Problem 7.1

Recall the mechanical system consisting of two coupled oscillators. The kinetic energy $T$ for the system is defined as usual ($T = \frac{1}{2}m(v_1^2 + v_2^2)$). The potential energy is denoted by $V(x_1, x_2)$ and is defined so that the force, $F_i$, on the $i^{th}$ particle ($i = 1, 2$) is given by

$$F_i = -\frac{\partial V}{\partial x_i}.$$ 

Find the form of $V$, and prove that the total energy $E = T + V$ is conserved, that is, $\frac{dE}{dt} = 0$ for solutions of the equations of motion.

Problem 7.2

Solutions to the wave equation have a conserved momentum. The momentum density for a wave $q(x, t)$ is defined by

$$\rho = \frac{\partial q}{\partial t} \frac{\partial q}{\partial x}.$$ 

Find the corresponding momentum current density $j$ for the wave. (*Hint: Use the continuity equations.*)

Problem 7.3

Recall the Gaussian wave

$$q(x, t) = A \left[ e^{-(x- vt)^2} + e^{-(x+ vt)^2} \right].$$ 

Compute the total energy contained in this wave by integrating the energy density $\rho(x, t)$ over all $x$ and show that the result does not depend upon the time $t$.

Problem 7.4

In the previous problem, it is shown that the total energy of the Gaussian wave is time independent. Explain this result by showing that the energy current density $j$ vanishes as $x \to \infty$.

Problem 7.5
Verify that
\[ \rho(r, t) = \frac{1}{2} \left[ \left( \frac{\partial q}{\partial t} \right)^2 + v^2(\nabla q)^2 \right], \]
and
\[ j(r, t) = -v^2 \frac{\partial q}{\partial t} \nabla q. \]
satisfy the continuity equation when \( q \) satisfies the (3-d) wave equation.

**Problem 7.6**

If \( q(r, t) \) depends only upon \( x \) and \( t \) (i.e., \( q \) is independent of \( y \) and \( z \)) show that the 3-dimensional forms for the energy density, energy current density, and continuity equation reduce to the 1-dimensional results.

**Problem 7.7**

Use the divergence theorem (14.33) to verify (14.28).

**Problem 7.8**

Verify (14.17). Show that the time rate of change of energy in the region is the net flux of energy into the region (14.18).

**Problem 7.9**

Derive the approximate formula (14.40).

**Problem 7.10**

Show that the quantity
\[ \Pi(t) = \int_{-\infty}^{\infty} dx \frac{\partial q(x, t)}{\partial t} \]
is independent of \( t \) (i.e., is a conserved quantity) for all solutions \( q \) of the one-dimensional wave equation whose first derivatives vanish at infinity,
\[ \lim_{x \to \pm \infty} \frac{\partial q(x, t)}{\partial x} = 0. \]