k \leq i \leq -1$. We can easily see that $\alpha(-k-1) \in F^{-k-1}(\alpha(0))$. $\square$

In what follows $\Theta : Prim(A) \times \mathbb{Z} \to \mathcal{P}(Prim(A))$ is a discrete multivalued dynamical system on $Prim(A)$ with generator F.

Definition 4.12. Let α be a solution for F and $\{I_i\}_{-k\leq i\leq m}$ be a trajectory for F passing through $\alpha(0) = I_0$. The ideal I_0 is called a fixed point for F if there exist $W \subseteq Prim(A)$ such that for all n, $F^n(I_0) = W$. Consequently for all n, $\alpha(n) \in W$.

Definition 4.13. The unstable manifolds of F at point $I \in Prim(A)$ is defined by

$$W^u(I, F) = \bigcup_{n\geq 1} F^n(I).$$

In the same way the stable manifold of F at I is defined by

$$W^s(I, F) = \bigcup_{n\geq 1} F^{-n}(I) = \bigcup_{n\geq 1} G^n(I).$$

Proposition 4.14. *Let $I, J \in Prim(A)$ and $W^u(I, F) \bigcap W^s(J, F) \neq \varnothing$. Then there exists a trajectory $\{L_i\}_{0\leq i\leq m}$ for F from I to J, i.e. $L_0 = I, L_m = J$.*

Proof. Let $L \in W^u(I, F) \bigcap W^s(J, F)$. Then $L \in W^u(I, F) = \bigcup_{n\geq 1} F^n(I)$. So there exists $n_0 \geq 1$ such that

$$L \in F^{n_0}(I) = F^{n_0-1}(F(I)) = \bigcup\{F^{n_0-1}(D) : D \in F(I)\}$$

So there exists $L_1 \in F(I)$ such that

$$L \in F^{n_0-1}(L_1) = F^{n_0-2}(F(L_1)) = \bigcup\{F^{n_0-2}(D) : D \in F(L_1)\}$$

So there exists $L_2 \in F(L_1)$ with $L \in F^{n_0-2}(L_2)$. Continuing in this process we obtain a sequence $\{L_1, ..., L_{n_0}\}$ with the property that $L_{i+1} \in F(L_i)$ and $L \in F^{n_0-i}(L_i)$ for all $1 \leq i \leq n_0 - 1$.

Now the sequence $\{L_0, L_1, ..., L_{n_0}\}$ with $L_0 = I, L_{n_0} = L$ is a trajectory for F from I to L.

On the other hand we have $L \in W^s(J, F) = \bigcup_{n\geq 1} F^{-n}(J) = \bigcup_{n\geq 1} G^n(J)$. So there exists $m \geq 1$ such that

$$L \in G^m(J) = G^{m-1}(G(J)) = \bigcup\{G^{m-1}(D) : D \in G(J)\}$$

So there exists $D_{m-1} \in G(J)$ such that

$$L \in G^{m-1}(D_{m-1}) = G^{m-2}(G(D_{m-1})) = \bigcup\{G^{m-2}(D) : D \in G(D_{m-1})\}$$

So there exists $D_{m-2} \in G(D_{m-1})$ with $L \in G^{m-2}(D_{m-2})$. Continuing in this process we obtain a sequence $\{D_1, ..., D_{m-1}\}$ with the property that $D_{m-i} \in G(D_{m-i+1})$, for all $2 \leq i \leq m - 1$ and $L \in G^{m-i}(D_{m-i})$, for $1 \leq i \leq m - 1$. This means that $D_{m-i+1} \in F(D_{m-i})$, for all $2 \leq i \leq m - 1$. Set $D_m = J$. From $D_{m-1} \in G(J)$, it follows that $J \in F(D_{m-1})$.

Now the sequence $\{D_0, D_1, ..., D_{m-1}, D_m\}$ with $D_0 = L, D_m = J$ is a trajectory for F from L to J. If we rename $D_i = L_{n_0+i}$, for $0 \leq i \leq m$, then the sequence $\{L_0, L_1, ..., L_{n_0+m}\}$ is a trajectory for F from I to J. $\square$

In the next section we go through an example to have a better understanding of the constructions of this section.

5. Dynamical system on $C(T^2)$

In [12] we explained how the CW complex structure for a compact topological space X induces a noncommutative CW complex structure on the algebra $C(X)$ of continuous functions on X. In this part we apply the techniques of the previous section to introduce a discrete multivalued dynamical system on the noncommutative CW complex structure of $C(T^2)$: the algebra of continuous functions on the 2-dimensional torus. We compute the stable and nonstable manifolds and explain the geometry of the noncommutative CW complex by its stable and unstable manifolds.

Consider the following CW complex structure for the torus T^2.

$X_0 = \{0\}$ is the one point set, $X_1 = \{\alpha, \beta\}$, where α, β are closed curves homeomorphic images of the circle S^1, starting at point 0 and $X_2 = T^2$.

The noncommutative CW complex decomposition on $C(T^2)$ is induced as: $A_0 = C(X_0), A_1 = C(X_1), A = A_2 = C(T^2)$. To each $x \in X$ there corresponds an ideal $I_x \in Prim(C(X))$ defined by

$$I_x := \{f \in C(X) : f(x) = 0\}.$$

We can partition $Prim(A)$ into three classes of ideals:

There is only one 0-ideal defined by $I_0 := \{f \in A : f(0) = 0\}$.

There are two 1-ideals defined by

$$I_\alpha := \{f \in A : f(x) = 0; x \in \alpha\} = \bigcap_{x\in\alpha} I_x,$$

$$I_\beta := \{f \in A : f(x) = 0; x \in \beta\} = \bigcap_{x\in\beta} I_x.$$

There is one 2-ideal defined by $I := \{f \in A : f(x) = 0; x \in T^2\} = \{0\}$.

Obviously $I \subseteq I_\alpha, I_\beta \subseteq I_0$. We have

$$W_0 = \{J \in Prim(A) : J \supseteq I_0\} = \{I_0\}$$
$$W_I = \{J \in Prim(A) : J \supseteq I\} = \{I_0, I_\alpha, I_\beta, I\}$$
$$W_\alpha = \{J \in Prim(A) : J \supseteq I_\alpha\} = \{I_0, I_\alpha\}$$
$$W_\beta = \{J \in Prim(A) : J \supseteq I_\beta\} = \{I_0, I_\beta\}$$

And $W_0 \subseteq W_\alpha, W_\beta \subseteq W_I$.

On the other hand we have

$$O_0 = \{J \in Prim(A) : J \subseteq I_0\} = \{I_0, I_\alpha, I_\beta, I\}$$
$$O_I = \{J \in Prim(A) : J \subseteq I\} = \{I\}$$
$$O_\alpha = \{J \in Prim(A) : J \subseteq I_\alpha\} = \{I, I_\alpha\}$$
$$O_\beta = \{J \in Prim(A) : J \subseteq I_\beta\} = \{I, I_\beta\}$$

And $O_I \subseteq O_\alpha, O_\beta \subseteq O_0$.

Now we start our computations.

• The ideal I_0: We have $\Theta(I_0,1) = F(I_0) = W_0 = \{I_0\}$. Also

$$\Theta(I_0,2) = F^2(I_0) = F(F(I_0)) = F(W_0) = \bigcup\{F(J) : J \in W_0\} = F(I_0) = \{I_0\}.$$

Continuing with this process, we see that for each $n \geq 1$ we have

$$\Theta(I_0,n) = F^n(I_0) = \{I_0\} = W_0.$$

We have

$$\Theta(I_0,-1) = G(I_0) = O_0 = \{I_0, I_\alpha, I_\beta, I\}$$
$$\Theta(I_0,-2) = G^2(I_0) = G(G(I_0)) = \bigcup\{G(J) : J \in G(I_0)\}$$
$$= \bigcup\{G(J) : J = I_0, I_\alpha, I_\beta, I\} = G(I_o)\bigcup G(I_\alpha)\bigcup G(I_\beta)\bigcup G(I)$$
$$= O_0 = \{I_0, I_\alpha, I_\beta, I\}$$

In the same way we see that for all $n \geq 1$,

$$\Theta(I_0,-n) = F^{-n}(I_0) = G^n(I_0) = \{I_0, I_\alpha, I_\beta, I\} = O_0$$

• The ideal I_α: We have $\Theta(I_\alpha,1) = F(I_\alpha) = W_\alpha = \{I_0, I_\alpha\}$. Also

$$\Theta(I_\alpha,2) = F^2(I_\alpha) = F(F(I_\alpha)) = F(W_\alpha) = \bigcup\{F(J) : J \in W_\alpha\}$$
$$= \bigcup\{F(J) : J = I_0, I_\alpha\} = F(I_0)\bigcup F(I_\alpha) = W_\alpha.$$

Continuing with this process, we see that for each $n \geq 1$ we have

$$\Theta(I_\alpha,n) = F^n(I_\alpha) = W_\alpha.$$

For the negative part We have

$$\Theta(I_\alpha,-1) = G(I_\alpha) = O_\alpha = \{I_\alpha, I\}$$
$$\Theta(I_\alpha,-2) = G^2(I_\alpha) = G(G(I_\alpha)) = \bigcup\{G(J) : J \in G(I_\alpha)\}$$
$$= \bigcup\{G(J) : J = I_\alpha, I\} = G(I_\alpha)\bigcup G(I)$$
$$= O_\alpha = \{I_\alpha, I\}$$

In the same way we see that for all $n \geq 1$,

$$\Theta(I_\alpha,-n) = F^{-n}(I_\alpha) = G^n(I_\alpha) = O_\alpha = \{I_\alpha, I\}$$

• The ideal I_β: For this ideal as in the case of I_α we can see that for all $n \in \mathbb{Z}^+$,

$$\Theta(I_\beta,n) = F^n(I_\beta) = \{I_\beta, I_0\}$$
$$\Theta(I_\beta,-n) = F^{-n}(I_\beta) = \{I_\beta, I\}$$

• The ideal I: We have $\Theta(I,1) = F(I) = W_I = \{I_0, I_\alpha, I_\beta, I\}$. Also

$$\Theta(I,2) = F^2(I) = F(F(I)) = F(W_I) = \bigcup\{F(J) : J \in W_I\}$$
$$= F(I_0)\bigcup F(I_\alpha)\bigcup F(I_\beta)\bigcup F(I) = \{I_0, I_\alpha, I_\beta, I\}.$$

Continuing with this process, we see that for each $n \geq 1$ we have

$$\Theta(I,n) = F^n(I) = W_I = \{I_0, I_\alpha, I_\beta, I\}$$

We have

$$\Theta(I,-1) = G(I) = O_I = \{I\}$$

$$\Theta(I,-2) = G^2(I) = G(G(I)) = \bigcup\{G(J) : J \in G(I)\} = G(I) = \{I\}$$

In the same way we see that for all $n \geq 1$,

$$\Theta(I,-n) = F^{-n}(I) = G^n(I) = \{I\} = O_I$$

Now we explain the trajectories of the system and find the fixed points and observe the behavior of the stable and unstable manifolds at the fixed points.

First we consider the sequence $\{I, I_\alpha, I_0\}$. For this sequence we have

$$I_\alpha \in F(I), \quad I_0 \in F(I_\alpha)$$

Therefore the sequence defines a trajectory for the system. Now if we define a curve $\sigma : [0,2] \subseteq \mathbb{Z} \to Prim(A)$ by

$$\sigma(0) = I, \quad \sigma(1) = I_\alpha, \quad \sigma(2) = I_0$$

then we have $\sigma(n) \in F(\sigma(n-1))$ and $\sigma(n) \in F^n(\sigma(0))$ for $n = 1,2$. On the other hand $F^n(\sigma(0)) = F^n(I) = W_I$. So I is a fixed point for this trajectory. The unstable and stable manifolds would be

$$W^u(I,F) = \bigcup_{n=1,2} F^n(I) = W_I = \{I_0, I_\alpha, I_\beta, I\}.$$

$$W^s(I,F) = \bigcup_{n\geq 1} F^{-n}(I) = \bigcup_{n=1,2} G^n(I) = O_I = \{I\}.$$

From the above calculations we can conclude that the whole $Prim(A)$ is unstable and the ideal I corresponded to the critical chain W_I of the modified discrete function on $prim(A)$ is stable, we refer to [12] for details on critical chains. This critical chain corresponds to the maximum point of the Morse height function on T^2. Since the compact torus T^2 is homeomorphic to the space $Prim(A)$ [7], this is a natural conclusion comparing to the unstability of torus.

We have another beautiful interpretation:

$$W^u(I_0,F) = \bigcup_n F^n(I_0) = W_0 = \{I_0\}.$$

$$W^s(I_0,F) = \bigcup_{n\geq 1} F^{-n}(I_0) = \bigcup_{n\geq 1} G^n(I_0) = O_0 = \{I_0, I_\alpha, I_\beta, I\}.$$

Which means that the stable and unstable manifolds are interchanged along suitable trajectories.

References

[1] M. Allili, D. Corriveau, S. Deriviere, T. Kaczynski and A. Trahan, *Discrete dynamical system framework for construction of connections between critical regions in lattice height data*, J. Math. Imaging and Vision **28** (2007), 99–111.

[2] A. Connes, *Noncommutative Geometry* (Academic Press, San Diego 1994).

[3] J. Cuntz, *Noncommutative Simplicial Complexes and the Baum-Connes Conjecture*, GAFA, Geom. Funct. Anal. **12** (2002), 307–329.

[4] J. Cuntz, *Quantum Spaces and Their Noncommutative Topology*, AMS Notices **8** (2001), 793–799.

[5] A. Duval, *A combinatorial decomposition of simplicial complexes*, Isreal J. of Math. **87** (1994), 77–87.

[6] S. Eilers, T. A. Loring and G. K. Pedersen, *Stability of anticommutation relations: an application to NCCW complexes*, J. Reine Angew Math. **99** (1998), 101–143.

[7] J. M. Gracia-Bondia, J. C. Varilly and H. Figueroa, *Elements of Noncommutative Geometry*, (Birkhauser, 2001).

[8] A. Hatcher, *Algebraic Topology*, (Cambridge Univ. Press, 2002).

[9] T. Kaczynski and M. Mrozek, *conley index for discrete multivalued dynamical systems*, Topology and its Applications **65** (1995), 83–96.

[10] G. Landi, *An Introduction to Noncommutative Spaces and Their Geometry*, Lecture notes in physics monographs **51** (Springer , 1998).

[11] J. Milnor, *Morse Theory*, Annals of Math. Studies, (Princeton Univ. Press, 1963).

[12] V. Milani, A. A. Rezaei and S. M. H. Mansourbeigi, *Morse theory for C*-Algebras: A geometric interpretation of some noncommutative manifolds*, Applied General Topology **12** (2011), 175–185.

[13] G. K. Pedersen, *Pull back and pushout constructions in C*-Algebras*, J. Funct. Analysis **167** (1999), 243–344.

[14] J. H. C. Whitehead, *Combinatorial Homotopy*, I. Bulletin of the American Society **55** (1949), 1133–1145.

