How To Find A Levi Decomposition of a Lie Algebra

Synopsis

• If \( g \) is a finite-dimensional Lie algebra, then the radical \( r \) of \( g \) is the largest solvable subalgebra contained in \( g \). A fundamental theorem in the structure theory of Lie algebras asserts that there exists a complementary subalgebra \( \mathfrak{s} \) such that \( g \) is a semi-direct sum \( g = \mathfrak{s} \oplus r \). This decomposition is called a Levi decomposition. The subalgebra \( \mathfrak{s} \) is semi-simple.
• In this worksheet we show how to compute the Levi decomposition of a Lie algebra using the command \texttt{LeviDecomposition}.

Example

Load in the required packages.

\[ > \text{with(DifferentialGeometry): with(LieAlgebras):} \]

We shall calculate the Levi decomposition for a 7-dimensional Lie algebra \( g \). The structure equations of \( g \) are

\[
\begin{align*}
\text{StructureEquations} &:= [[x1, x2] = 2 \times x2, [x1, x3] = 2 \times x3, [x1, x6] = -2 \times x6, [x1, x7] = -2 \times x7, [x2, x4] = 2 \times x3, [x2, x6] = x1, [x2, x7] = x4, [x3, x5] = x3, [x3, x6] = -x4, [x4, x5] = x4, [x4, x6] = -2 \times x7, [x5, x7] = -x7];
\end{align*}
\]

\[ \text{StructureEquations} \vdash [[x1, x2] = 2 \times x2, [x1, x3] = 2 \times x3, [x1, x6] = -2 \times x6, [x1, x7] = -2 \times 7, [x2, x4] = 2 \times x3, [x2, x6] = x1, [x2, x7] = x4, [x3, x5] = x3, [x3, x6] = -x4, [x4, x5] = x4, [x4, x6] = -2 \times x7, [x5, x7] = -x7]; \quad (1) \]

Initialize this Lie algebra with the \texttt{LieAlgebraData} and \texttt{DGsetup} commands.

\[ > \text{LD} := \text{LieAlgebraData(StructureEquations, [x1, x2, x3, x4, x5, x6, x7], alg1)}; \]

\[ > \text{DGsetup(LD);} \]

\[ \text{Lie algebra: alg1} \quad (2) \]

Calculate the Levi decomposition. In the output, the first list of vectors defines the radical and the second list of vectors defines the semi-simple subalgebra.

\[ \text{alg1} > \text{Levi} := \text{LeviDecomposition()}; \]

\[ \text{Levi} := [[e3, e4, e5, e7], [e1, e2, e6]] \quad (3) \]
We can use the \texttt{Query} command to verify that the span of the first set of vectors is a solvable ideal and that the span of the second list is a semi-simple algebra.

\begin{verbatim}
\texttt{alg1 > Query(Levi[1], "Solvable");
true}
\texttt{alg1 > Query(Levi[1], "Ideal");
true}
\texttt{alg1 > Query(Levi[2], "Semisimple");
true}
\end{verbatim}

\textbf{Remark.} The structure of the Lie algebra becomes more transparent if we make a change of basis adapted to the Levi decomposition.

\begin{verbatim}
\texttt{alg1 > NewBasis := [e1, e2, e6, e5, e7, e4, e3];
NewBasis := [e1, e2, e6, e5, e7, e4, e3]}
\end{verbatim}

Calculate the structure equations of the Lie algebra in the new basis.

\begin{verbatim}
\texttt{alg1 > NewLD := LieAlgebraData(NewBasis, alg2):}
\end{verbatim}

Initialize the Lie algebra in the new basis. The new basis elements are labeled by X (with dual 1-forms labeled by omega).

\begin{verbatim}
\texttt{alg1 > DGsetup(NewLD, [X], [omega]);
Lie algebra: alg2}
\end{verbatim}

\begin{verbatim}
\texttt{alg2 > MultiplicationTable("LieTable");}
\begin{tabular}{|c|cccccccc|}
\hline
 & X1 & X2 & X3 & X4 & X5 & X6 & X7 \\
\hline
X1 & 0 & 2X2 & 2X3 & 0 & 0 & 0 & 2X7 \\
X2 & 2X2 & 0 & 0 & 0 & 0 & 2X5 & X6 \\
X3 & 2X3 & 2X1 & 0 & 0 & 0 & 0 & X6 \\
X4 & 2X5 & 2X6 & 0 & 0 & 0 & 0 & X7 \\
X5 & 2X5 & 2X6 & 0 & 0 & 0 & 0 & X7 \\
X6 & 2X7 & 0 & 0 & 0 & 0 & 0 & 0 \\
X7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\end{verbatim}

We see that the semi-simple part of the Levi decomposition, spanned by \{X1, X2, X3\}, is \(sl(2)\) with Cartan subalgebra defined by \(X1\). We see that the radical, \{X4, X5, X6, X7\}, decomposes into a 1-dimensional invariant subalgebra and a 3-dimensional irreducible subalgebra with highest weight vector \(X7\). The vectors \{X5, X6, X7\} span an abelian subalgebra and define the nilradical. These statements can all be verified with commands in
the LieAlgebra package.

Commands Illustrated
• DGsetup, LieAlgebraData, LeviDecomposition, Query

Related Commands
• Decompose, CartanSubalgebra, Nilradical, Series

References
• N. Jacobson, Lie Algebras, Dover, 1979
• V. S. Varadarajan, Lie Groups, Lie Algebras and their Representations Graduate Texts in Mathematics 102, Springer
• http://en.wikipedia.org/wiki/Levi_decomposition

Release Notes
• The illustrated commands are available in Maple 11 and subsequent releases.

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