Rainich-type Conditions for Null Electrovacuum Spacetimes II

Synopsis

• In this second of two worksheets I continue describing local Rainich-type conditions which are necessary and sufficient for the metric to define a null electrovacuum. In other words, these conditions, which I will call the null electrovacuum conditions, guarantee the existence of a null electromagnetic field such that the metric and electromagnetic field satisfy the Einstein-Maxwell equations. When it exists, the electromagnetic field is easily constructed from the metric. The results illustrated here are based upon [1].

• In this worksheet I consider the null electrovacuum conditions which apply when a certain null geodesic congruence determined by the metric is twisting. I shall illustrate these conditions using a couple of pure radiation spacetimes taken from the literature [3, 4].

• A companion worksheet (Rainich-type Conditions for Null Electrovacuum Spacetimes I) treats the twist-free case, which is considerably simpler.

Theory

• Let \((M, g)\) be a spacetime – a 4-dimensional manifold \(M\) endowed with a Lorentz-signature metric \(g\). The Rainich conditions are geometric conditions on \(g\) such that there exists an electromagnetic field \(F\) with \((g, F)\) satisfying the Einstein-Maxwell equations.

\[
G_{ab} = F_a^c F_{bc} - \frac{1}{4} g_{ab} F_{mn} F^{mn}, \quad \nabla [a F_{bc}] = 0, \quad \nabla_a F^{ab} = 0.
\]

The classical Rainich conditions involve the metric, the Ricci tensor \(R_{ij}\), the covariant derivative \(\nabla\), and the volume form \(\epsilon_{ijhk}\) of the metric, and are given by

\[
R^i_h R^h_j - \frac{1}{4} \delta^i_j R^{hk} R_{hk} = 0, \quad R^i_i = 0, \quad R_{ij} \not\approx 0, \quad \nabla [j \alpha_i] = 0, \quad \text{where} \quad \alpha_i = \frac{\epsilon_{ijhk} R^l_m \nabla^h R^{nk}}{R^n_l R^l_j}.
\]

Here \(i^j\) is any timelike vector field. When these conditions are satisfied there is a straightforward procedure for constructing the electromagnetic field (see RainichElectromagneticField), which is determined by the metric up to a duality rotation:
\[ F_{ab} \rightarrow \cos(\theta) \, F_{ab} - \sin(\theta) \ast F_{ab}, \quad \theta \in \mathbb{R}, \]

where \( \ast \) denotes the Hodge dual on 2-forms determined by the metric. If a metric satisfies the Rainich conditions we say that it determines an electrovacuum spacetime.

• A spacetime admits a non-null electromagnetic source if and only if it satisfies the Rainich conditions.

• The Rainich conditions are not defined for null electrovacua, i.e., solutions of the Einstein-Maxwell equations with a null electromagnetic field,

\[ F_{ab} F^{ab} = 0, \quad F_{ab} \ast F^{ab} = 0 \]

because such fields have a null energy-momentum tensor and hence a null Ricci tensor:

\[ R_{ab} R^{ac} = 0. \]

The Rainich conditions do not provide local, geometric criteria for null electrovacua.

• A geometric description of null electrovacua is as follows [1]. A metric with a null Ricci tensor – a pure radiation spacetime – determines a null vector field \( k^a \) via

\[ G_{ab} = R_{ab} = \frac{1}{4} k^a k_b. \]

(1)

The contracted Bianchi identity implies

\[ k^b \nabla_b k_a = \left( \nabla_b k^b \right) k_a, \]

which implies that the congruence generated by this vector field \( k^a \) is a geodesic congruence. The vector field \( k^a \) determines a family of 2-forms,

\[ f_{ab} = k_{[a} s_{b]}, \]

(2)

where \( s^b \) is any spacelike unit vector orthogonal to \( k^a \). The energy-momentum tensor of \( f_{ab} \) satisfies the Einstein equations in the sense that its energy-momentum tensor equals the Ricci tensor (which is the same as the Einstein tensor in the null case). Therefore, if there is a solution \( F_{ab} \) to the Einstein-Maxwell equations, at each point of \( M \) it must be related to \( f_{ab} \) by a duality rotation. Thus there will exist a function \( \varphi : M \rightarrow \mathbb{R} \) such that the electromagnetic field takes the form

\[ F_{ab} = \cos(\varphi) f_{ab} - \sin(\varphi) \ast f_{ab}. \]

(3)

The Maxwell equations for \( F_{ab} \) impose a number of conditions on \( k^a \) and \( \varphi \). In particular,
the vector field \( k^a \) defines a shear-free, null, geodesic congruence. In terms of a null tetrad whose first leg is \( k^a \), and using the associated Newman-Penrose formalism [2], these conditions on \( k^a \) take the form

\[
\sigma = 0 = \kappa \quad \frac{1}{2} (\rho + \overline{\rho}) = \epsilon + \overline{\epsilon} .
\]  

To analyze the conditions imposed by the Maxwell equations on \( \varphi \), one must distinguish two cases: the congruence tangent to \( k^a \) is (i) twisting or (ii) is twist-free. The twist, denoted \( \omega \), is defined by

\[
\omega = - \text{Im}(\rho).
\]

The solution spaces for \( \varphi \) – and hence for the electromagnetic field – are significantly different in these two cases. This worksheet considers the twisting case only (see the companion worksheet for the twist-free case).

- In the twisting case the function \( \varphi \) determining the duality rotation must satisfy

\[
\frac{1}{i} \delta \varphi + \tau - 2 \beta = 0, \quad \frac{1}{i} \overline{\delta \varphi} - \overline{\tau} + 2\overline{\beta} = 0, \quad \frac{1}{i} D\varphi - \epsilon + \overline{\epsilon} = 0, \quad \omega \Delta \varphi + i \text{Im}(\mu) (\rho - 2 \epsilon)
\]

\[
+ \text{Re}[(\delta + \overline{\beta} - \alpha)(\tau - 2 \beta)] = 0 .
\]

(5)

Here the letters \( \alpha, \tau, \beta, \epsilon, \delta, \rho, \mu, D, \Delta \) denote the standard Newman-Penrose quantities [2], which are determined once \( k^a \) has been incorporated into the first leg of a null tetrad. As shown in [1], there are two non-trivial integrability conditions for these equations. We express them as a single complex-valued condition:

\[
\mathcal{J} = 0,
\]

where

\[
\mathcal{J} = \omega \delta(\text{Re}(\overline{\delta} - 2\beta)) - \delta \omega + \omega (\tau - \alpha - \beta) \left[ \text{Re}\{(\delta + \overline{\beta} - \alpha)(\tau - 2 \beta)\} \right]
\]

\[
+ i \text{Im}(\mu)(\rho - 2 \epsilon) + \frac{\omega}{2} \left[ \beta \delta(2 \alpha + \tau - 4 \beta) + \beta \delta(2 \alpha + \overline{\tau} - 4 \overline{\beta}) + 2 i \delta \left[ \text{Im}(\mu)(\rho - 2 \epsilon) + \tau \delta(\overline{\beta} - \alpha) + \overline{\tau} \delta(\beta - \alpha) - \alpha \delta(\tau - 2 \beta) - \overline{\alpha} \delta(\overline{\tau} - 2 \overline{\beta}) \right] - i \omega^2 \Delta(\tau - 2 \beta)
\]

\[
+ \omega^2 \left[ \nu(\omega + 2 \text{Im}(\epsilon)) - (\tau - 2 \beta)(2 \text{Im}(\gamma) + i \mu) + i \lambda(\overline{\tau} - 2 \overline{\beta}) \right].
\]

(6)

- Conditions (4) – (6) are suitably invariant under the set of local Lorentz transformations which fix \( k^a \); they represent invariant conditions which are defined independently of the choice of null tetrad adapted to \( k^a \).
Conditions (4) and (6) on the null congruence determined by the Ricci tensor (1) provide geometric conditions on the spacetime geometry which are necessary and sufficient for the existence of a (null) electromagnetic source [1]. Thus we obtain Rainich-type conditions for null electrovacuum spacetimes.

Any two solutions to (5) differ by a solution to \( \delta \phi = \delta \phi = D \phi = \Delta \phi = 0 \), i.e., a constant. A metric satisfying (4) and (6) therefore determines the electromagnetic field up to a duality rotation.

### Procedures for computing the Maxwell equations and their integrability conditions.

The following procedure computes the Maxwell equations (3) for the function \( \phi \) and isolates the 4 coordinate derivatives. The input is a table of spin coefficients and a table of directional derivatives, both computed from a null tetrad adapted to the principal null vector \( \mathbf{k} \) using the commands \texttt{NPSpinCoefficients} and \texttt{NPDirectionalDerivatives}.

\[
\texttt{NullMaxwellEquations} := \texttt{proc}(\texttt{NPS}, \texttt{NPD})
\]

The following procedure computes the integrability conditions \( \mathcal{J} \) in (4). The input is a table of spin coefficients and a table of directional derivatives, both computed from a null tetrad adapted to the principal null vector \( \mathbf{k} \).

\[
\texttt{NullElectrovacuumConditions} := \texttt{proc}(\texttt{NPS}, \texttt{NPD})
\]

If the command \texttt{NullElectrovacuumConditions} is executed with no arguments, the formula for the integrability conditions is displayed. The quantity \( \delta \mathcal{J} \) is the conjugate operator to \( \delta \).

The symbols \texttt{conj(x)}, \texttt{im(x)}, and \texttt{re(x)} denote the complex conjugate, imaginary part, and real part of \( x \), respectively.

\[
\begin{align*}
\omega \delta (r e(\delta \mathcal{J}(\tau - 2 \beta))) + \omega^2 \text{conj}(\nu) (\omega + 2 \text{im}(\epsilon)) - 1 \omega^2 A(\tau - 2 \beta) - (\delta(\omega) \\
+ \omega (\tau - \text{conj}(\alpha) - \beta)) (r e(\delta \mathcal{J}(\tau - 2 \beta) + (\text{conj}(\beta) - \alpha) (\tau - 2 \beta)) \\
+ 1 \text{im}(\mu) (\rho - 2 \epsilon) + \frac{1}{2} (\omega \text{conj}(\beta) \delta(2 \text{conj}(\alpha) + \tau - 4 \beta) + \beta \delta(2 \alpha) \\
+ \text{conj}(\tau) - 4 \text{conj}(\beta)) + 2 \delta(\text{im}(\mu) (\rho - 2 \epsilon)) + \tau \delta(\text{conj}(\beta) - \alpha) \\
+ \text{conj}(\tau) \delta(\beta - \text{conj}(\alpha))) + \omega^2 ((\tau - 2 \beta) (-2 \text{im}(\gamma) - 1 \mu) \\
+ 1 \text{conj}(\lambda) \text{conj}(\tau - 2 \beta)) - \frac{\omega (\alpha \delta(\tau - 2 \beta) + \text{conj}(\alpha) \delta(\text{conj}(\tau - 2 \beta)))}{2}
\end{align*}
\]
These two bits of code must be executed (e.g., by clicking on the code edit region) before running the following examples.

### Example 1: Nurowski-Tafel electrovacuum

Nurowski and Tafel have constructed a class of algebraically special solutions of the Einstein-Maxwell equations [4] with null electromagnetic field. In particular, they have found the only known solutions which have twisting rays and a purely radiative electromagnetic field. These solutions are built from a freely specifiable holomorphic function \( b(\xi) \) and two parameters, \( \alpha \) and \( c2 \). We consider a special case of these solutions, specialized to Petrov type III \( (c2 = 0) \) and with \( b(\xi) = b/\xi \), where \( b \) is a real constant. To avoid confusion with the Newman-Penrose spin coefficients, we relabel \( \alpha \) as the parameter \( a \). We verify that the integrability conditions (6) are satisfied and construct the electromagnetic field from the metric using (5) and (3).

### Set-up.

We begin by defining the spacetime manifold and various auxiliary quantities needed to define the metric.

```math
> with(DifferentialGeometry): with(Tensor): with(Tools):
> DGsetup([u, r, xi, xil], M, complexconjugatepairs=[[xi, xil]])

frame name: M

(3.1.1)

\[
M > \text{bet} := \text{evalDG}( (r+I*\Sigma)/b(xi)/(1+xi*xil) * dxi);
\]

\[
\text{bet} := \frac{r + I \Sigma}{b(\xi)} \frac{d\xi}{(\xi \xi l + 1)}
\]

(3.1.2)

\[
M > \text{bet1} := \text{simplify(evalDG}((r - I* \Sigma)/b1(xi1)/(1+ xi*xi1) * dxil), \text{symbolic});
\]

\[
\text{bet1} := -\frac{I \Sigma - r}{b1(\xi l)} \frac{d\xi l}{(\xi \xi l + 1)}
\]

(3.1.3)

\[
M > \theta3 := \text{evalDG}(du + L* dxi + L1 * dxil);
\]

\[
\theta3 := du + L \frac{d\xi}{\xi} + L1 dxil
\]

(3.1.4)

\[
M > \Sigma := _\alpha*(1 - xi*xil)/(1+ xi*xil);
\]

\[
\Sigma := \frac{\alpha (-\xi \xi l + 1)}{\xi \xi l + 1}
\]

(3.1.5)

\[
M > L := I* _\alpha*xi/(b(xi)*b1(xil)*(1 + xi * xil)^2) + I* _\alpha* \int (xil*diff(ln(b1(xil)), xil)/(b(xi)*b1(xil)*(1+ xi*xil)^2), xil);
\]

\[
L := \frac{1 - \alpha \xi l}{b(\xi) b1(\xi l) (\xi \xi l + 1)^2} + I - \alpha \left( \begin{array}{c}
\xi l \left( \frac{d}{d\xi l} b1(\xi l) \right) \\
\frac{b1(\xi l)^2 b(\xi)}{b1(\xi l)^2 b(\xi) (\xi \xi l + 1)^2} d\xi l
\end{array} \right)
\]

(3.1.6)

\[
M > L1 := -I* _\alpha*xi/(b1(xil)*b(xi)*(1 + xi * xil)^2) - I
\]

```
Here is a general form of the metric considered in [4].

\[ g_0 := \text{evalDG}(2*(r^2 + \Sigma^2)/(b(\xi)*b1(\xi1)*(1+ \xi*\xi1)^2) * d\xi \wedge d\xi1 - 2*\theta3 \wedge (dr + 2*\alpha*I/(1 + \xi*\xi1)^2*(\xi*d\xi1 - \xi1 * d\xi) + (b(\xi)*b1(\xi1) - c2*r/(r^2 + \Sigma^2))*\theta3)): \]

Here we specialize to the case \( b(\xi) = \frac{b}{\xi} \). We also fix \( c2=0 \) which means the spacetime is Petrov type III. The metric \( g \) is quite complicated, so we do not display it.

\[ ch := \{b(\xi) = b/\xi, b1(\xi1) = b/\xi1, c2=0\}; \]

\[ ch := \left[ b(\xi) = \frac{b}{\xi}, b1(\xi1) = \frac{b}{\xi1}, c2 = 0 \right] \]

\[ M > g := \text{factor}(\text{eval}(g0, ch)):\]

Adapted tetrad, spin coefficients and the integrability conditions.

Our next task is to identify the tangent vector \( k \) to the preferred congruence and construct a tetrad adapted to it. To this end, we begin by constructing a convenient null tetrad which is used in [4].

\[ M > \omega_0 := \text{evalDG}((dr + 2*\alpha*I/(1 + \xi*\xi1)^2*(\xi*\xi1 - \xi1 * d\xi) + (b(\xi)*b1(\xi1) - c2*r/(r^2 + \Sigma^2))*\theta3)): \]

\[ M > \omega_1 := \text{eval}(\omega_0, ch); \]

\[ M > \text{coframe} := \text{factor}(\text{eval}((\omega_0, \theta3, \beta, \beta1), ch)); \]

\[ \text{coframe} := \left[ \frac{b^2}{\xi \xi l} du + dr - \frac{1}{\xi^2 \xi l (\xi l + 1)^2} \left( 1 - \alpha \left( \ln(\xi l + 1) \xi^2 \xi l^2 + \xi l^2 \xi^2 + 2 \ln(\xi l + 1) \xi l + \xi l + \ln(\xi l + 1) + 1 \right) \right) d\xi \right. \]

\[ \left. + \frac{1}{\xi \xi l^2 (\xi l + 1)^2} \left( 1 - \alpha \left( \ln(\xi l + 1) \xi^2 \xi l^2 + \xi l^2 \xi^2 + 2 \ln(\xi l + 1) + 1 \right) \right) d\xi \right. \]

\[ \left. + \frac{1}{\xi \xi l^2 (\xi l + 1)^2} \left( 1 - \alpha \left( \ln(\xi l + 1) \xi^2 \xi l^2 + \xi l^2 \xi^2 + 2 \ln(\xi l + 1) + 1 \right) \right) d\xi \right. \]
+ 1) \xi \xi I + \xi \xi I + \ln(\xi \xi I + 1) + 1 \right) \ dxI, \ du

- \frac{1}{\xi b^2 (\xi \xi I + 1)^2} \left( 1 \left( \ln(\xi \xi I + 1) \xi^2 \xi I^2 - \xi I^2 \xi^2 + 2 \ln(\xi \xi I + 1) \xi \xi I

+ \xi \xi I + \ln(\xi \xi I + 1) + 1 \right) \right) \ d\xi + \frac{1}{\xi I b^2 (\xi \xi I + 1)^2} \left( 1 \left( \ln(\xi \xi I

+ 1) \xi^2 \xi I^2 - \xi I^2 \xi^2 + 2 \ln(\xi \xi I + 1) \xi \xi I + \xi \xi I + \ln(\xi \xi I + 1) + 1 \right) \right) \ d\xi

\ d\xiI, - \frac{1}{(\xi \xi I + 1)^2} \frac{1}{b} \left( \frac{\alpha \xi \xi I + 1 r \xi \xi I + \alpha + 1 r}{\xi \xi I + 1} \right) \ d\xi,

+ \frac{1}{(\xi \xi I + 1)^2} \frac{1}{b} \left( \frac{\alpha \xi \xi I + 1 r \xi \xi I + \alpha + 1 r}{\xi \xi I + 1} \right) \ d\xiI \right]

Here we check that this coframe does define a null tetrad.

\textbf{N} > \texttt{simplify(map(expand,TensorInnerProduct(g, coframe, coframe)), symbolic)};

\begin{bmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}

(3.2.2)

\textbf{M} > \texttt{Fr := simplify(DualBasis(coframe), symbolic)};

\begin{align*}
\text{Fr} &= \left[ D_r, D_u - \frac{b^2}{\xi \xi I} D_r, - \frac{1}{b \xi^2 \left( \alpha \xi \xi I + 1 r \xi \xi I - \alpha + 1 r \right)} \left( \alpha \left( \ln(\xi \xi I

+ 1) \xi^2 \xi I^2 - \xi I^2 \xi^2 + 2 \ln(\xi \xi I + 1) \xi \xi I + \xi \xi I + \ln(\xi \xi I + 1) + 1 \right) \right) D_u

- \frac{2}{\left( \alpha \xi \xi I + 1 r \xi \xi I - \alpha + 1 r \right)} \frac{b}{\xi} \ D_r

+ \frac{1}{\left( \alpha \xi \xi I + 1 r \xi \xi I - \alpha + 1 r \right)} \frac{b}{\xi} \ D_\xi,

\frac{1}{b \xi I^2 \left( 1 r \xi \xi I - \alpha \xi \xi I + 1 r + \alpha \right)} \left( \alpha \left( \ln(\xi \xi I + 1) \xi^2 \xi I^2 - \xi I^2 \xi^2

+ 2 \ln(\xi \xi I + 1) \xi \xi I + \xi \xi I + \ln(\xi \xi I + 1) + 1 \right) \right) D_u
\end{align*}
We verify that the frame satisfies the reality conditions of a null tetrad.

\[
M > \text{DGIm}(Fl[1]); \quad 0 \text{ D}_u \\
M > \text{DGIm}(Fl[2]); \quad 0 \text{ D}_u \\
M > \text{DGconjugate}(Fl[3]) \&\text{minus Fl}[4]; \quad 0 \text{ D}_u
\]

Next we compute the Ricci tensor; it is of the form \( R_{ab} = \frac{1}{4} k_a k_b \), where \( k_a \) is the 1-form \( \theta^a \) defined above. This is also the second 1-form in the null coframe.

\[
\begin{align*}
M > \text{Ric1} &:= \text{factor}(\text{RicciTensor}(g)); \\
M > \text{TensorInnerProduct}(g, \text{Ric1}, \text{Ric1}, \text{tensorindices}=[1]); \quad 0 \text{ du} \otimes \text{du} \\
M > \text{R22} &:= \text{op}(\text{factor}(\text{GetComponents}(\text{Ric1}, [\text{coframe}[2] \&\text{coframe}[2]])))); \\
\end{align*}
\]

\[
R_{22} := \frac{2 b^4 (\xi \xi l + 1)^4}{(-\alpha^2 \xi l^2 + r^2 \xi l^2 - 2 \alpha^2 \xi \xi l + 2 r^2 \xi \xi l + \alpha^2 + r^2) \xi^3 \xi^3 l^3}
\]

For this to be a pure radiation spacetime (a necessary condition for null electrovacuum) we must have \( \chi > 0 \).

We now construct the vector field \( k \) tangent to the preferred congruence.

\[
\begin{align*}
M > \Phi &:= \text{simplify}(\sqrt{R22}, \text{symbolic}) \text{ assuming } \chi > 0, \xi^* \xi l > 0, b > 0; \\
\Phi &:= \frac{\sqrt{2 b^2 (\xi \xi l + 1)^2}}{\xi^3 l^2 \sqrt{-\alpha^2 \xi^2 \xi l^2 + r^2 \xi^2 \xi l^2 - 2 \alpha^2 \xi \xi l + 2 r^2 \xi \xi l + \alpha^2 + r^2}} \\
M > Kdn &:= \text{simplify}(\text{evalDG}(2*Phi*coframe[2])) \text{ assuming } \chi > 0, \xi^* \xi l > 0, b > 0;
\end{align*}
\]
\[ d\xi = - \frac{2 \sqrt{2} b^2 (\xi \xi l + 1)^2}{\xi^3 l^3 + \sqrt{-\alpha^2 \xi^2 \xi l^2 + r^2 \xi^2 \xi l^2 - 2 \alpha^2 \xi l + 2 r^2 \xi l + \alpha^2 + r^2}} \]

\[ du - \left( 2 \sqrt{2} \left( \ln(\xi \xi l + 1) \xi^2 \xi l^2 - \xi^2 \xi l^2 + 2 \ln(\xi \xi l + 1) \xi \xi l + \xi \xi l \right) \right) \]

\[ \left( \xi^3 l^3 + \sqrt{-\alpha^2 \xi^2 \xi l^2 + r^2 \xi^2 \xi l^2 - 2 \alpha^2 \xi l + 2 r^2 \xi l + \alpha^2 + r^2} \right) \]

\[ d\xi l \]

\[ D_r \]

We verify that Kup is a principal null direction, as follows from the Goldberg-Sachs theorem.

\[ \text{true} \]

We extend Kup into an adapted null tetrad, NT, by boosting the original null frame.
\[ D_r, \]
\[
- \frac{1}{4 b^2 (\xi \xi l + 1)^2} \left( \sqrt{2} \xi^2 \xi l^2 \right|^2 - \alpha^2 \xi \xi l^2 + 2 r^2 \xi \xi l + \alpha^2 + r^2 \right) D_u
\]
\[
+ \frac{1}{4 (\xi \xi l + 1)^2} \left( \sqrt{2} \sqrt{\xi} \sqrt{\xi l} \right)
\]
\[
\sqrt{\alpha^2 \xi^2 \xi l^2 + r^2 \xi^2 \xi l^2 - 2 \alpha^2 \xi \xi l + 2 r^2 \xi \xi l + \alpha^2 + r^2} \right) D_r,
\]
\[
- \frac{1}{b \xi^2 (\alpha \xi \xi l + 1 r \xi \xi l - \alpha + 1 r)} \left( \ln(\xi \xi l + 1) \xi^2 \xi l^2 - \xi l^2 \xi^2 + 2 \ln(\xi \xi l + 1) \xi \xi l + \xi \xi l + \ln(\xi \xi l + 1) + 1) \right) D_u
\]
\[
- \frac{2 \alpha \xi l b}{(\alpha \xi \xi l + 1 r \xi \xi l - \alpha + 1 r) \xi} D_r
\]
\[
+ \frac{1 (\xi \xi l + 1)^2 b}{(\alpha \xi \xi l + 1 r \xi \xi l - \alpha + 1 r) \xi} D_{\xi l},
\]
\[
\frac{1}{b \xi l^2 (1 r \xi \xi l - \alpha \xi \xi l + 1 r + \alpha)} \left( \ln(\xi \xi l + 1) \xi^2 \xi l^2 - \xi l^2 \xi^2 + 2 \ln(\xi \xi l + 1) \xi \xi l + \xi \xi l + \ln(\xi \xi l + 1) + 1) \right) D_u
\]
\[
+ \frac{2 \alpha \xi b}{(1 r \xi \xi l - \alpha \xi \xi l + 1 r + \alpha) \xi l} D_r
\]
\[
+ \frac{1 (\xi \xi l + 1)^2 b}{(1 r \xi \xi l - \alpha \xi \xi l + 1 r + \alpha) \xi l} D_{\xi l}.\]

We check that NT has all the required properties.

\[
\text{M > evalDG(NT[1] - convert(Kup, DGvector))};
\]
\[
0 \quad \text{(3.2.17)}
\]
\[
\text{M > evalDG(EinsteinTensor(g) - 1/4*NT[1] &t NT[1])};
\]
Compute the spin coefficients and directional derivatives.

\[
\begin{align*}
M &> NPS01 := \text{map(factor,} \text{simplify(NPSpinCoefficients(NT)))}; \\
M &> NPS1 := \text{simplify(NPS01) assuming } \chi > 0, \xi * \xi_1 > 0, b > 0; \\
M &> NPD1 := \text{NPDirectionalDerivatives(NT)}; \\
\end{align*}
\]

Check that the congruence defined by \( k \) is shear-free, geodesic and parametrized according to (2).

\[
\begin{align*}
M &> \text{NPS1["sigma"]}; \\
M &> \text{NPS1["kappa"]}; \\
M &> \text{factor(simplify(DGRe(NPS1["rho"])) - 2*DGRe(NPS1["epsilon"]))) assuming } \chi > 0, \xi * \xi_1 > 0, b > 0; \\
\end{align*}
\]

Here we compute the twist of the congruence, which is non-vanishing in general.

\[
\begin{align*}
M &> \text{factor(simplify(DGIm(-NPS1["rho"]))) assuming } \chi > 0, \xi * \xi_1 > 0, b > 0; \\
&\left(2 (\xi \xi I + 1) ^3 b^2 \sqrt{2} (\xi \xi I - 1) \_\alpha \right) / \\
&\left(\xi^3 / 2 \xi_1 ^3 / 2 \sqrt{\_\alpha^2 \xi^2 \xi_1 ^2 + r^2 \xi^2 \xi_1 ^2 - 2 \_\alpha^2 \xi \xi I + 2 r^2 \xi \xi I + \_\alpha^2 + r^2} \\
&\_\alpha ^2 \xi \xi I \_\alpha + 1 r) (1 r \xi \xi I - \_\alpha \xi \xi I + 1 r + \_\alpha) \right) \\
\end{align*}
\]

Therefore this spacetime is an electrovacuum if and only if \( J = 0 \).

\[
\begin{align*}
M &> \text{NullElectrovacuumConditions(NPS1, NPD1)}; \\
\end{align*}
\]
This spacetime is an electrovacuum.

The electromagnetic field.

We now compute the electromagnetic field by solving equations (5) for $\phi$. These equations can be put into the following form.

\[
EQ := \text{NullMaxwellEquations}(NPS1, NPD1);
\]

\[
EQ := \begin{cases}
\frac{\partial}{\partial r} \phi(u, r, \xi, \xi I) \\
\frac{\partial}{\partial u} \phi(u, r, \xi, \xi I) \\
\frac{\partial}{\partial \xi} \phi(u, r, \xi, \xi I)
\end{cases}
\]

\[
\xi I = 0, \frac{\partial}{\partial \xi} \phi(u, r, \xi, \xi I) = \left( -\frac{1}{2} \left( -3 \alpha^2 \xi^2 \xi I^2 - 3 r^2 \xi^2 \xi I^2 \right) \\
+ 41 \alpha r \xi \xi I + 6 \alpha^2 \xi I - 6 r^2 \xi I - 3 \alpha^2 - 3 r^2 \right) \bigg/ ((1 r \xi \xi I \\
- \alpha \xi \xi I + 1 r + \alpha) (-\alpha \xi \xi I + 1 r \xi \xi I - \alpha + 1 r) \xi), \frac{\partial}{\partial \xi} \phi(u, r, \xi, \xi I)
\]

\[
\frac{\partial}{\partial u} \phi(u, r, \xi, \xi I) = \left( -\frac{1}{2} \left( 3 \alpha^2 \xi^2 \xi I^2 + 3 r^2 \xi^2 \xi I^2 + 41 \alpha r \xi \xi I - 6 \alpha^2 \xi \xi I + 6 r^2 \xi \xi I \\
+ 3 \alpha^2 + 3 r^2 \right) \bigg/ ((-\alpha \xi \xi I + 1 r \xi \xi I - \alpha + 1 r) (1 r \xi \xi I - \alpha \xi \xi I + 1 r \\
+ \alpha) \xi I) \right) \bigg/ ((1 r \xi \xi I \\
- \alpha \xi \xi I + 1 r + \alpha) (-\alpha \xi \xi I + 1 r \xi \xi I - \alpha + 1 r) \xi)
\]

The solution to this system is as follows.

\[
\phi_{\text{sol}} := \text{combine}(\text{simplify}(\text{combine}(\text{pdsolve}(EQ, \{\phi(u, r, \xi, \xi I)\}), \text{symbolic}), \text{symbolic}), \text{symbolic});
\]

\[
\phi_{\text{sol}} := \left\{ \phi(u, r, \xi, \xi I) = -\frac{3 I \ln \left( \frac{-1}{\xi I} \right)}{2} + \arctan \left( \frac{r \left( \xi I + \xi I \right)}{\left( \xi I - 1 \right) \xi I + 1 r} \right) \right\}
\]
To compute the electromagnetic field it is convenient to work with an anholonomic frame, defined as follows.

\[
M \rightarrow \text{DGsetup}([u, r, xi, xil], M); \\
\text{frame name: } M
\]

\[
M > \text{FD} := \text{FrameData}([\text{NT[1]}, \text{NT[2]}, 1/\sqrt{2}*(\text{NT[3]} + \text{NT[4]}), 1/\sqrt{2}/I^*(\text{NT[3]} - \text{NT[4]})], \text{null}): \\
The \text{third leg of this tetrad is the unit vector field } s^a \text{ orthogonal to } k^a, \text{ used in (2).}
\]

\[
M > \text{DGsetup}(\text{FD}): \\
The \text{metric in this frame is the following.}
\]

\[
\text{null} > \eta := \text{evalDG}(-2*\Theta1 \&s \Theta2 + \Theta3 \&t \Theta3 \\
+ \Theta4 \&t \Theta4); \\
\eta := -\Theta1 \otimes \Theta2 - \Theta2 \otimes \Theta1 + \Theta3 \otimes \Theta3 + \Theta4 \otimes \Theta4
\]

The following 2-form, defined in (2), solves the Einstein equations, but not the Maxwell equations.

\[
\text{null} > f := \text{evalDG}(-1/2*\Theta2 \&w \Theta3); \\
f := -\frac{1}{2} \Theta2 \wedge \Theta3
\]

\[
\text{null} > \text{Gnull} := \text{EinsteinTensor}(\eta); \\
\text{null} > \text{Tnull} := \text{EnergyMomentumTensor("Electromagnetic",} \\
\text{eta, f}); \\
\text{null} > \text{factor(evalDG(Gnull - Tnull));} \\
0 \ E1 \otimes \ E1
\]

We now construct from \( f \) and \( \phi \) the 2-form \( F \), according to (3), which solves the Einstein-Maxwell equations.

\[
\text{null} > \text{fd} := \text{HodgeStar}(\eta, f, \text{detmetric=-1}); \\
\text{fd} := -\frac{1}{2} \Theta2 \wedge \Theta4
\]

\[
\text{null} > \text{Cphi} := \text{eval(cos(phi(u, r, xi, xil)), phisol)}; \\
Cphi := \cos \left( -\frac{3}{2} \ln \left( \frac{\xi}{\xi l} \right) + \arctan \left( \frac{r (\xi \xi l + 1)}{(\xi \xi l - 1) \_\alpha} \right) + -C1 \right)
\]

\[
\text{null} > \text{Sphi} := \text{eval(sin(phi(u, r, xi, xil)), phisol)}; \\
Sphi := \sin \left( -\frac{3}{2} \ln \left( \frac{\xi}{\xi l} \right) + \arctan \left( \frac{r (\xi \xi l + 1)}{(\xi \xi l - 1) \_\alpha} \right) + -C1 \right)
\]

\[
\text{null} > F := \text{evalDG(Cphi*f - Sphi*fd)};
\]
Because $\frac{\xi}{\xi I}$ is a phase, its natural logarithm is pure imaginary, therefore $F$ is real.

We verify the Maxwell equations:

$$null > \text{MatterFieldEquations("Electromagnetic", } \eta, \ F); \quad 0 \ E1, 0 \ \Theta 1 \wedge \Theta 3$$

We verify the Einstein equations:

$$null > \text{evalDG}(Gnull - \text{EnergyMomentumTensor} \ ("Electromagnetic", \ \eta, \ F)); \quad 0 \ E1 \otimes \ E1$$

\section*{Example 2: A pure radiation spacetime}

This example is taken from a paper by Lewandowski and Nurowski [3]. They relate pure radiation spacetimes with a shear-free, null, geodesic congruence to CR geometry, and exhibit all solutions with a minimum of three conformal Killing vector fields. One class of solutions has Petrov type II and a 3-dimensional conformal symmetry group of type VI (in the Bianchi classification of three-dimensional groups), and it is this class of solutions we consider here. We shall see that these pure radiation spacetimes cannot be electrovacua (for any values of the parameters).

\section*{Set-up.}

Initialize the manifold and define various quantities which are used in [3] to construct the spacetime. We put an underscore in front of some of the quantities used in [3] so they don't get confused with Newman-Penrose spin coefficients later on.
NullElectrovacuumConditions := proc(NPS, NPD)

> DGsetup([u, r, x, y], M);

frame name: M

Omega1 := evalDG(1/y*dx + I/y*dy + d/(d+1)*(y^d*du - 1/y*dx));

\(\Omega I := \frac{dy^d}{d + 1} du + \frac{1}{(d + 1) y} dx + \frac{1}{y} dy\)

Omega := evalDG(-2/(d+1)*(y^d*du - 1/y*dx));

\(\Omega := -\frac{2y^d}{d + 1} du + \frac{2}{(d + 1) y} dx\)

_alpha := -I/2*(1-d); _beta := -1/4*d; _theta := -I/4*d; a := I/8; b := 1/4; C := -1/4*I; w := w0*y^(3/4); W := 2*I*a*exp(I*r) + b;

_\alpha := -\frac{1}{2}(1 - d)

_\beta := -\frac{d}{4}

_\theta := -\frac{1}{4}d

a := \frac{1}{8}

b := \frac{1}{4}

C := -\frac{1}{4}

w := w0*y^(3/4)

W := -\frac{e^{Ir}}{4} + \frac{1}{4}

h := -6*a*DGconjugate(a) + _alpha * DGconjugate(_alpha) - \frac{1}{2}*\beta (I*DGconjugate(_alpha)*b - _alpha*DGconjugate(b));

\(h := -\frac{11}{32} + \frac{(1 - d)^2}{4} + \frac{3d}{8}\)

G := G1 + I*G2;

\(G := G1 + I G2\)

G1 := simplify(1/2*(h + 2 *DGRe(a*(DGconjugate(_alpha - I* b))) + 4*C*DGconjugate(C)));

\(G1 := -\frac{1}{64} + \frac{d^2}{8}\)

H := factor(simplify(2*DGRe(G*exp(2*I*r)) + 2*DGRe((2*G - (DGconjugate(_alpha) + I*DGconjugate(b)))*a)*exp(I*r)) + h,
Here we check the reality properties of the coframe: $e^i$ is the conjugate of $e^i$, $e^i$ and

\[ e^4 \text{ are real.} \]

\[
\begin{align*}
\text{M} & > \text{evalDG(DGconjugate(e1) - e2);} & (4.1.13) \\
\text{0} & \\
\text{M} & > \text{DGIm(e3);} & (4.1.14) \\
\text{0 du} & \\
\text{M} & > \text{DGIm(e4);} & (4.1.15) \\
& \\
\end{align*}
\]

This is the dual basis of vector fields.

\[
\begin{align*}
\text{M} & > \text{Frame := simplify(factor(convert(DualBasis(coFrame), exp), symbolic));} \\
Frame & := \begin{bmatrix}
\begin{array}{c}
\cos\left(\frac{r}{2}\right) \frac{y^{-3/4} - d}{2 w0} D_u + \frac{1}{2} \sin\left(\frac{r}{2}\right) \cos\left(\frac{r}{2}\right) e^{\frac{1}{2} r} \\
\frac{y^{1/4}}{2 w0} D_x - \frac{1}{2} y^{1/4} \cos\left(\frac{r}{2}\right) \frac{\cos\left(\frac{r}{2}\right) y^{-3/4} - d}{2 w0} D_u \\
- \frac{1}{2} \sin\left(\frac{r}{2}\right) \cos\left(\frac{r}{2}\right) e^{\frac{1}{2} r} \\
+ \frac{1}{2} y^{1/4} \cos\left(\frac{r}{2}\right) \frac{\cos\left(\frac{r}{2}\right) y^{-3/4} - d}{2 w0} D_u \\
- \frac{1}{y^{3/2} w0^2} \left( \frac{1}{64} \left( -8 I e^{-21r} d^2 - 16 I d^2 - 16 I e^{1r} d^2 - 16 I e^{-1r} d^2 - 4 I e^{1r} + 4 I e^{-1r} - 6 I e^{21r} G2 - 64 e^{-21r} G2 + 128 e^{1r} G2 - 128 e^{-1r} G2 - 8 I e^{21r} d^2 + 8 I d + 4 I e^{1r} d + 4 I e^{-1r} d + 1 e^{21r} + 6 I \right) \\
\end{array}
\end{bmatrix}
\end{align*}
\]
From this frame we can construct a null tetrad which matches the conventions used in DifferentialGeometry. We will use this as an anholonomic frame. The anholonomic vector basis is denoted $\text{NT}= [E1, E2, E3, E4]$ and the dual basis of 1-forms is denoted $[\Theta1, \Theta2, \Theta3, \Theta4]$.  

\begin{verbatim}
M > NT := [Frame[4], Frame[3], Frame[1], Frame[2]]:
M > FD := simplify(factor((map(convert,FrameData(NT, null),
                      exp))), symbolic):
M > DGsetup(FD);

frame name: null

\end{verbatim}  (4.1.17)

This is the spacetime metric expressed in the null anholonomic coframe.  

\begin{verbatim}
M > eta := evalDG(-2*Theta1 &s Theta2 + 2*Theta3 &s Theta4);
   \eta := -\Theta1 \otimes \Theta2 - \Theta2 \otimes \Theta1 + \Theta3 \otimes \Theta4 + \Theta4 \otimes \Theta3

\end{verbatim}  (4.1.18)

Here we verify the Petrov type. (This computation takes a little time.)  

\begin{verbatim}
null > PetrovType([E1, E2, E3, E4], [u = 0, r = 0, x = 0, y
   = 1]);

"II"

\end{verbatim}  (4.1.19)

\section*{Adapted tetrad, spin coefficients and the integrability conditions.}

Now we compute the Einstein tensor to verify that this is a pure radiation spacetime with the preferred null vector field $k$ being parallel to $E1$. (This computation takes a little time.)  

\begin{verbatim}
null > Ein := EinsteinTensor(eta):
null > Ein := factor(DGsimplify(simplify(convert(Ein, exp),
                      symbolic)));

Ein := \frac{d (4 d + 1) (2 d - 1) \cos \left(\frac{r}{2}\right)^6}{32 \, y^3 \, w0^4} \, E1 \otimes E1

\end{verbatim}  (4.2.1)

\begin{verbatim}
null > chi := op(DGinfo(Ein, "CoefficientList", [E1 &t E1])
   );

\chi := \frac{d (4 d + 1) (2 d - 1) \cos \left(\frac{r}{2}\right)^6}{32 \, y^3 \, w0^4}

\end{verbatim}  (4.2.2)

For this to be pure radiation spacetime the function $\chi$ must be positive.

The vector field $k$ we use to define the null congruence is given by the following.  

\begin{verbatim}
null > Kup := simplify(evalDG(2*sqrt(chi)*E1), symbolic);

\end{verbatim}
We can create a null tetrad adapted to $k$ by applying a suitable \texttt{boost} to the original null frame.

\begin{equation}
\text{null > ANT} := \text{simplify}(\text{NullTetradTransformation}([E1, E2, E3, E4], \text{"boost"}, \text{evalDG}(2*\text{sqrt(chi))), \text{symbolic}}; \\
\quad \text{ANT} := \begin{bmatrix}
\frac{\sqrt{2} \cos \left( \frac{r}{2} \right)^3 \sqrt{4 d + 1} \sqrt{2 d - 1}}{4 y^3 |^2 w0^2} & E1, \\
\frac{2 \sqrt{2} y^3 |^2 w0^2}{\cos \left( \frac{r}{2} \right)^3 \sqrt{4 d + 1} \sqrt{2 d - 1}} & E2, E3, E4
\end{bmatrix}
\end{equation}

We check that this is in fact a null tetrad and that the first leg of this tetrad satisfies $G^{ab} = \frac{1}{4} k^a k^b$.

The spin coefficients and directional derivatives for this adapted tetrad are computed in the following.

\begin{align}
\text{null > TensorInnerProduct(eta, ANT, ANT);} & \begin{bmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix} \\
\text{null > evalDG(Ein - 1/4*ANT[1] \&t ANT[1]);} & \begin{bmatrix}
0 & E1 \otimes E1
\end{bmatrix}
\end{align}

We verify that the congruence generated by $k$ is geodesic, shear-free and parametrized as in (2):

\begin{align}
\text{null > NPS2 := simplify(map(convert, NPSpinCoefficients (ANT), exp), symbolic);} & \text{null > NPD2 := NPDirectionalDerivatives(ANT);} \\
\text{null > NPS2["kappa"];} & 0 \\
\text{null > NPS2["sigma"];} & 0
\end{align}
null > DGRe(NPS2["rho"] - 2*NPS2["epsilon"]) = 0

(4.2.9)

The twist of the congruence is given by $\omega = -\text{Im}(\rho)$:

$$\text{null} > \text{simplify}(-\text{DGIm}(\text{NPS2}["rho"]))$$

$$\frac{\sqrt{2} \cos\left(\frac{r}{2}\right)^3 \sqrt{d} \sqrt{4d + 1} \sqrt{2d - 1}}{8 y^3 \frac{1}{2} w_0^2}$$

(4.2.10)

The twist is non-vanishing since $\chi > 0$. Therefore, this spacetime is a null electrovacuum if and only if $\mathcal{J} = 0$, with $\mathcal{J}$ defined in (4).

$$\text{null} > \text{IC} := \text{NullElectrovacuumConditions}(\text{NPS2}, \text{NPD2})$$

$$\text{null} > \text{factor(simplify(convert(IC, exp), symbolic))}$$

$$\frac{-\frac{1}{256} d^3 \frac{1}{2} \sqrt{2} \sqrt{4d + 1} \sqrt{2d - 1} (4d - 1) \cos\left(\frac{r}{2}\right)^6}{y^{15} \frac{1}{4} w_0^5}$$

(4.2.11)

This family of spacetimes does not admit a null electrovacuum.

References

Release Notes
- The illustrated commands are all available in Maple 17 and subsequent releases.

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