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11 Separation of Variables

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11. Separation of variables.

There is yet another way to find the general solution to the wave equation which is valid in 1, 2, or 3 (or more!) dimensions. This method is quite important and, as we shall see, can often be used for other linear homogeneous differential equations. This technique for solving the wave equation is called the method of *separation of variables.*

We begin by assuming that the solution of the wave equation is a superposition of elementary solutions $q(\vec{r}, t)$, each of which can be expressed as a product of four functions of one variable:

$$q(\vec{r}, t) = T(t)X(x)Y(y)Z(z). \tag{11.1}$$

We shall show that each of these 4 functions satisfies a relatively simple ordinary differential equation. To this end, we insert the trial solution (11.1) into the wave equation to obtain (*exercise*):

$$\frac{1}{v^2}T''XYZ - TX''YZ - TXY''Z - TXYZ'' = 0, \tag{11.2}$$

where the prime on a function denotes a derivative with respect to the argument of that function, *e.g.,* $X' = \frac{dX(x)}{dx}$. Away from points where $q = 0$, we can divide both sides of this equation by q to get

$$\frac{1}{v^2}\frac{T''}{T} - \frac{X''}{X} - \frac{Y''}{Y} - \frac{Z''}{Z} = 0. \tag{11.3}$$

Note that each term in (11.3) is a function, respectively, *only* of the *different* variables (t, x, y, z). Because of this property, we say that the wave equation is "separable", or "separates" in Cartesian coordinates. As we shall see, the property of being separable really refers not just to the equation, but also to the coordinate system employed. Typically, one can only expect linear, homogeneous partial differential equations to separate.

Because the variables are separated in (11.3), this equation represents a very stringent requirement; it amounts to saying that each of the functions T''/T, X''/X, Y''/Y, Z''/Z is in fact a constant! To see this, let us isolate the T''/T term on one side of the equation. We we then get an equation of the form

$$f(t) = g(x, y, z), \tag{11.4}$$

where we are denoting

$$f(t) = \frac{1}{v^2}\frac{T''}{T} \tag{11.5}$$

and

$$g(x, y, z) = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{11.6}$$

Because we can vary x, y, z independently of t, (11.4) holds if and only if $f(t) = a = g(x, y, z)$, where a is a constant. To see this directly, simply take the partial derivative of (11.4) with respect to t to find $f' = 0$ *(exercise).*

So, at this point we know that for some constant a

$$\frac{1}{v^2}\frac{T''}{T} = a, \tag{11.7}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = a. \tag{11.8}$$

But we can play the same game again! We write

$$\frac{X''}{X} = a - \frac{Y''}{Y} - \frac{Z''}{Z}, \tag{11.9}$$

and we see that both sides must equal a constant, say, b:

$$\frac{X''}{X} = b. \tag{11.10}$$

Applying this logic repeatedly, we conclude that (*exercise*)

$$\begin{aligned} \frac{1}{v^2}\frac{T''}{T} &= a,\\ \frac{X''}{X} &= b,\\ \frac{Y''}{Y} &= c,\\ \frac{Z''}{Z} &= d, \end{aligned} \tag{11.11}$$

where a, b, c, d are constants ("separation constants") and

$$a - b - c - d = 0, \quad i.e., \quad a = (b + c + d). \tag{11.12}$$

The single *partial* differential equation (9.7) has thus been turned into 4 *ordinary* differential equations (11.11) involving 3 arbitrary separation constants.

It is now a simple matter to solve the quartet (11.11) of linear, homogeneous ordinary differential equations for T, X, Y, Z. They are all of the form

$$f'' = -\alpha^2 f, \tag{11.13}$$

provided we permit α to be imaginary, if necessary. At least formally, we have found the harmonic oscillator equation again! This equation has (complex) solutions (*exercise*)

$$f(u) = \beta e^{\pm i\alpha u},$$

where α and β are (complex) constants. To recover our plane wave "basis" for the general solution to the wave equation we set

$$b = -k_x^2, \quad c = -k_y^2, \quad d = -k_z^2, \tag{11.14}$$

and

$$a = -k_x^2 - k_y^2 - k_z^2 = -k^2. \tag{11.15}$$

Thus our separation of variables (complex) solution is a plane wave of the form

$$q(\vec{r}, t) = Ae^{ik_x x}e^{ik_y y}e^{ik_z z}e^{i\omega t} = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{11.16}$$

where

$$\omega = \pm kv \tag{11.17}$$

as before. Thus the "separation constants" make up the frequency of the wave and the components of the wave vector.

We can build up the general solution to the wave equation by superposition of these elementary separation of variables solutions and arrive again at the Fourier expansions of §9 (*exercise*).*

12. Cylindrical Coordinates.

We have seen how to build solutions to the wave equation by superimposing plane waves with various choices for amplitude, phase and wave vector $\vec{k}$. In this way we can build up solutions which need not have the plane symmetry (*exercise*), or any symmetry whatsoever. Still, as you know by now, many problems in physics are fruitfully analyzed when they are modeled as having various symmetries, such as cylindrical symmetry or spherical symmetry. For example, the magnetic field of a straight wire carrying a steady current can be modeled as having cylindrical symmetry in regions close to the wire (compared to the length of the wire).* Likewise, the sound waves emitted by a compact source are nicely approximated as spherically symmetric in regions far from the source (compared to the size of the source).** Now, using the Fourier expansion in plane waves we can construct such symmetric solutions — indeed, we can construct any solution to the wave equation. But, as you also know, we have coordinate systems that are adapted to a variety of symmetries, *e.g.,* cylindrical coordinates, spherical polar coordinates, *etc.* When looking for waves with some chosen symmetry it is advantageous to get at the solutions to the wave equation directly in these coordinates, without having to express them as a superposition of plane waves. Our task now is to see how to express solutions of the wave equations in a useful fashion in terms of such *curvilinear* coordinate systems.

12.1 The Wave Equation in Cylindrical Coordinates

Our first example will involve solutions to the wave equation in cylindrical coordinates (ρ, ϕ, z), which we shall now define. To begin, we point out that the purpose of coordinates

* This justifies the choice of negative separation constants in (11.14).
* This situation is usually called "the magnetic field due to a long straight wire".
** This situation is usually called "the sound emitted by a point source".