How To Find Killing Vectors

Synopsis

• Let \( g \) be a (pseudo) Riemannian metric on a manifold \( M \). Let \( \nabla_a \) be the covariant derivative defined by the Christoffel connection of the metric \( g \), and let \( K_a = g_{ab} K^b \) be the dual vector corresponding to the vector field \( K^a \). Then \( K^a \) is a Killing vector field if and only if it solves the Killing equations:

\[
\nabla_a K_b + \nabla_b K_a = L_K g_{ab} = 0.
\]

• The flow of a Killing vector field is a 1-parameter family of isometries of \((M, g)\). The set of Killing vectors of a given metric form a Lie algebra using the commutator of vector fields as the Lie bracket.

• In this worksheet we show how to compute the Lie algebra of Killing vector fields of a metric using the commands \texttt{KillingVectors} and \texttt{LieAlgebraData}.

Example

Here we compute the Lie algebra of Killing vector fields of the Gödel metric, a metric featuring in the general theory of relativity.

Load in the required packages. Set typesetting preferences (optional).

\[
> \texttt{with(DifferentialGeometry): with(Tensor): with(LieAlgebras):}
> \texttt{Preferences("TensorDisplay", 1):}
\]

Define a coordinate chart.

\[
> \texttt{DGsetup([t, x, y, z], M);}
\]

Here is the Gödel metric.

\[
> \texttt{g := evalDG(dx \&t dx + dy \&t dy +1/2*exp(2*x)*dz \&t dz -(dt + exp(x)*dz) \&tensor (dt + exp(x)*dz));}
\]

\[
\begin{align*}
g & := -dt \otimes dt - e^x dt \otimes dz + dx \otimes dx + dy \otimes dy - e^x dz \otimes dt - \frac{e^{2x}}{2} dz \otimes dz
\end{align*}
\]
The command \texttt{KillingVectors} computes a basis for the vector space of solutions to the Killing equations.

\begin{verbatim}
> KV := KillingVectors(g);
KV := \begin{bmatrix} -4 e^{-x} D_t + 2 z D_x - (z^2 - 2 e^{-2x}) D_z, 2 D_x - 2 z D_z, -2 D_z, -D_t, D_y \end{bmatrix}
\end{verbatim}

We see that the Gödel metric admits a 5-dimensional vectors space of Killing vectors.

\begin{verbatim}
> nops(KV);
5
\end{verbatim}

We can verify that the vector fields in the list \( KV \) are Killing vectors with the \texttt{LieDerivative} command.

\begin{verbatim}
> LieDerivative(KV, g);
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{verbatim}

We can also verify that the Killing vectors form a Lie algebra using the command \texttt{LieAlgebraData}. The algebra elements corresponding to the vector fields are labeled \( e_1, e_2, ..., e_5 \). Only the non-zero brackets are displayed.

\begin{verbatim}
> LAD := LieAlgebraData(KV);
LAD := \begin{bmatrix} [e_1, e_2] = 2 e_1, [e_1, e_3] = 2 e_2, [e_2, e_3] = 2 e_3 \end{bmatrix}
\end{verbatim}

The Lie algebra defined by \( LAD \) can be analyzed using commands in the \texttt{LieAlgebras} package.

**Commands Illustrated**
- \texttt{DGsetup}, \texttt{evalDG}, \texttt{KillingVectors}, \texttt{LieAlgebraData}, \texttt{LieDerivative}

**Related Commands**
- \texttt{KillingTensors}, \texttt{KillingSpinors}

**References**
- \url{http://en.wikipedia.org/wiki/Killing_vector_field}

**Release Notes**
- The illustrated commands are available in Maple 13 and subsequent releases.

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