10 Why "Plane" Waves?

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where $c(\vec{k})$ are determined by the initial data (exercise). Physically, you can think of this integral formula as representing a (continuous) superposition of plane waves over their possible physical attributes. To see this, consider a plane wave of the form

$$q(\vec{r}, t) = \text{Re} \left[ c e^{i(k \cdot \vec{r} - \omega t)} \right], \quad (9.26)$$

where $c$ is a complex number. You can check that this is a wave traveling in the direction of $\vec{k}$, with wavelength $\frac{2\pi}{k}$, and with amplitude $|c|$. The phase $\frac{\omega}{|c|}$ of the complex number $c$ adds a constant to the phase of the wave (exercise). The integral in (9.21) is then a superposition of waves in which one varies the amplitudes ($|c|$), relative phases ($c/|c|$), wavelengths ($2\pi/k$), and directions of propagation ($\vec{k}/k$) from one wave to the next.

Equivalently, every solution to the wave equation can be obtained by superimposing real plane wave solutions of the form

$$q(\vec{r}, t) = A \cos(\vec{k} \cdot \vec{r} - \omega t + \phi). \quad (9.27)$$

Here the (continuous) superposition takes place by varying the amplitude $A$, the wave vector $\vec{k}$ and the phase $\phi$ (exercise).

Exactly as we did for the space of solutions to the one-dimensional wave equation, we can view the space of solutions of the three-dimensional wave equation as a vector space (exercise). From this point of view, the plane waves form a basis for the vector space of solutions.

10. Why “plane” waves?

Let us now pause to explain in more detail why we called the elementary solutions (9.9) and (9.27) plane waves. The reason is that the displacement $q(\vec{r}, t)$ has the symmetry of a plane. To see this, fix a time $t$ (take a “snapshot” of the wave) and pick a location $\vec{r}$. Examine the wave displacement $q$ (at the fixed time) at all points in a plane that is (i) perpendicular to $\vec{k}$, and (2) passes through $\vec{r}$. The wave displacement will be the same at each point of this plane. To see this most easily, simply choose, say, the $x$-axis to be along the vector $\vec{k}$. The planes perpendicular to $\vec{k}$ are then parallel to the $y$-$z$ plane. In these new coordinates the wave (9.27) takes the simple form (exercise)

$$q(\vec{r}, t) = A \cos(kx - \omega t + \phi). \quad (10.1)$$

Clearly, at a fixed $t$ and $x$, $q(\vec{r}, t)$ is the same anywhere on the plane obtained by varying $y$ and $z$.

A more formal — and perhaps more instructive — way to see the plane wave symmetry of (9.27) is to fix a time $t$ and ask for the locus of points upon which the wave displacement
is constant. At a fixed time, the wave displacement \( q \) is a function of 3 variables. As you know, the locus of points where a function takes the same values generically defines a surface. Since the spatial dependence of the plane wave is via the combination \( \vec{k} \cdot \vec{r} \), when \( t = \text{constant} \) the surfaces of fixed \( q \) are given by \( \vec{k} \cdot \vec{r} = \text{constant} \) (exercise). But the equation \( k_x x + k_y y + k_z z = \text{constant} \) (with each of \( k_x, k_y, k_z \) a constant) is the equation for a plane (exercise)! This plane is everywhere orthogonal to the wave vector \( \vec{k} \), which can be viewed as a constant vector field (i.e., a vector field whose Cartesian components are the same everywhere). To see this, we recall from our discussion in §9 that the gradient of a function is always perpendicular to the surfaces upon which the function is constant. We just saw that the plane of symmetry (where \( q \) doesn’t change its value) arises when the function \( \vec{k} \cdot \vec{r} \) is constant. Thus, the plane of symmetry for the plane wave is orthogonal to the (constant) vector field

\[
\nabla (\vec{k} \cdot \vec{r}) = \vec{k}.
\]

(10.2)

The wave vector is thus normal to the planes of symmetry of a plane wave. As time evolves, the displacement profile on a given plane of symmetry moves along \( \vec{k} \). In this way \( \vec{k} \) determines the propagation direction of the plane wave. The wave vector thus determines the wavelength, the direction of motion, and the plane of symmetry of a plane wave.