The Riemann curvature tensor, its invariants, and their use in the classification of spacetimes

Jesse Hicks
Utah State University

Follow this and additional works at: https://digitalcommons.usu.edu/dg_pres

Part of the Cosmology, Relativity, and Gravity Commons, Geometry and Topology Commons, and the Other Applied Mathematics Commons

Recommended Citation
https://digitalcommons.usu.edu/dg_pres/8

This Presentation is brought to you for free and open access by DigitalCommons@USU. It has been accepted for inclusion in Presentations and Publications by an authorized administrator of DigitalCommons@USU. For more information, please contact digitalcommons@usu.edu.
The Riemann curvature tensor, its invariants, and their use in the classification of spacetimes

Jesse Hicks
Utah State University
March 20, 2015
“Spacetime tells matter how to move; matter tells spacetime how to curve” - John Wheeler

The Einstein field equations:

\[ R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R + \Lambda g_{\alpha \beta} = \kappa T_{\alpha \beta} \]

- Solutions are components \( g_{\alpha \beta} \) of metric tensor \( g \)
- Equivalence Problem: \( g_1 \equiv g_2 \)?
Let $M$ be an $n$-dimensional differentiable manifold.

**Definition**

A **metric tensor field** $g$ on $M$ is a mapping $p \mapsto g_p$, where $p \in M$ and

$$g_p : T_p M \times T_p M \to \mathbb{R}$$

is a symmetric, non-degenerate, bilinear form on the tangent space to $p$ on $M$. 
Given a basis \( \{ X_\alpha (p) \} \) of \( T_p M \), \( g(p) \) has components

\[
g(p)_{\alpha\beta} = g_p (X_\alpha (p), X_\beta (p))
\]

In a neighborhood \( U \) of a point,

\[
g_{\alpha\beta}
\]

are \( C^\infty \) functions of coordinates

**Definition**

A **spacetime** is a 4-dimensional manifold with metric tensor having signature \((3, 1)\), i.e. a Lorentzian signature.
We refer to $g_{\alpha\beta}$ as the metric.

The metric defines a unique connection

$$\Gamma_{\alpha}^{\lambda}_{\beta} = \frac{g^{\lambda\mu}}{2} \left( \partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta} \right)$$

$$= \Gamma_{\beta}^{\lambda}_{\alpha}$$

$\Gamma_{\alpha}^{\lambda}_{\beta}$ allows us to define a derivative, called the covariant derivative

...which returns a new tensor when applied to a given tensor.
On a 4-dimensional manifold $M$

$$X : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$$

$$X = X^1(x)\partial_{x^1} + X^2(x)\partial_{x^2} + X^3(x)\partial_{x^3} + X^4(x)\partial_{x^4}$$

The components of the covariant derivative of $X$ are

$$X^\nu;_\alpha = \partial_\alpha X^\nu + \Gamma^\nu_\alpha^\beta X^\beta$$

When performing repeated covariant differentiation and taking the difference, we get

$$X^\nu;_\alpha;_\beta - X^\nu;_\beta;_\alpha = (\partial_\beta \Gamma^\nu_\mu^\alpha - \partial_\alpha \Gamma^\nu_\mu^\beta + \Gamma^\nu_\lambda^\beta \Gamma^\lambda_\mu^\alpha - \Gamma^\nu_\lambda^\alpha \Gamma^\lambda_\mu^\beta) X^\mu$$
Define

\[ R_{\mu}^{\nu}{}_{\alpha\beta} = \partial_\beta \Gamma_{\mu}^{\nu}{}_{\alpha} - \partial_\alpha \Gamma_{\mu}^{\nu}{}_{\beta} + \Gamma_{\lambda}^{\nu}{}_{\beta} \Gamma_{\mu}^{\lambda}{}_{\alpha} - \Gamma_{\lambda}^{\nu}{}_{\alpha} \Gamma_{\mu}^{\lambda}{}_{\beta} \]

- \( R_{\mu}^{\nu}{}_{\alpha\beta} \) are the components of the **Riemann curvature tensor**!

- \( R_{\mu}^{\nu}{}_{\alpha\beta} \neq 0 \) indicates **curvature**.

- If all \( R_{\mu}^{\nu}{}_{\alpha\beta} = 0 \), the spacetime is **flat**.
The $R_{\mu}^{\nu}{}_{\alpha\beta}$ change when changing coordinates,

BUT certain contractions and products and combinations of its contractions do NOT!

Called **Riemann invariants**

**Example**

*From* $R_{\mu\alpha} := R_{\mu}^{\nu}{}_{\alpha\nu}$ *is created the* **Ricci scalar** $R := g^{\mu\alpha} R_{\mu\alpha}$
From J. Carminati and R. McLennagh’s paper “Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space”
continued...

\[ m_1 = \Phi_{ABCD} \Phi_{CD} \Phi^{ABCD} = \frac{1}{4} C_{acdb} S^{cd} S^{ab}, \]

\[ m_2 = \Phi_{ABCD} \Phi_{CD} \Phi^{ABEF} \Phi^{EFCD} \]

\[ = \frac{1}{4} C_{acdb} S^{cd} C_{ef}^{ab} S^{ef}, \]

\[ m_3 = \Phi^{AB} \Phi_{CD} \Phi^{ABCD} \Phi_{ABCD} \]

\[ = \frac{1}{4} C_{acdb} S^{cd} C_{ef}^{ab} S^{ef}, \]

\[ m_4 = \Phi^{B} \Phi_{DE} \Phi_{ABC} \Phi^{ABC} \Phi_{CDE} \Phi^{A} \]

\[ = -\frac{1}{8} C_{acdb} S^{cd} C_{efg}^{ab} S^{ef} S^{ag}, \]

\[ m_5 = \Phi^{AB} \Phi_{CD} \Phi^{EF} \Phi^{EFCD} \Phi_{AB} \]

\[ = \frac{1}{4} C_{aeef} C_{cd}^{ab} S^{cd} C_{ef}^{gh} S^{ef}. \]
**THE KEY:** Obvious when two metrics are NOT equivalent
The Gödel metric SHOULD be in Petrov’s classification, as it’s a homogeneous space with isometry dimension 5.
The G"odel Metric

\[ g = -a^2 \, dt \, dt - a^2e^x \, dt \, dz + a^2 \, dx \, dx + a^2 \, dy \, dy - a^2e^x \, dz \, dt - 1/2 \, a^2e^{2x} \, dz \, dz \]

\[ g = 1/4 \, a^2 \, dr \, dr + 1/4 \, a^2 \, dr \, ds + 3/4 \, a^2 \, dr \, du + 1/4 \, a^2(2e^{s/2+u/2+v/2+r/2} + 3) \, dr \, dv + 1/4 \, a^2 \, ds \, dr + 1/4 \, a^2 \, ds \, ds - 1/4 \, a^2 \, ds \, du - 1/4 \, a^2(1 + 2e^{s/2+u/2+v/2+r/2}) \, ds \, dv + 3/4 \, a^2 \, du \, dr - 1/4 \, a^2 \, du \, ds + 1/4 \, a^2 \, du \, du - 1/4 \, a^2(2e^{s/2+u/2+v/2+r/2} - 1) \, du \, dv + 1/4 \, a^2(2e^{s/2+u/2+v/2+r/2} + 3) \, dv \, dr - 1/4 \, a^2(1 + 2e^{s/2+u/2+v/2+r/2}) \, dv \, ds - 1/4 \, a^2(2e^{s/2+u/2+v/2+r/2} - 1 + 2e^{s+u+v+r}) \, dv \, dv \]

Using \( t = \frac{-r+s+u+v}{2}, x = \frac{r+s+u+v}{2}, y = \frac{r-s+u+v}{2}, z = v \)
Here Gödel is again:

\[ g = -a^2 \, dt \, dt - a^2 e^x \, dt \, dz + a^2 \, dx \, dx + a^2 \, dy \, dy - a^2 e^x \, dz \, dt - \frac{1}{2} a^2 e^{2x} \, dz \, dz \]

It’s been shown that (33.17) with \( \epsilon = -1 \) is the ONLY metric in Petrov with equivalent Killing vectors:

\[ \tilde{g} = 2 \, dx^1 \, dx^4 + k_{22} e^{-2x^3} \, dx^2 \, dx^2 - e^{-x^3} \, dx^2 \, dx^4 + k_{22} \, dx^3 \, dx^3 + 2 \, dx^4 \, dx^1 - e^{-x^3} \, dx^4 \, dx^2 \]
Compute their respective Riemann invariants:

- Gödel, we have $R = -1$ and

\[
\begin{align*}
  r_1 &= \frac{3}{16a^4} \\
  r_2 &= \frac{3}{64a^6} \\
  r_3 &= \frac{21}{1024a^8} \\
  w_1 &= \frac{1}{6a^6} \\
  w_2 &= \frac{1}{36a^6} \\
  m_1 &= 0 \\
  m_2 &= \frac{1}{96a^8} \\
  m_3 &= \frac{1}{96a^8} \\
  m_4 &= -\frac{1}{768a^{10}} \\
  m_5 &= \frac{1}{576a^{10}}
\end{align*}
\]
Petrov’s (33.17), \( \epsilon = -1 \), we have \( R = \frac{-2}{k_{22}} \) and

\[
\begin{align*}
  r_1 &= \frac{1}{4k_{22}^2} \\
  r_2 &= 0 \\
  r_3 &= \frac{1}{64k_{22}^4} \\
  w_1 &= \frac{1}{6k_{22}^2} \\
  w_2 &= \frac{1}{36k_{22}^3} \\
  m_1 &= -\frac{1}{12k_{22}^3} \\
  m_2 &= \frac{1}{36k_{22}^4} \\
  m_3 &= \frac{1}{36k_{22}^4} \\
  m_4 &= 0 \\
  m_5 &= -\frac{1}{108k_{22}^5}
\end{align*}
\]

These are different metrics. Petrov incorrectly normalized.
A corrected and complete classification is in a database.

A classifier has been coded in Maple that makes comparisons against database.

Many metrics from Stephani’s compilation have been identified in Petrov.

Software has been written to help find explicit equivalences.

Using that software, all homogeneous spaces of dimension 3-5 have explicit equivalences to metrics in Petrov.
Many thanks to my advisor Dr. Ian Anderson and the Differential Geometry Group at USU!

Thank You!