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Finite Difference Solutions of Axisymmetric Infiltration Through Partially Saturated Porous Media

Chi-Yuan Wei

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FINITE DIFFERENCE SOLUTIONS OF AXISYMMETRIC INFILTRATION THROUGH PARTIALLY SATURATED POROUS MEDIA

by

Chi-Yuan Wei
and
Roland W. Jeppson

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April 1971
PREFACE

This report examines the problem of partially saturated moisture movement from a circular infiltrometer to a water table. While actually three-dimensional, the problem can be described by a two-directional coordinate system by taking advantage of the axis of symmetry.

The report begins with a discussion of Darcy's law and functional equations which have been proposed by a number of investigators for defining an effective hydraulic conductivity for partially saturated soils for use in Darcy's law. Readers familiar with this background may wish to begin their reading with the section “Inverse formulation,” and only scan the sections “Influence of Parameters in Permeability-Capillary Pressure Relationship” and “Relative Permeability as a Function of Saturation.”

The mathematical description of the physical problem considers the magnitudes of the radial and the axial coordinates as the dependent variables and the potential function (hydraulic head) and Stokes' stream function as the independent variables or the coordinates which define the plane of the partial differential equation boundary value problem. The resulting inverse formulation reverse the usual role of the variables, and has several advantages, which are made apparent in the report, over the more conventional formulation.

A computer program has been developed for solving the resulting two nonlinear partial differential equations with their associated boundary conditions simultaneously by utilizing techniques of finite differences. A number of solutions are obtained by varying parameters which describe the hydraulic properties of a range of soil types. Analyses from the results from these solutions are presented in a number of graphs toward the later part of the report which permit one to predict a number of features of the steady state flow under varying conditions.
ABSTRACT

Solutions are obtained to the problem of steady-state partially saturated infiltration of moisture applied over a horizontal source circle which moves through homogeneous soils toward a water table. A commonly accepted relationship between relative permeability and capillary pressure has been utilized in conjunction with Darcy’s law to formulate the mathematical model. The solutions have utilized an inverse formulation and have been obtained by finite difference. The inverse formulation considers the magnitudes of the cylindrical coordinates r and z as the dependent variables and the potential function \( \Phi \) and Stokes' stream function \( \psi \) as the independent variables (i.e. the problem is solved for r and z in the \( \Phi\psi \) plane). The approach used for solving the problems is practical with modern digital computers. The computer output gives the r and z coordinates at each finite difference grid point. These values can readily be plotted in flownet form to show the characteristics of the flow pattern at a glance. From the solution results, the distribution of capillary pressure, relative permeability, or effective saturation over any surface or plane of interest can be obtained. The solutions indicate that significant radial movement of moisture (or spreading effect) occurs causing higher infiltration rate at the edge of the source circle than near the center. The infiltration rate is closely related to various soil parameters which characterize the hydraulic properties of soils. Also presented are several distributions of the relative permeability or effective saturation on the surface, along the axis of symmetry, and on the plane including the axis of symmetry and how these distributions are related to the soil parameters.
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Distribution of permeability parameter, $K_r'$, along the axis of symmetry for different $p_b/\gamma$-values, with $\eta = 2.50$, and $p_c/\gamma = 1.0'$ over the source circle.

Distribution of permeability parameter, $K_r'$, along the axis of symmetry for different $\eta$-values, with $p_b/\gamma = 3.50'$, and $b = 1.0$, and $p_c/\gamma = 1.0'$ over the source circle.

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Distribution of equi-permeability curves for $\eta = 3.50$, $p_b/\gamma = 3.50'$, and $b = 1.0$.

Distribution of equi-permeability curves for $\eta = 2.50$, $p_b/\gamma = 3.50'$, and $b = 1.0$.

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Distribution of equi-permeability curves for $\eta = 2.50$, $p_b/\gamma = 3.50'$, and $b = 1.0$.

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Nomogram which relates the ratio of the axisymmetric infiltration rate divided by the one-dimensional rate to the soil parameters.
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<th>Meaning</th>
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<tr>
<td>A</td>
<td>soil parameter;</td>
</tr>
<tr>
<td>a'</td>
<td>soil parameter;</td>
</tr>
<tr>
<td>a</td>
<td>soil parameter;</td>
</tr>
<tr>
<td>b'</td>
<td>soil parameter;</td>
</tr>
<tr>
<td>b</td>
<td>soil parameter;</td>
</tr>
<tr>
<td>C₁</td>
<td>correction factor for obtaining infiltration rate;</td>
</tr>
<tr>
<td>D</td>
<td>depth of the water table from the ground surface;</td>
</tr>
<tr>
<td>D</td>
<td>Jacobian determinant;</td>
</tr>
<tr>
<td>d</td>
<td>½ ΔΦ₁;</td>
</tr>
<tr>
<td>eₛ</td>
<td>an orthogonal unit vector in s-direction of the orthogonal curvilinear coordinates;</td>
</tr>
<tr>
<td>eₙ</td>
<td>an orthogonal unit vector in n-direction of the orthogonal curvilinear coordinates;</td>
</tr>
<tr>
<td>g</td>
<td>acceleration of gravity;</td>
</tr>
<tr>
<td>H</td>
<td>potential energy per unit pound of water at the source surface;</td>
</tr>
<tr>
<td>h</td>
<td>total head defined as the height above a standard elevation datum of water column in a manometer;</td>
</tr>
<tr>
<td>l</td>
<td>infiltration rate per unit area (the volume flow of water per unit area);</td>
</tr>
<tr>
<td>Iᵣ</td>
<td>relative infiltration rate per unit area;</td>
</tr>
<tr>
<td>Iᵢ</td>
<td>relative infiltration rate for one-dimensional case;</td>
</tr>
<tr>
<td>I₃</td>
<td>relative infiltration rate for axisymmetric case;</td>
</tr>
<tr>
<td>J</td>
<td>the Jacobian of the inverse transformation;</td>
</tr>
<tr>
<td>K</td>
<td>coefficient of permeability;</td>
</tr>
<tr>
<td>Kₛ/b</td>
<td>Kₛ = permeability of saturation;</td>
</tr>
<tr>
<td>Kᵣ</td>
<td>relative permeability;</td>
</tr>
<tr>
<td>Kᵣ</td>
<td>bKᵣ = permeability parameter;</td>
</tr>
<tr>
<td>Kᵣ</td>
<td>equivalent coefficient of permeability for anisotropic soils;</td>
</tr>
<tr>
<td>Kᵣ</td>
<td>permeability in the radial direction;</td>
</tr>
<tr>
<td>K₂</td>
<td>permeability in the axial direction;</td>
</tr>
<tr>
<td>k</td>
<td>the effective intrinsic permeability;</td>
</tr>
<tr>
<td>m</td>
<td>½ Δξₖ;</td>
</tr>
<tr>
<td>n</td>
<td>porosity;</td>
</tr>
<tr>
<td>p</td>
<td>the pressure of water;</td>
</tr>
<tr>
<td>p_b</td>
<td>the bubbling pressure;</td>
</tr>
<tr>
<td>p_c</td>
<td>-p = the capillary pressure;</td>
</tr>
<tr>
<td>p_o</td>
<td>p_c/p_b;</td>
</tr>
<tr>
<td>q</td>
<td>the volume flow of water per unit area in z direction;</td>
</tr>
<tr>
<td>q₀</td>
<td>q/Kᵣ;</td>
</tr>
<tr>
<td>R</td>
<td>Δψ/ΔΦ₁;</td>
</tr>
<tr>
<td>Rₙ</td>
<td>remainder term of a Taylor's series expansion;</td>
</tr>
<tr>
<td>R₀</td>
<td>the radius of the source circle;</td>
</tr>
<tr>
<td>r</td>
<td>the radial distance in the horizontal direction from the origin;</td>
</tr>
<tr>
<td>S</td>
<td>the saturation of the soil;</td>
</tr>
<tr>
<td>Sₑ</td>
<td>the effective saturation;</td>
</tr>
<tr>
<td>Sᵣ</td>
<td>the residual saturation;</td>
</tr>
<tr>
<td>V</td>
<td>seepage velocity (flow rate per unit area);</td>
</tr>
<tr>
<td>Vᵣ</td>
<td>the velocity component in the radial direction;</td>
</tr>
<tr>
<td>Vₙ</td>
<td>the velocity component in the axial direction;</td>
</tr>
<tr>
<td>Vₛ</td>
<td>seepage velocity vector;</td>
</tr>
<tr>
<td>z</td>
<td>the vertical distance in the axial direction from the origin;</td>
</tr>
<tr>
<td>z₀</td>
<td>z/(ρ_b/γ);</td>
</tr>
<tr>
<td>z₁</td>
<td>the equivalent z for anisotropic soils;</td>
</tr>
<tr>
<td>(ζ₀,ζ₂)</td>
<td>The directional cosines of the unit vector eₛ;</td>
</tr>
</tbody>
</table>
\( (\alpha_1', \alpha_2') \) = the directional cosines of the unit vector  
\( \vec{v}_n \).  
\( \beta = \) the angle between an equipotential line and a streamline at a grid point of a mesh;  
\( \gamma = \rho g = \) the specific weight of water;  
\( \eta = \) the pore-size distribution index;  
\( \theta = \) the angle between the direction of flow and the horizontal \( r \)-axis;  
\( \lambda = \) a soil parameter similar to the pore-size distribution index;  
\( \lambda' = \) a soil parameter;  
\( \mu = \) the absolute viscosity of water;  
\( \nu = \) the kinematic viscosity of water;  
\( \rho = \) the mass density of water;  
\( \tau = \) a soil parameter;  
\( \Phi = \) the potential energy per unit pound of water;  
\( \Phi_\lambda = \) Stokes' stream function;  
\( \psi = \) over relaxation factor.  

\( K_n \Phi \):
INTRODUCTION

Water movement through soils and other porous media has been of interest to mankind since early history. Approximately one hundred years ago, the scientific basis upon which to predict water movement in soils began to develop. The most important contribution to the quantitative study of the groundwater movement was made by Henry Darcy who, in 1856, observed the characteristics of downward flow of water through sand filters and postulated the now famous basic linear law of flow of water through porous media. His postulate states that the volume of water crossing a unit area in unit time is directly proportional to the difference between heights of water in manometers terminated above and below the sand, and inversely proportional to the thickness of the sand. In other words, the velocity is proportional to the negative hydraulic gradient. This relation has since become known as Darcy’s law. Darcy’s law applies to porous media that are saturated and contain only a single parameter that characterizes the media known as the coefficient of permeability.

Although Darcy’s original experiment was made on water flowing vertically downward through a sand, Darcy’s law has been shown to be invariant with respect to the direction of the flow in the earth’s gravity field. The generalization of Darcy’s equation for saturated flow in three-dimensional space can be written as $\nabla \cdot \mathbf{v} = -K \nabla h$ (Muskat, 1937, and Hubbert, 1956) in which $\mathbf{v}$ is the velocity vector, and $h$ (a scalar quantity) is the total head defined as the height above a standard elevation datum of the water column in a manometer, and $K$ is a coefficient with units of velocity depending on the permeability of the porous media. While Darcy’s law is stated macroscopically, the volume elements to which the velocity and pressure refer are supposed to contain a large number of pores and the dynamical variables are averages over a large number of pores.

On a small scale there are large variations in size of individual pores in a disordered porous media such as soil. The fluid motions within the individual pores obey the Navier-Stokes equations of motion. Darcy’s law is really a statistical result giving the empirical equivalent of the Navier-Stokes equations averaged over a large number of individual pores. Thus, Darcy’s law is a statistical macroscopic equivalent of the Navier-Stokes equations of motion for the flow of water through porous media (Hall, 1956, and Hubbert, 1956).

Darcy’s law itself does not permit the solution of a particular problem of flow of water through porous media. Rather it is useful in formulating a partial differential equation for the problem. The partial differential equation is obtained by substituting Darcy’s law in the continuity equation. If the porous media is homogeneous and isotropic and the flow is assumed to be steady, the equation reduces to the well-known Laplace equation.

Before attempting to apply Darcy’s law to any porous media flow problem, it is essential to know the limitations of the law. Darcy’s law is subject to the following limitations: (1) The fluid is homogeneous. (2) There is no interaction of fluid and the porous media. (3) The flow rates are relatively small so that the inertia effects are negligible. The presence of clay in a porous media which interacts with the fluid could well reduce the validity of the Darcian proportionality. Good agreement with Darcy’s equation occurs for seepage flow in nonswellling silt-sized particles down to the 0.002 mm size, generally considered the upper limit of the clay range (Olsen, 1965 and 1966). However, should the porous medium contain particles within the clay range, greater-than-proportional response of flow velocity to hydraulic gradient may be observed. Indications suggest the deviation from linearity is mainly due to non-Newtonian liquid behavior caused by clay water interaction (Swartzendruber, 1962a, b, 1963). Darcy’s law is also valid only for low rates of flow. The limiting Reynolds number using the effective diameter of the soil, $d_{10}$, as the length parameter ($R_e = V d_{10}$) in which $V$ is the flow rate per unit area, $d_{10}$, is the effective diameter of the opening of the sieve, through which 10 percent of the soil sample by weight passes, and $v$ is the kinematic viscosity of water has been found within the range of 3 to 10. The limiting Reynolds number is approximately unity when using the average grain diameter, defined by $d = \Sigma n_i d_i^2 / \Sigma n_i$ in which $d_i$ is the arithmetic mean of the opening of the consecutive sieves of the Taylor or U.S. Standard series, and $n_i$ is the number of grains of diameter $d_i$, as found by a sieve analysis.

Many saturated porous media flow problems have been solved using Darcy’s law and the equation of continuity. Among these are a number of analytic solutions to a variety of porous media flow problems. (See Muskat, 1937; Harr, 1962; Polubarinova-Kochina, 1962.) With the development of high-speed digital computers, more complicated problems can be solved by numerical methods. A number of approximate solutions have been obtained. (See Shaw and Southwell, 1941; Fayers and Sheldon, 1962; Freeze and Witherspoon, 1966; Zienkiewicz, Mayer, and Cheung, 1966; Terzidis, 1968; Jeppson, 1969a; Taylor and Luthin, 1969.)

The problem of steady infiltration of rainfall into the surface of a watershed is difficult to solve. The flow is partially saturated, and is effected by capillary tensions at the air-water interface as well as by reduced effective soil voids. Modification of Darcy’s law to apply to partially saturated flow in porous media has been accomplished by past research.

An extension of Darcy’s law to partially saturated flow in soils was made by Buckingham (1907) who studied the capillary flow of water through a soil with the analogy of thermal and electrical conduction. Gardner (1936) and Gardner and Gardner (1950) suggested that Darcy’s equation be modified for partially saturated flow by including a function of the moisture content, f, giving $\nabla \cdot \mathbf{v} = -f K \nabla h$. Hall (1956) showed, by analytical derivation, that the general form of Darcy’s equation for anisotropic porous media is valid for all saturated and partially saturated porous media systems if the inertia forces of the system are negligible, the liquid films are continuous, and a volume element of the fluid-porous medium system can be selected which is small compared to the gross dimension of the system, yet large enough that the surface area of the matrix therein can be considered to be uniformly distributed throughout the volume element. He also stated that the permeability $K$ is a scalar quantity whose magnitude may vary with time, location, and moisture content. The evaluation of this quantity by analytical means is difficult since in partially saturated porous media flow, capillary forces are present at each air-water interface in the
is the capillary pressure, and $P_c$ and $P_b$ are parameters depending on the liquid, the soil, and the capillary pressure history. This equation has been used to solve partially saturated drainage and subirrigation problems (Sewell and Schilfgaarde, 1963). Further investigations made by Scott and Corey (1961) and Brooks and Corey (1964, 1966) suggested an equation of the form $K_r = (p/p_c)\beta$ for $p_c > p_b$ in which $K_r$ is relative permeability defined as the ratio of permeability at any given saturation to the permeability at complete saturation, $p_c = p$ is the capillary pressure, and $p_b$ and $\eta$ are parameters depending upon the liquid, the soil, and the capillary pressure history of the system. The equation ignores the zone of relatively constant permeability for the range of capillary pressure less than $p_b$. King (1965) noted that Gardner’s equation is dimensionally inconsistent and proposed a dimensionless form of the equation

$$K_r = \frac{1}{(p/p_c)\beta + b}$$

in which $K_r$ is a dimensionless permeability parameter equal to the relative permeability divided by $b$, and $p_b$ (a positive quantity) is a parameter having the same dimension as capillary pressure, and $\eta$ and $b$ are dimensionless parameters. Equation 1 fits imbibition and desaturation experimental data quite well (King, 1965). The magnitudes of the parameters in Eq. 1 are related to hydraulic properties of soils. Furthermore, Eq. 1 can readily be used for solutions of steady-state porous media flow problems. Also Eq. 1 is algebraic and differentiable and consequently its use simplifies the computer programming for numerical solutions over that needed for a non-algebraic equation. The equation is also simple, can be solved repeatedly without excessive computer execution time, and does not require core storage as would be the case if actual data were stored. Hence solutions of larger problems such as the general three-dimensional infiltration problems are feasible.

Efficient watershed management requires accurate information on the infiltration rate at which different soils will take water under different conditions. Infiltrimeters are often used to determine the final infiltration rate of a soil on small watersheds or on experimental or sample areas within large watersheds. However, these measurements are subject to an unknown effect due to the lateral movement of water in soils. Musgrave (1942) pointed out the importance of lateral movement of water which occurs to some extent with any method of applying water to a small plot by rainfall simulators of various kinds and sizes and with different sizes of concentric rings. Duley and Domingo (1943), working with spray applications, found that the mean total intake of water on isolated small plots (16 by 72 inches) having no protected border protection was 75 percent higher than for the large plots protected with a buffer strip of a saturated belt of soil, and 92 percent greater than for the small enclosed plots. The results indicated that lateral movement of moisture had allowed these small plots to take in more water than would be possible if it were raining over the entire surface of a large area. Marshall and Stirk (1950) used both buffered and unbuffered rings ranging from 1 to 10 feet in diameter to study the effect of size of plots on the lateral movement of water and showed that the steady minimum infiltration rate of a given soil decreased with increasing size of plot. Wooding (1968), upon assuming that permeability can be defined as an exponential of the pressure, linearized the non-linear differential equation, and obtained solutions for steady infiltration from a shallow, circularly inundated area on the horizontal surface of a semi-infinite porous medium, and showed varying degrees of lateral movement of moisture for varying soils. Jeppson and Nelson (1970), using Eq. 1 to describe hydraulic properties of soils for partially saturated seepage from canals, obtained solutions and found that a more pronounced spreading effect of moisture movement can be detected for the partially saturated solutions than for solutions based on the saturated flow assumption.

Investigations on the spreading effect and infiltration characteristics through the application of partially saturated flow theory to infiltration is important to the development of watershed hydrology. To investigate spreading characteristics is one of the objectives of this study.

The problem of infiltration of rainfall on the surface of a watershed can be formulated mathematically by non-linear partial differential equations. The general three-dimensional unsteady problem thus formulated is difficult to solve. The axisymmetric case has been solved by Jeppson (1970c). The problem of steady-state axisymmetric infiltration resulting from moisture applied at a horizontal surface has been selected for this study to determine the influence of various soil properties (i.e. the parameters in Eq. 1 which describe a wide variety of soils and their hydraulic properties) on the flow patterns as well as the magnitude of the “spreading effect,” the steady-state solutions will provide a check on the validity of unsteady-state solutions.

The partial differential equations, which result from substituting Eq. 1 into the steady state continuity equation as well as the equivalent equation using the stream function as the dependent variable, are of elliptic type. With proper boundary conditions, axisymmetric flow from an infiltrimeter becomes a mathematical boundary value problem. Reisenauer (1963), using the methods outlined by Nelson (1962) has obtained a numerical solution to the problem of steady-state partially saturated axisymmetric flow from a circular pond by a formulation in the physical plane. An inverse formulation (see Jeppson, 1966, 1967, 1968a, 1968b, 1969a, 1969b; Jeppson and Nelson, 1970) has been used in this study to construct the mathematical problem. This inverse formulation considers the coordinates in the radial and axial directions (r and z respectively) as the dependent variables and the potential function $\Phi$ (proportional to the hydraulic head) and the stream function $\Psi$ as the independent variables (i.e. the problem is solved for $r$ and $z$ in the $\Phi$, $\Psi$ plane). Finite difference techniques are used in solving the problem by an iterative procedure and a computer program has been written. The solution results consist of the $r$ and $z$ coordinates at the intersection of each streamline with each potential line. These values can readily be used to construct the flownet with a computer driven plotter. The resulting flownets show characteristics of the flow patterns at a glance. In addition, the capillary tension can be obtained at each point from the values of z and the potential $\Phi$ from the solution, and the relative permeability at each point can be obtained by substituting capillary tension at each point, and the soil parameters specified to obtain the solution, into Eq. 1.
Saturation is another item of interest. The functional relationship between relative permeability and saturation can be obtained analytically, if a suitable algebraic expression for the effective saturation $S_e = f(p_e)$ can be defined, which involves the soil parameters in Eq. 1. While this algebraic expression is not available, the first order ordinary differential equation obtained from Burdine's equation (Burdine, 1953) for relative permeability can be solved for the relationship $K_r$ vs. $S_e$. The solution of the differential equation cannot be obtained analytically, thus, numerical approximation methods have been used to obtain values for $S_e$ corresponding to $K_r$.

In the following sections, the mathematical formulation and the procedure followed in obtaining the inverse finite difference operators for the problem and the numerical techniques involved in obtaining solutions for different soil types are discussed in detail. Solution results are presented in flownet form, and effects due to different soil parameters are given.
The general form of the Darcy's equation is assumed to apply for partially saturated flow

\[ \mathbf{v} = -K \text{grad } \Phi \]  \hspace{1cm} (2)

in which \( \mathbf{v} \) is the velocity vector, \( K(p(r,z)) \) is the permeability coefficient of the soil with dimension of velocity, and \( \Phi \) is the potential energy per unit pound of water with dimension of length and is defined by

\[ \Phi = \frac{p}{\rho g} + z \]  \hspace{1cm} (3)

in which \( p \) is the pressure of water, \( \rho \) is the mass density of water, \( g \) is the acceleration of gravity, and \( z \) is the vertical coordinate. The effects of the soil and water are incorporated in the coefficient of permeability

\[ K = \frac{k (p(r,z))}{\mu} \]  \hspace{1cm} (4)

in which \( k(p(r,z)) \) is the effective intrinsic permeability with dimensions of length squared, and \( \mu \) is the absolute viscosity of water. The velocity components also result from the definition of Stokes' stream function. Thus the components of velocity in the radial and axial directions are given by

\[ V_r = -K \frac{\partial \Phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial z} \]  \hspace{1cm} (5)

and

\[ V_z = -K \frac{\partial \Phi}{\partial z} = -\frac{1}{r} \frac{\partial \psi}{\partial r} \]  \hspace{1cm} (6)

In Eqs. 5 and 6, it is assumed that the flow is axisymmetric and the soil is isotropic, i.e. the permeability is the same in the radial and axial directions. For anisotropic soils, with the permeability in the radial direction equal to \( K_r \) and that in the vertical direction \( K_z \), Eqs. 5 and 6 apply if \( z \) is replaced by \( z \) and \( K \) by \( K_r \) from the following two equations (Muskat, 1937).

\[ z_t = \sqrt{\frac{K_r}{K_z}} z \]  \hspace{1cm} (7a)

\[ K_t = \sqrt{\frac{K_r}{K_z}} K \]  \hspace{1cm} (7b)

The differential equation for either the potential function or the stream function can be obtained from Eqs. 5 and 6. Upon differentiating Eq. 5 with respect to \( r \) and Eq. 6 with respect to \( z \) and combining the resulting equations to eliminate the common term \( \frac{\partial^2 \psi}{\partial r \partial z} = \frac{\partial^2 \psi}{\partial z \partial r} \) gives

\[ K \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) + \frac{\partial K}{\partial r} \frac{\partial \phi}{\partial r} + \frac{\partial K}{\partial z} \frac{\partial \phi}{\partial z} = 0 \]  \hspace{1cm} (8)

The general form of Eq. 8, not associated to a particular coordinate system, can be obtained by substituting Darcy's Eq. 2 into the steady-state, continuity equation.

\[ \nabla \cdot (K \text{grad } \phi) = K \nabla^2 \phi + \text{grad } K \cdot \text{grad } \phi = 0 \]  \hspace{1cm} (9)

The equation for the stream function which results from differentiating Eq. 5 with respect to \( z \) and Eq. 6 with respect to \( r \) and combining the resulting equations is

\[ K \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial K}{\partial z} \frac{\partial \psi}{\partial r} - \frac{\partial K}{\partial r} \frac{\partial \psi}{\partial z} = 0 \]  \hspace{1cm} (10)

Equations 8 and 10 apply in both the saturated and the partially saturated regions of flow. In saturated regions, since \( K \) is constant for homogeneous soil, the terms including derivatives of \( K \) vanish. In partially saturated regions, the derivatives with respect to \( K \) must be evaluated from the relationship between permeability and capillary pressure. The equation used for this relationship is described later.

Instead of solving Eqs. 8 and 10 on the physical plane as mentioned earlier, the problem in this study is solved on the \( \phi \psi \) plane (a rectangular region).

**Inverse formulation**

The inverse partial differential equations which describe the flow in terms of \( r(\phi, \psi) \) and \( z(\phi, \psi) \) can be derived from the implicit-function theory (Taylor, 1955) following the approach used by Jeppson (1968b). A pair of basic equations similar to Eqs. 5 and 6 can be obtained;

\[ K \frac{\partial \psi}{\partial r} = -\frac{1}{r} \frac{\partial \phi}{\partial r} \]  \hspace{1cm} (11)

and

\[ K \frac{\partial \phi}{\partial \psi} = 1 \frac{\partial z}{\partial \psi} \]  \hspace{1cm} (12)
These equations are important for developing the governing partial differential equations in the $\Phi_\Psi$ plane. Upon differentiating Eq. 11 with respect to $\Phi$ and Eq. 12 with respect to $\Psi$, and combining the results to eliminate the common term $\partial^2 \Phi/\partial \Phi^2 = \partial^2 \Psi/\partial \Psi^2$ gives

$$K_r \left( K^2 r^2 \frac{\partial^2 \chi}{\partial \Phi^2} + \frac{\partial^2 \chi}{\partial \Psi^2} \right) + r \left( K^2 r^2 \frac{\partial K}{\partial \Psi} \frac{\partial \Phi}{\partial \Phi} - \frac{\partial K}{\partial \Phi} \frac{\partial \Psi}{\partial \Psi} \right)$$

$$+ K \left[ K^2 r^2 \left( \frac{\partial \chi}{\partial \Phi^2} - \frac{(\partial \chi)^2}{\partial \Phi^2} \right) = 0 \ldots \ldots (13) \right.$$  

Also differentiating Eq. 11 with respect to $\Psi$ and Eq. 12 with respect to $\Phi$ and combining the results gives

$$K \left( K^2 r^2 \frac{\partial^2 \chi}{\partial \Psi^2} + \frac{\partial^2 \chi}{\partial \Phi^2} \right) + \left( K^2 r^2 \frac{\partial K}{\partial \Phi} \frac{\partial \Psi}{\partial \Phi} - \frac{\partial K}{\partial \Phi} \frac{\partial \Psi}{\partial \Psi} \right)$$

$$+ 2 K^2 \frac{\partial \chi}{\partial \Phi} \frac{\partial \chi}{\partial \Psi} = 0 \ldots \ldots (14)$$

Since both dependent variables $r(\Phi, \Psi)$ and $\chi(\Phi, \Psi)$ appear in Eqs. 13 and 14, these two equations must be solved simultaneously. In partially saturated regions $K$ is a function of pressure, and consequently, also a function of $z$. The methods used for evaluating $K$ and its derivatives will be discussed in the following section.

Evaluation of the derivatives of the permeability

Equation 1 proposed by King (1965) which describes the relationship between capillary pressure and permeability at any given saturation for imbibition as well as drainage was adopted for this study. For convenience Eq. 1 is rewritten

$$K = \frac{K_o}{\frac{-p}{P_b} + b} \ldots \ldots (1)$$

in which $K_o/b$ is the saturated permeability, $p(r,z)$ is the water pressure and is negative for partially saturated flow, $P_b$ is the bubbling pressure and $\eta$ is the pore size distribution index defined as the negative slope of the straight-line portion of the log-log plot of $K_r$ versus $-p/P_b$ and $b$ is a dimensionless constant. The magnitude of the parameters in Eq. 1 depends on the characteristics of the soil and the capillary pressure history of the soil water system.

Using the definition of the total energy per pound of water from Eq. 3

$$-p = \rho g (z - \Phi) \ldots \ldots (15)$$

Substituting Eq. 15 into Eq. 1 gives

$$K_r = \frac{1}{\frac{-\rho g}{P_b} (z - \Phi)^n + b} \ldots \ldots (16)$$

in which $K_r$ is the permeability parameter defined earlier and given by

$$K_r = \frac{K}{K_o} \ldots \ldots$$

Differentiating Eq. 16 with respect to $\Phi$ and $\Psi$ gives respectively

$$K_r \frac{\partial K}{\partial \Phi} \left[ \frac{-\rho g}{P_b} (z - \Phi)^n + b \right] = \frac{\partial K_r}{\partial \Phi} \ldots \ldots (17)$$

Since both dependent variables $r(\Phi, \Psi)$ and $\chi(\Phi, \Psi)$ appear in Eqs. 13 and 14, these two equations must be solved simultaneously. In partially saturated regions $K$ is a function of pressure, and consequently, also a function of $z$. The methods used for evaluating $K$ and its derivatives will be discussed in the following section.

Eqs. 15 through 19b apply only in those regions of flow which are partially saturated (i.e. when $p < 0$). Whenever the soils become saturated, the coefficient of permeability remains constant and equals $K_o/b$.

Scaling of variables

Since permeability is generally a relatively small quantity, Eqs. 13 and 14 can be made more amenable to solution by finite difference methods by making the coefficients of each term closer to unity. To accomplish this, the total potential energy per pound of water $\Phi$ has been scaled by

$$\Phi_t = \frac{\Phi}{K_o} \ldots \ldots (20)$$

From Eq. 20, the following relationships can be obtained

$$\frac{\partial r}{\partial \Phi_t} = \frac{K_r}{K_o} \frac{\partial r}{\partial \Phi} \ldots \ldots (21)$$

$$\frac{\partial^2 r}{\partial \Phi_t^2} = \frac{K_r^2}{K_o} \frac{\partial^2 r}{\partial \Phi^2} \ldots \ldots (22)$$
\[
\frac{\partial z}{\partial \phi} = K_0 \frac{\partial z}{\partial \Phi} t \quad (23)
\]

\[
\frac{\partial^2 z}{\partial \phi^2} = K_0^2 \frac{\partial^2 z}{\partial \Phi^2} t \quad (24)
\]

\[
\frac{\partial K}{\partial t} = K_0^2 \frac{\partial^2 R}{\partial \Phi^2} t \quad (25)
\]

Upon substituting from Eqs. 21 through 25, Eq. 13 becomes

\[
\frac{K_r}{r} \left( \frac{K_r}{r}^2 r^2 \left( \frac{\partial^2 z}{\partial \phi^2} + \frac{\partial^2 z}{\partial \Phi^2} t \right) \right) + \frac{r}{K_r} \left( \frac{K_r}{r}^2 r^2 \left( \frac{\partial^2 K_r}{\partial \phi \partial \Phi} + \frac{\partial^2 K_r}{\partial \Phi^2} t \right) \right) \]

\[
+ \frac{K_r}{r} \left( \frac{K_r}{r}^2 r^2 \left( \frac{\partial^2 K_r}{\partial \phi \partial \Phi} - \frac{\partial K_r}{\partial \phi} \frac{\partial r}{\partial \Phi} \right) \right) = 0 \quad (26)
\]

Likewise, Eq. 14 becomes

\[
\frac{K_r}{r} \left( \frac{K_r}{r}^2 r^2 \left( \frac{\partial^2 z}{\partial \phi^2} + \frac{\partial^2 z}{\partial \Phi^2} t \right) \right) \quad (27)
\]

In the case of saturated soils, the permeability parameter \( K_r \) remains constant and equals 1/b. The derivatives of \( K_r \) with respect to both \( \Phi \) and \( \psi \) are consequently zero whenever \( p \neq 0 \) and Eqs. 26 and 27 reduce respectively to the inverse equation for axisymmetric seepage through saturated homogeneous porous media

\[
r^3 \frac{\partial^2 z}{\partial \phi^2} + 2r \left( \frac{\partial r}{\partial \phi} \right)^2 + \frac{\partial^2 r}{\partial \phi^2} t = \left( \frac{\partial r}{\partial \Phi} \right)^2 = 0 \quad (28)
\]

and

\[
r^2 \frac{\partial^2 z}{\partial \phi^2} + \frac{\partial^2 z}{\partial \Phi^2} t + 2 \frac{\partial z}{\partial \phi} \frac{\partial z}{\partial \Phi} = 0 \quad (29)
\]

Equations 26, 27 and/or 28, 29 are final equations for governing the partially saturated-saturated flow system in the transformed \( \Phi, \psi \) plane.

**Boundary conditions**

Equations 26 and 27 are elliptic types. (See, e.g., Courant and Hilbert, 1962, for definition of elliptic type equation.) Consequently, the problem being studied is a boundary value problem. For a well posed boundary value problem, boundary conditions must be established on all boundaries enclosing the region of flow.

The problem which is described herein is that of steady state axisymmetric infiltration from a circular rainfall simulator through homogeneous soil to a water table at a known depth (see Fig. 1). Since the rainfall is applied over a circular area of the soil surface, the region of infiltration below the circular area is symmetric about the vertical centerline. The boundary value problem can be set up for only one-half of any vertical plane containing the axis of symmetry such as the vertical rectangular area shown in Fig. 1.

The boundary conditions are shown in Fig. 2 in which the equivalent flow region to that shown in Fig. 1 has been plotted on the \( \Phi, \psi \) plane. These boundary conditions are used to establish values of variable \( r \) along the water table and the soil surface including the circular area of surface infiltration and the values of the variable \( z \) along the axis of symmetry. These conditions are obtained as follows (the number notations for the boundaries are shown on Figs. 1 and 2).

**Axis of symmetry**

The value of \( r \) is zero along the axis of symmetry and since equipotential lines are horizontal in the physical plane at the axis of symmetry, the boundary conditions along the axis of symmetry are \( r = 0 \) and \( \frac{\partial z}{\partial \psi} = 0 \).

Special consideration must be given in obtaining the solution near the axis of symmetry, since as the radius \( r \) approaches zero, the inverse transformation used to obtain Eqs. 11 and 12 is not valid. Failure of the transformation at \( r = 0 \) can be checked by noting that the value of the Jacobian \( J \) for the transformation, which is given by

\[
J = \frac{\partial (\Phi, \psi)}{\partial (r, z)} = \frac{1}{\partial (r, z)} \frac{\partial (\Phi, \psi)}{\partial (r, z)} = 1 \quad (30)
\]

approaches zero as \( r \) approaches zero, with \( \frac{\partial z}{\partial \psi} = 0 \), i.e.

\[
J = \frac{1}{\partial (r, z)} \frac{\partial (\Phi, \psi)}{\partial (r, z)} = -\frac{K_r}{2} = 0 \quad (31)
\]

Equation 31 shows that a line singularity exists along the axis of symmetry and the derivative \( \partial r/\partial \psi \) is discontinuous at the axis. The manner in which this line singularity has been handled in the finite difference solution is described in a later section.
Fig. 1. Axisymmetric infiltration through partially saturated porous media from surface application of moisture over a circular area to a water table.

Fig. 2. Formulation of the boundary value problem for steady state axisymmetric infiltration from a source circle through a homogeneous porous media to a water table.
Source surface 1 - 3. The source surface is assumed to be horizontal and consequently at a constant height $D$ above the water table, the condition for $z$ along this boundary is

$$z = D,$$  \hspace{1cm} (32)

provided that the origin of the coordinates is selected at the axis of symmetry on the water table. A constant pressure head is also specified over the source surface resulting in a constant value for $K$. Consequently the potential energy per pound of fluid at the surface is also constant, i.e.

$$\phi_{NH} = H,$$  \hspace{1cm} (33)

The value of $H$ is read as an input parameter in the computer program. If the value of $H$ is selected to be greater than $D$, ponded water with depth $H-D$ exists over the source area, and the problem becomes a combined saturated-unsaturated flow case. Also since the streamlines leave the source surface vertically, the boundary condition for $r(\Phi, \psi)$ can be expressed by

$$\frac{\partial r}{\partial \psi} = 0,$$  \hspace{1cm} (34)

Values of $r(\Phi, \psi)$ along the source surface boundary may be established, alternatively, by integrating Eq. 12 (Jeppson, 1968b). After transforming $\Phi$ to $\Phi_t$ and integrating, Eq. 12 yields

$$r = \int_{\Phi_t}^{\Phi} \frac{1}{K_r} \frac{\partial z}{\partial \Phi_t} d\psi,$$  \hspace{1cm} (35)

The subscript of the integral indicates the integration is to be carried out along an equipotential line, i.e. along the boundary. This implicit equation for $r$ (notice that $r$ appears on both sides of the equal sign) can be solved by finite differences.

Ground surface 3 - 4. The ground surface is horizontal and at a constant height $D$ above the water table, hence, the condition for $z$ along this boundary is $z = D$. Since the ground surface is a streamline, with each equipotential line entering the boundary perpendicularly, the condition for $r$ is $\partial r/\partial \psi = 0$. The boundary condition for $r$ can also be obtained, utilizing the solution for $z$, by integrating Eq. 11 along the boundary (i.e. the last $\psi$ = constant line), giving

$$r = -\int_{\psi} K_r r \frac{\partial z}{\partial \psi} d\Phi_t \hspace{1cm} (36)$$

The subscript by the integral indicates the integration is to be carried out along a streamline. This implicit equation for $r$ is solved by finite differences.

Vertical outer boundary 3 - 4. This vertical boundary is taken at a great enough distance from the axis of symmetry so that no moisture movement will occur along the boundary. The boundary is considered as a streamline and equipotential lines are essentially horizontal at this boundary. Under this assumption the condition becomes $\partial z/\partial \psi = 0$. An alternative approach may be used to obtain values of $z$ on and adjacent to part of this boundary. This approach has not been used in this study, but reduces the partial differential equation to an ordinary differential equation by noting that $K_r$ is very small along most of this boundary. Consequently terms in Eq. 27 which contain $K_r^2$ or $K_r^3$ can be omitted. Equation 27 then becomes

$$\frac{K_r}{\Phi_t} \frac{d^2 z}{d\Phi_t^2} - \frac{dK_r}{d\Phi_t} \frac{dz}{d\Phi_t} = 0 \hspace{1cm} (37)$$

Values of $z$ along the outer vertical boundary come from the solution of Eq. 37.

Water table 2 - 4. The water table is specified horizontal. The boundary condition for $z$ along this boundary is $z = 0$. Since the streamlines enter the water table vertically, the boundary condition for $r(\Phi, \psi)$ can be expressed by $\partial r/\partial \psi = 0$. The alternative way to establish the values of $r(\Phi, \psi)$ along the boundary by integration as mentioned in the last section has not been used, because the rapid variations in $K_r$ in the vicinity of the boundary have caused difficulty in obtaining a solution of $r$ on the boundary. Instead, a parabolic extrapolation has been used to meet the normal condition $\partial r/\partial \psi = 0$ and will be described later.
INFLUENCE OF PARAMETERS IN PERMEABILITY-CAPILLARY PRESSURE RELATIONSHIP

In partially saturated flow in porous media, capillary forces are present at each air-water interface in the interior of the flow system.

Several empirical equations have been proposed for representing the permeability as a function of capillary pressure in terms of some soil parameters. The manner in which the permeability varies with the parameters in two of these equations is described in this section.

Scott and Corey (1961) proposed the equations

\[ K_r = \left( \frac{p_b}{p_c} \right)^\eta \text{ for } p_c \geq p_b \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \quad (38) \]

and

\[ K_r = 1 \text{ for } p_c < p_b \]

in which \( K_r \) is the relative permeability (the ratio of permeability at any given saturation to the permeability at complete saturation), \( p_c \) is the capillary pressure (the absolute value of the pressure difference between air and water at the air-water interface in the pores of soil), \( p_b \) is bubbling pressure and was found by Brooks and Corey (1964) to be closely related to the largest pores forming a continuous network within a porous medium, and \( \eta \) is the pore-size distribution index which is expressed by the negative slope of the straight line portion of the log-log plot of \( K_r \) versus \( p_c \). Equation 38 defines a straight line on log-log paper having an intercept at \( K_r = 1.0 \). This equation assumes the permeability is constant for the range of capillary pressure less than \( p_b \) (Brooks and Corey, 1964, and King, 1965).

King has defined this zone of relatively constant permeability as a "plateau" which may exist both drainage and imbibition (King, 1965). King has also proposed Eq. 1 as a dimensionless form

![Fig. 3. Permeability parameter, \( K_r \), as a function of capillary pressure head for different \( p_b/\gamma \) values, with \( \eta = 2.50 \) and \( b = 1.0 \).](image-url)
of Gardner's equation as mentioned earlier. Equation 1 and not Eq. 38 was used to represent the permeability-capillary pressure relationship in this study and is discussed in detail in this section.

The functional relationship between $K_r$ and capillary pressure head obtained from Eq. 1 is shown in Fig. 3 for different values of $p_b/\gamma$ with $\eta = 2.50$ and $b = 1.0$. The five curves on the graph are nearly parallel for larger values of $p_c/\gamma$. The lengths of plateaus are different depending on $p_b/\gamma$ and to a lesser extent on $b$. The length of the plateau decreases as the value of bubbling pressure decreases. Curves for $K_r$ as a function of capillary pressure are shown in Fig. 4 for different values of $b$ with $p_b/\gamma = 3.50^1$ and $\eta = 2.50$. The slopes of the straight line portion of the curves are the same. Furthermore, the capillary pressures obtained by extrapolation of the straight line portion of the curves at the top of the graph where $K_r = 1.0$ are the same. The main difference in these curves is the height of plateau, which decreases with the increasing value of $b$. Five $K_r$ versus $p_c$ curves obtained from Eq. 1 are shown in Fig. 5 for different $\eta$-values with $p_b/\gamma = 3.50^1$ and $b = 1.0$. The plateaus of these curves have the same heights, and the capillary pressures obtained by extrapolation of the straight line portion of the curves to the value of $K_r = 1.0$ are also the same, and equal to the bubbling pressure. On the graph, all the curves have passed through a common point at $p_c = p_b$ and $K_r = 1/(1+b)$. For $p_c > p_b$, the permeability parameter ($K_r$) increases as the value of $\eta$ decreases for constant capillary pressure heads greater than $p_b/\gamma$ and $K_r$ decreases as the value of $\eta$ decreases for $p_c < p_b$. Also the radius of curvature of the curved portion of the curve increases as the value of $\eta$ decreases.

The function $K_r$ vs. $p/\gamma$ will approach a step function as the value of $\eta$ becomes very large.

The properties of Eq. 1 depicted in Figs. 3, 4, and 5 have influences on the flow patterns resulting from specification of different values of the parameters $b$, $p_b$, and $\eta$. These influences are discussed in later sections.

![Fig. 4. Permeability parameter, $K_r$, as a function of capillary pressure head for different $b$ values, with $p_b/\gamma = 3.50^1$ and $\eta = 2.50$.](image-url)
Fig. 5. Permeability parameter, $K_r$, as a function of capillary pressure head for different $\eta$ values, with $p_b/\gamma = 3.50$ and $b = 1.0$. 

\[ K_r = \frac{1}{1 + b} \] 

\[ K_r = \frac{1}{\eta \left( \frac{p_b}{\gamma} \right) + b} \] 

\[ p_b = p_b \]
RELATIVE PERMEABILITY AS A FUNCTION OF SATURATION

If air and water are the nonwetting and wetting fluids, respectively, experimental data of effective saturation vs. capillary pressure plots close to a straight line on log-log graph paper for capillary pressure, \( p_c \), greater than bubbling pressure, \( p_b \). A simple equation defining this relationship for the drainage cycle was given by Brooks and Corey (1964)

\[
S_e = \left( \frac{p_b}{p_c} \right)^{\lambda}, \quad \text{for} \quad p_c \geq p_b \tag{39}
\]

where \( \lambda = (\eta - 2)/3 \) is a number closely related to the pore-size distribution index, \( \eta \), and the effective saturation is defined by

\[
S_e = \frac{S - S_r}{1 - S_r} \quad \tag{40}
\]

in which \( S \) is the saturation and \( S_r \) is the residual saturation. (See Brooks and Corey, 1964, for the determination of \( S_r \).) A more general approximation equation proposed by Brutsaert (1968) is

\[
K_r = \left( S_e \right)^\lambda \tag{43}
\]

where \( A \) and \( \lambda \) are the parameters whose magnitude depends on the soil types. Where \( A \) is zero, the Brooks-Corey approximation results.

According to the analysis of Burdine (1953), the relative permeability can be expressed as a function of \( S_e \) (Brooks and Corey, 1964)

\[
K_r = S_e^2 \int_0^1 \frac{S_e}{p_c^2} \frac{dS_e}{p_c} \tag{42}
\]

where \( K_r \) is the relative permeability which has a maximum value of unity as \( p_c \) approaches zero.

The necessary Burdine integration has been carried out for Eq. 39 (or Eq. 41 with \( A = 0 \)) by Brooks and Corey (1964) to obtain the approximation equation for relative permeability, i.e. Eq. 38. And the relationship for \( K_r \) vs. \( S_e \) becomes a function of the pore-size distribution index, \( \eta \), only, and the following approximation equation for \( K_r \) vs. \( S_e \) is obtained

\[
K_r = \left( \frac{3 \lambda + 2}{S_e} \right)^\lambda \tag{44}
\]

When \( A \) is not zero, the limit for \( S_e \) approaches the constant value \( 1/A \) as \( p_c \) approaches zero. Generally, letting \( A \) take on a value gives a better fit to experimental data than if \( A = 0 \). With a non-zero \( A \) in Eq. 41 there is no direct method such as integrating the Burdine equation for evaluating the magnitudes of \( \lambda \) and \( A \) (Eq. 42) from the magnitudes of the parameters in Eq. 1 as is the case if \( A = 0 \). Consequently, the Burdine equation has been solved numerically.

If Eq. 42 is differentiated with respect to \( S_e \), a first order differential equation can be obtained

\[
\frac{dK_r}{dS_e} = \frac{2K_r}{S_e} + \frac{1}{C} \left( \frac{S_e}{p_c} \right)^2 \tag{44}
\]

in which

\[
C = \int_0^1 \frac{dS_e}{p_c^2} \frac{S_e}{p_c} \tag{45}
\]

is a constant depending on the relationship of saturation to capillary pressure. While Eq. 44 cannot be solved analytically the solutions of the equation can be generated numerically if a correct starting value can be found. The expression for \( K_r \) is given by

\[
K_r = \frac{b}{\eta} = bK_r \tag{45}
\]

An approximate value of \( C \) can be obtained by integrating Eq. 39 giving

\[
C = \frac{\lambda}{\lambda + 2} \frac{1}{p_b^2} \tag{46}
\]
With this approximate value, Eq. 44 can be solved approximately and thereafter the value of C can be improved by using the resulting relationship for \( S_e \) and \( p_c \). This improved value of C can, in turn, again be used to obtain a better solution to Eq. 44. By repeating this procedure until the value of C does not change between consecutive cycles, the solution is obtained. And the solution to \( K_r \) vs. \( S_e \) is obtained from the solution to Eq. 44 when \( K_r = K_r' / b \).

To solve Eq. 44 it is necessary to start with a value of \( K_r \) which corresponds to the starting value of \( S_e \). When the capillary pressure approaches infinity, the permeability parameter, \( K_r' \), as well as the effective saturation approaches zero. If, however, \( K_r = 0 \) and \( S_e = 0 \) are selected as the starting values, the first term on the right side of the equal sign in Eq. 44 becomes indeterminate, and the generation of the solution cannot be carried out. To prevent this indeterminacy, both \( K_r \) and \( S_e \) have been assigned small values with \( K_r \) less than \( S_e \). The magnitude of these small values must be improved by a trial and error procedure so that \( K_r = 1.0 \) when \( S_e = 1.0 \). The solution might also be obtained starting with \( K_r = 1.0 \) and \( S_e = 1.0 \) and using a negative increment for \( S_e \). If this approach is used, however, it is necessary to modify Eq. 44 to prevent division by zero because \( p_c = 0 \) when \( S_e = 1.0 \). To prevent the integral from becoming infinite as \( p_c \) approaches zero, the numerical integration must stop (or start) with \( S_e \) less than unity. Alternatively, a constant, \( p_c \), might be added to \( p_c \) (Jeppson, 1970b). If \( p_c \) is added to \( p_o \) only for a portion of the solution where \( 1 \geq S_e \geq S_o \) in which \( S_o \) is a constant ranging from 0.9 to 1.0, Eq. 44 becomes

\[
\frac{dK_r'}{S_e} = \frac{2K_r'}{S_e} + \frac{1}{C'} \left( \frac{S_e}{p_c + p_o} \right)^2,
\]

for \( 1 \geq S_e \geq S_o \) ......... (47)

in which

\[
C' = \int_0^1 \frac{dS_e}{(p_c + p_o)^2}
\]

is a constant depending on the relationship of saturation to capillary pressure.

A semi-log plot is shown in Fig. 6 for the permeability parameter, \( K_r' \), vs. effective saturation, \( S_e \), obtained by solving Eq. 44 numerically for different \( \eta \) values with \( p_b / \gamma = 3.50^\circ \) and \( b = 1.0 \) using the first approach mentioned earlier, that is, starting with small values of \( K_r \) and \( S_e \). The permeability parameter increases as the value of the pore-size distribution index, \( \eta \), increases at a given effective saturation. In other words, at a given saturation, when the pore-size distribution index increases, the resistance of the soils to the flow is reduced and the permeability is also increased.

Curves for the permeability parameter, \( K_r' \), vs. effective saturation, \( S_e \), for different bubbling pressure heads, \( p_b / \gamma \), with \( \eta = 2.50 \) and \( b = 1.0 \) using the first approach were also obtained. Within the accuracy of the numerical methods, no differences in the \( K_r \), \( S_e \) relationships for different values of \( p_b / \gamma \) could be detected and consequently it is concluded that the effect of bubbling pressure upon the \( K_r' \) vs. \( S_e \) curves is very small. In other words, the relationship between relative permeability or the permeability parameter and effective saturation is quite independent of the bubbling pressure. The same conclusion can be drawn from Eq. 43.

Figure 7 shows curves for relative permeability vs. effective saturation for different \( b \) values with \( p_b / \gamma = 3.50^\circ \), and \( \eta = 2.50 \) (using the first approach to obtain the solution curves). The curves on this figure show that for \( S_e \) constant, \( K_r' \) decreases as the value of \( b \) decreases. When the value of \( K_r' \) is scaled to \( K_r \) as defined by Eq. 45, curves similar to those on Fig. 7 would almost coincide with each other. The only parameter which significantly affects the curves for \( K_r' \) vs. \( S_e \) is the pore-size distribution index \( \eta \).

The distribution of effective saturation for each solution can be obtained from the \( r \) and \( z \) coordinates given as computer output by using Eq. 16 and the curves obtained for \( K_r' \) vs. \( S_e \).
Fig. 6. Theoretical curves for permeability parameter, $K_r$, as a function of effective saturation for different $\eta$ values, with $p_b/\gamma = 3.50^\circ$ and $b = 1.0$.

Fig. 7. Theoretical curves for permeability parameter, $K_r$, as a function of effective saturation for different $b$ values, with $p_b/\gamma = 3.50^\circ$ and $\eta = 2.50$. 
The finite difference operator for Eq. 26 can be obtained by
replacing the derivatives by second order central differences and
writing the result as a polynomial in terms of \( r \) at the grid point in
question. The operator for Eq. 26 is

\[
\begin{align*}
\phi_1 (r_o, z_o) &= 2 \left( [K(r_o)]^2 r_o^2 \frac{\partial^2}{\partial r^2} \right) (r_o^2 + r_4^2) r_o^3 \\
&- \left[ \frac{1}{2} R \left( [K(r_o)]^2 \left( \frac{p}{\rho_b} \right)^n \right) \right] \eta (z_o - \phi_o)^{n-1} \left( r_1^2 - r_3^2 \right) (r_2^2 - r_4^2) \\
&+ \frac{1}{4} \left( [K(r_o)]^2 \right) \left( r_2^2 - r_4^2 \right)^2 \left( 2 R^2 \right) r_o^2 - \left[ R^2 (r_1^2 + r_3^2) \right] \eta (z_o - \phi_o)^{n-1} \left( r_1^2 - r_3^2 \right) r_o^2 \\
&+ \frac{1}{4} R^2 (r_1^2 - r_3^2)^2 = 0 \ldots \ldots (48)
\end{align*}
\]

The operator for Eq. 27 is

\[
\begin{align*}
\phi_2 (r_o, z_o) &= \left( [K(r_o)]^2 \right) (z_2 + z_4 - 2z_o) \\
&- \frac{1}{4} \left( [K(r_o)]^2 \right) \left( \frac{p}{\rho_b} \right)^n \eta (z_o - \phi_o)^{n-1} \left( z_2 - z_4 \right)^2 \right) \frac{r_o}{r_o^2} \\
&+ \frac{1}{4} \left( [K(r_o)]^2 \right) \left( r_1 - r_3 \right) \left( \frac{\Delta \psi}{K_o} \right) \left( z_1 - z_3 \right) \left( z_1 + z_3 - 2z_o \right) \\
&+ \frac{1}{4} R^2 (z_1 + z_3 - 2z_o) + \frac{1}{4} R (z_1 - z_3) (z_2 - z_4) = 0 \ldots \ldots (49)
\end{align*}
\]

In Eqs. 48 and 49 the subscript notation has been used to denote
the value of the grid points as follows:

\[
r_o = r_{i,j}, \quad r_1 = r_{i+1,j}, \quad r_2 = r_{i,j+1}, \quad r_3 = r_{i-1,j},
\]

and \( r_4 = r_{i,j-1} \)

Similar notations apply for \( z \). The subscript \( i \) denotes the number of constant \( \phi \) grid lines (i.e. \( \phi_i = (i-1) \Delta \phi_t \)) and \( j \) denotes the number of the constant \( \psi \) grid lines (i.e. \( \psi = (j-1) \Delta \phi_o \)). The ratio of the spacing of streamlines to the transformed equipotential lines is denoted by \( R = \Delta \psi / \Delta \phi_t \).

For axisymmetric infiltration through saturated homogeneous porous media, the operators (Eqs. 48 and 49) reduce to

\[
\begin{align*}
\phi_1 (r_o, z_o) &= \frac{1}{2} R \left( r_2^2 + r_4^2 \right) r_o^3 \\
&+ \left[ R^2 - \frac{1}{8} (r_2^2 - r_4^2) \right] r_o^2 - \frac{1}{2} R^2 (r_1^2 + r_3^2) r_o \\
&\frac{1}{8} R^2 (r_1^2 - r_3^2)^2 = 0 \ldots \ldots . \ldots (50)
\end{align*}
\]

and

\[
\begin{align*}
z_o &= \frac{1}{2} \left[ R^2 (z_1 + z_3) + (z_2 + z_4) r_o^2 + \frac{1}{2} R (z_1 - z_3) (z_2 - z_4) \right] \\
&\ldots \ldots \ldots \ldots \ldots \ldots (51)
\end{align*}
\]

These latter two operators can also be obtained from Eqs. 28 and 29. Only Eq. 51 of the four operators, Eqs. 48 through 51, can be solved explicitly for the variable \( z \) at the grid point \( (r_o, z_o) \). To evaluate \( r \) and \( z \) at the grid point in question in the other operators, it is necessary to solve the operators by implicit methods. Jeppson (1968b, 1968c) used the Newton-Raphson iterative method for solving implicit functions for inner iteration and successive over relaxation for the outer iteration to settle the values of the variables throughout the flow field. The same approach has been used to solve Eqs. 48 and 49 simultaneously throughout the entire flow field. The Newton-Raphson method (Ralston, 1965) for the inner iteration can be expressed as

\[
\begin{align*}
\begin{bmatrix}
 r_o^{(n+1)} \\
z_o^{(n+1)}
\end{bmatrix} &=
\begin{bmatrix}
 r_o^{(n)} \\
z_o^{(n)}
\end{bmatrix} - \begin{bmatrix}
 \frac{\partial f_1}{\partial r_o}^{(n)} & \frac{\partial f_1}{\partial z_o}^{(n)} \\
-\frac{\partial f_2}{\partial r_o}^{(n)} & \frac{\partial f_2}{\partial z_o}^{(n)}
\end{bmatrix} \begin{bmatrix}
 f_1^{(n)} \\
f_2^{(n)}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
 f_1^{(n)} \\
f_2^{(n)}
\end{bmatrix} &=
\begin{bmatrix}
 \frac{\partial f_1}{\partial r_o}^{(n)} & \frac{\partial f_1}{\partial z_o}^{(n)} \\
-\frac{\partial f_2}{\partial r_o}^{(n)} & \frac{\partial f_2}{\partial z_o}^{(n)}
\end{bmatrix}^{-1} \begin{bmatrix}
 r_o^{(n+1)} \\
z_o^{(n+1)}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
 \frac{\partial f_1}{\partial r_o}^{(n)} & \frac{\partial f_1}{\partial z_o}^{(n)} \\
-\frac{\partial f_2}{\partial r_o}^{(n)} & \frac{\partial f_2}{\partial z_o}^{(n)}
\end{bmatrix}^{-1} \begin{bmatrix}
 r_o^{(n+1)} \\
z_o^{(n+1)}
\end{bmatrix} =
\begin{bmatrix}
 f_1^{(n)} \\
f_2^{(n)}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
 f_1^{(n)} \\
f_2^{(n)}
\end{bmatrix} &=
\begin{bmatrix}
 \frac{\partial f_1}{\partial r_o}^{(n)} & \frac{\partial f_1}{\partial z_o}^{(n)} \\
-\frac{\partial f_2}{\partial r_o}^{(n)} & \frac{\partial f_2}{\partial z_o}^{(n)}
\end{bmatrix}^{-1} \begin{bmatrix}
 r_o^{(n+1)} \\
z_o^{(n+1)}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
 f_1^{(n)} \\
f_2^{(n)}
\end{bmatrix} &=
\begin{bmatrix}
 \frac{\partial f_1}{\partial r_o}^{(n)} & \frac{\partial f_1}{\partial z_o}^{(n)} \\
-\frac{\partial f_2}{\partial r_o}^{(n)} & \frac{\partial f_2}{\partial z_o}^{(n)}
\end{bmatrix}^{-1} \begin{bmatrix}
 r_o^{(n+1)} \\
z_o^{(n+1)}
\end{bmatrix}
\end{align*}
\]
where $D$ is the Jacobian determinant:

$$
D = \begin{vmatrix}
\frac{\partial f_1(n)}{\partial r_0} & \frac{\partial f_1(n)}{\partial z_0} \\
\frac{\partial f_2(n)}{\partial r_0} & \frac{\partial f_2(n)}{\partial z_0}
\end{vmatrix}
$$

and the superscript $(n)$ refers to the iteration number. The successive over relaxation method (Forsythe and Wasow, 1960) for the outer iteration can be expressed as

$$
\begin{bmatrix}
r_0^{(m+1)} \\
\omega z_0^{(m+1)}
\end{bmatrix} = \begin{bmatrix}
r_0^{(m)} \\
\omega z_0^{(m)}
\end{bmatrix} - \omega \begin{bmatrix}
\frac{\partial f_1}{\partial r_0} \\
\frac{\partial f_2}{\partial z_0}
\end{bmatrix}
$$

where $r_0$ and $z_0$ are the new values of the variables $r$ and $z$ obtained from Eq. 52, $\omega$ is the over relaxation parameter with a value between 1 and 2, and the superscript $(m)$ refers to the outer iteration number.

In the case of Eq. 50 which involves only one unknown, a simple form of the Newton-Raphson iteration for nonlinear equations can be used (Kunz, 1957)

$$
r_0^{(n+1)} = r_0^{(n)} - \frac{f(r_0^{(n)})}{d(f(r_0^{(n)})/dr_0)}
$$

where the superscript $(n)$ refers to the iteration number and $f(r_0^{(n)})$ is the function given by Eq. 50. The value of $f(r_0^{(n)})$ will be zero for the correct value of $r_0$.

Expressions for the derivatives in Eq. 52 can be obtained by differentiating Eqs. 48 and 49 with respect to $r_0$ and $z_0$ respectively, as given below

$$
\frac{\partial f_1}{\partial r_0} = \left[ \frac{\eta(z_0, \phi_0)}{\eta(z_0, \phi_0) - \phi_0} \right] \left[ \frac{\eta(z_0, \phi_0)}{\eta(z_0, \phi_0) - \phi_0} \right] r_0 - \frac{1}{2} R(z_1, z_3) - 2 \frac{R(z_1, z_3)}{K_0} \frac{\Delta \psi}{K_0} (z_1 - z_3)
$$

where $\eta$ is the Jacobian determinant.

Finite difference operators for boundary conditions

Values for the variable $r$ must be determined along the boundaries of the source surface, the ground surface and the water table as part of the solution by satisfying appropriate finite difference approximations to the boundary conditions already described. Likewise, values for the variable $z$ must be established along the axis of symmetry and the cylindrical outer boundary.

**Axis of symmetry $\Omega$.** The finite difference approximation to the normal derivative condition $\partial z/\partial \psi = 0$ obtained by setting the value of $z$ at the nonexistent grid point outside the boundary equal to that of its reflection within the flow field and using the regular finite difference operator cannot be used at this boundary since the transformation to the $\Phi$ plane is not valid along this line. Consequently the solution near the axis is obtained on an additional row of grid points adjacent to the axis spaced one-tenth of the regular spacing from the axis (Jeppson, 1968b). The values of $z$ at these added points are denoted by $z_{\Omega}$. The derivatives in Eq. 27 with respect to $\psi$ at these added points can be obtained by using Taylor's series over an irregular grid network adjacent to the axis of symmetry as shown in Fig. 8 in which the dashed lines denoted by 4
and c' are obtained from the reflection of grid points 2 and c about the axis of symmetry. The finite difference expressions for first and second derivatives obtained from Taylor's series (see Jeppson, 1968b, p. 1283) are

\[
\left( \frac{\partial z}{\partial \psi} \right)_c = 0.2020202 \frac{z_{i,2} - z_{c,i}}{\Delta \psi} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldOTS
\]
The values of \( z \) on the axis of symmetry are obtained through the use of the finite difference expression for the condition that \( \partial z/\partial \psi = 0 \) in terms of the values \( z_{i1}, z_{e1}\) and \( z_{i2}\). A third degree polynomial approximation for \( z_{i1} \) gives

\[ z_{i1} = 1.01010101 z_{e1} - 0.0101010101 z_{i2}. \] (65)

Source surface ① - ③. A two step operation is needed to evaluate Eq. 35 for determining the boundary value of \( r \). The first step is to evaluate the derivatives \( \partial z/\partial \Phi \), and the second step is to perform numerical integration of Eq. 35. The derivatives have been evaluated at each grid point along the boundary by differentiating the backward Gregory-Newton interpolation formula (Wylie, 1960) truncating fourth order terms. After expressing the results in terms of the values of \( z \) at the interior points

\[
\left( \frac{\partial z}{\partial \Phi} \right)_{t NH,j} = \frac{1}{\Delta \Phi} \left[ \frac{11}{6} z_{NH,j} - 3 z_{NH,j} + \frac{3}{2} z_{NH2,j} - \frac{1}{3} z_{NH3,j} \right] \ldots \ldots \ldots . (66)
\]

in which \( NH \) is the total number of equipotential lines specified for the entire region, and this boundary is specified as the \( NH^{th} \) equipotential line, and \( NH_1=NH-1, NH_2=NH-2, \) and \( NH_3=NH-3 \). The second step for carrying out the numerical integration is based on central differences except at the end grid points of the boundary. The symmetric Bessel’s interpolation formula is used to evaluate the integration in terms of two pairs of derivatives on each side of the grid point over which the integration is to be performed. Eq. 35 becomes,

\[
r_{NH,j+1} - r_{NH,j} = \frac{2 \Delta \psi}{K_r (r_{NH,j+1} + r_{NH,j})} \left[ \frac{13}{24} \left( \frac{\partial z}{\partial \Phi} \right)_{t NH,j} + \frac{1}{24} \left( \frac{\partial z}{\partial \Phi} \right)_{t NH,j+1} \right] + \frac{1}{24} \left( \frac{\partial z}{\partial \Phi} \right)_{t NH,j+2} \ldots \ldots \ldots \ldots \ldots \ldots . (67)
\]

Upon multiplying both sides of Eq. 66 by \( (r_{NH,j+1} + r_{NH,j}) \), the value of \( r_{NH,j+1} \) can be obtained when the value of \( r_{NH,j} \) is known. Consequently, the integration of Eq. 67 begins at the axis of symmetry and proceeds toward the outer edge of the boundary. Since the values of the derivatives are changing until the entire field is settled, the values of \( r \) on the boundary must be recomputed during each outer iteration.

Ground surface ① - ③. A finite difference operator for \( r \) on this boundary is obtained by replacing the derivative \( \partial r/\partial \psi = 0 \) with second order central differences, combining the results with Eq. 48, and eliminating the value of \( r \) which lies outside the boundary at a nonexistent grid point. The operator is the same as Eq. 48 with \( r_g \) replacing \( r_s \). The two step operation described in the last section can also be applied to evaluate Eq. 36 for determining the value of \( r \) along this boundary.

Vertical outer boundary ② - ④. The normal derivative condition \( \partial z/\partial \Phi = 0 \) is replaced by second order central differences and the result is combined with Eq. 49 to eliminate the value of \( z \) which lies outside the boundary at a nonexistent grid point. The resulting operator is the same as Eq. 49 with \( z_g \) replacing \( z_s \). Caution is necessary so that a good initial value of \( z \) is supplied when this is solved by the Newton-Raphson method. Upon inspection of the function \( f_z(r_s, z_s) \) in Eq. 48, the slope of the function at the point where the solution exists was found to be negative but changed to positive with a small decrease in \( z \). Before the iteration begins, the slope of the function with given initial \( z \) is examined. If the slope is positive, the initial value of \( z \) is increased until the slope changes to negative. Subsequently, the regular iterative procedure is followed to obtain the solution.

Water table ② - ③. To satisfy the normal condition \( \partial r/\partial \Phi = 0 \) along the boundary a parabola is fit through three consecutive points on and adjacent to the boundary for each streamline as it enters the water table. The parabola is symmetric about the boundary. Two interior grid points on the streamline adjacent to the boundary are used to determine the parabola which passes through them. The radius at which this parabola intersects with the water table gives the value of \( r \) on the boundary.

Determination of the ratio \( R \) of horizontal to vertical grid spacing

The ratio of spacing between adjacent streamlines to the spacing between adjacent transformed equipotential lines is \( R = \Delta \psi/\Delta \Phi \). The number of \( \Phi \) constant lines, the number of \( \psi \) constant lines used in the finite difference solution, and the total head on the source surface are specified as input parameters in defining a problem to be solved. The value of the ratio \( \Delta \psi/\Delta \Phi \) is adjusted repeatedly during the iterative process to ensure that the boundary conditions of the specified problem are satisfied. To adjust \( R \), an expression for \( R \) in terms of \( \psi \) and \( \Phi \), or \( (\Phi, \psi) \), (where the subscripts \( s \) and \( n \) may be considered as curvilinear coordinates) is needed. A relationship commonly used for the general two-dimensional case can be expressed as

\[
\frac{\Delta \psi}{\Delta n} = K \frac{\Delta \phi}{\Delta s} \quad \text{or} \quad \frac{\Delta \phi}{\Delta s} = -K_r \frac{\Delta n}{\Delta \phi}, \quad \ldots \ldots \ldots \ldots \ldots . \quad (68)
\]

where \( s \) and \( n \) are considered as orthogonal curvilinear coordinates commonly called natural coordinates; \( s \) is taken along any streamline and \( n \) is normal to \( s \). The spacing \( \Delta s \) is the arc length along a \( \psi \) = constant streamline between two consecutive \( \Phi \) constant lines, and \( \Delta n \) is the arc length along a \( \Phi \) = constant line between adjacent streamlines. For the axisymmetric case, consider a set of orthogonal unit vectors \( e_s \) and \( e_n \) in which the subscript \( s \) and \( n \) denote the orthogonal curvilinear coordinates as shown in Fig. 9. Let \( (\alpha_s, \alpha_n) \) and \( (\alpha_n', \alpha_n') \) be the direction cosines of the unit vectors \( e_s \) and \( e_n' \) respectively. Then, since \( e_s \) is a unit vector in a normal direction obtained from \( e_s \) by counterclockwise rotation of 90°,
the direction cosines $a_1, a_2, a_1', a_2'$ can all be obtained from the angle between the horizontal and the direction of the streamline or

$$a_1 = \cos \theta; \quad a_2 = \sin \theta$$

and

$$a_1' = -\sin \theta; \quad a_2' = \cos \theta$$

From Eq. 69 it is obvious that

$$a_1' = -a_2$$

and

$$a_2' = a_1$$

Let $e_r$ and $e_z$ be the unit vectors in the $r$ and $z$ directions, respectively. Then the gradient of $\phi$, can be written as

$$\text{grad} \phi = \nabla \phi = \frac{\partial \phi}{\partial r} e_r + \frac{\partial \phi}{\partial z} e_z$$

The components of the directional derivatives of $\phi$ and $\psi$ in the directions of $e_n$ and $e_n'$ are

$$\frac{\partial \phi}{\partial n} = \psi \cdot e_n = \alpha_1 \frac{\partial \phi}{\partial r} + \alpha_2 \frac{\partial \phi}{\partial z}$$

and

$$\frac{\partial \psi}{\partial n} = \nabla \psi \cdot e_n = \alpha_1' \frac{\partial \psi}{\partial r} + \alpha_2' \frac{\partial \psi}{\partial z}$$

Substituting Eqs. 5, 6, and 70 into Eqs. 72 through 75 and rearranging the results gives

$$\begin{align*}
\frac{\partial \psi}{\partial s} &= -K_r \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial s} = -K_r \frac{\partial \phi}{\partial s} \\
\frac{\partial \phi}{\partial s} &= K_r \frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial s} \\
\frac{\partial \psi}{\partial s} &= -K_r \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial s}
\end{align*}$$

An approximation of Eq. 77 is

$$\Delta \psi \approx -K_r \frac{\Delta \phi}{\Delta s}$$

Rearranging Eq. 78 and solving for $R$ gives

$$R = \frac{\Delta \psi}{\Delta \phi} = -K_r \frac{\Delta \phi}{\Delta s}$$

The minus sign in Eq. 79 is due to the fact that $\phi$ decreases along the positive $s$ direction.

The arc lengths between the streamlines for each rectangular mesh of the orthogonal grid network $\Delta n_1$, and $\Delta n_2$ shown in Fig. 10 are approximated by straight lines

$$\Delta n_1 (i) = \left[ (r_{i,j+1} - r_{i,j})^2 + (z_{i,j+1} - z_{i,j})^2 \right]^{1/2}$$

and

$$\Delta n_2 (i) = \left[ (r_{i+1,j+1} - r_{i,j+1})^2 + (z_{i+1,j+1} - z_{i,j+1})^2 \right]^{1/2}$$
in which the parenthesized "I" denotes the Ith rectangular mesh
from the lower boundary (water table), and the arc lengths between
the equipotential lines $\Delta s_1$ and $\Delta s_2$ are approximated by

$$\Delta s_1(I) = \left[ (r_{i+1,j} - r_{i,j})^2 + (z_{i+1,j} - z_{i,j})^2 \right]^{1/2}$$

and

$$\Delta s_2(I) = \left[ (r_{i+1,j+1} - r_{i,j+1})^2 + (z_{i+1,j+1} - z_{i,j+1})^2 \right]^{1/2}$$

Values for $K_{rm}$ and $r_m$ in Eq. 79 are taken as the average permeability
parameter and average radial distance from the axis of symmetry for
the four grid points forming each mesh. These averages denoted by
$K_{rm}$ and $r_m$ respectively are given by the following equations

$$K_{rm} = \frac{1}{4} \left( K_{r_{i,j}} + K_{r_{i,j+1}} + K_{r_{i+1,j}} + K_{r_{i+1,j+1}} \right)$$

in which $K_{r_i}$'s can be obtained from Eq. 17, and

$$r_m = \frac{1}{4} \left( r_{i,j} + r_{i,j+1} + r_{i+1,j} + r_{i+1,j+1} \right)$$

Thus the ratio $\Delta \psi/\Delta \phi$ for each rectangular mesh is given by

$$R(I) = \frac{K_{rm}(I) r_m(I)}{\frac{\Delta n_1(I) + \Delta n_2(I)}{\Delta s_1(I) + \Delta s_2(I)}}$$

Finally the ratio $R$ is obtained by averaging the $R(I)$'s through two
selected consecutive streamlines

$$R = \frac{1}{NH_1} \sum_{I=1}^{NH_1} R(I)$$

or

$$R = \frac{1}{NH_1} \sum_{I=1}^{NH_1} K_{rm}(I) \left[ \frac{\Delta n_1(I) + \Delta n_2(I)}{\Delta s_1(I) + \Delta s_2(I)} \right]$$

in which $NH_1 = NH - 1$; $NH$ is the number of equipotential lines
specified as the input data parameter. The $\psi = constant$ lines
between which the ratios $R(I)$ are calculated are removed from the
axis of symmetry to avoid being affected by the line singularity at
the axis of symmetry which will be discussed in a later section.

**Determination of scaled volume flux rate**

In order to solve Eqs. 48 and 49 simultaneously as shown in
Eq. 52, the scaled volume flux rate $\Delta \psi/K_o$ must be known. The
rate of flow between any two consecutive stream surfaces $\Delta Q$ is
given by

$$\Delta Q = K \frac{\Delta \phi}{\Delta s} \Delta A$$

in which $A$ is the annular area perpendicular to the direction of flow
and surrounded by two consecutive stream surfaces. An approxima­
tion to this area using the average radii and normal increments $\Delta n$ is

$$\Delta A = \frac{\pi}{4} \left( r_{i,j} + r_{i,j+1} + r_{i+1,j} + r_{i+1,j+1} \right) (\Delta n_1 + \Delta n_2)$$

Substituting Eq. 89 into Eq. 88 gives

$$\Delta Q(I) = \frac{\Delta \phi}{\Delta s} \Delta A \left( \frac{\Delta n_1(I) + \Delta n_2(I)}{\Delta s_1(I) + \Delta s_2(I)} \right)$$

and the average value of $\Delta Q(I)$ is

$$\Delta Q = \frac{1}{NH_1} \sum_{I=1}^{NH_1} \Delta Q(I)$$
Since for three-dimensional axisymmetric flow $\Delta Q = 2\pi \Delta \psi$, 

$$
\frac{\Delta \psi}{K_0} = \frac{\Delta \phi}{NH_1} \sum_{I=1}^{NH_1} \frac{K_{r_m}(I)}{r_m(I)} \left[ \frac{\Delta n_1(I) + \Delta n_2(I)}{\Delta s_1(I) + \Delta s_2(I)} \right]
$$

and from Eq. 87 

$$
\frac{\Delta \psi}{K_0} = \Delta \phi R
$$

in which $\Delta \phi = H/NH_1$, ($H$ is the total fluid head at the surface inside the rainfall simulator and is specified as an input parameter). Finally Eq. 93 becomes 

$$
\frac{\Delta \psi}{K_0} = \frac{H}{NH_1} R
$$

and the scaled volume flux rate $\Delta \psi/K_0$ can be obtained from the ratio $R = \Delta \psi/\Delta \phi$. 

**Subdivided grid network**

In parts of the region of the problem values of $\Delta r$ and $\Delta z$ are greater than in other regions. Better accuracy can be expected from the finite difference solutions in regions directly underneath the source circle where $\Delta r$ and $\Delta z$ are the smallest. At the region closer to the outer boundary where the moisture content is very low, $\Delta r$ and $\Delta z$ will be large and poorer precision can be expected in finite difference solutions. While this outside region is not of major interest, the poor approximations there will affect the accuracy of the finite difference solution in other regions.

However, if the spacing of the grid network is made too small, unnecessarily large amounts of computer time are required for the solution. The convergence rate of the outer iteration of the system also decreases with increasing number of grid points. Better results are obtained by subdividing the grid network over part of the region.

In the present analysis, a finer grid network is used for the region enclosed by the last two $\psi =$ constant lines and the source surface and water table boundaries. The spacing of the subdivided grid network is one-quarter of the regular spacing. The values of $r$ and $z$ for the grid points in the subdivided network are related to the values at the regular grid points as shown in Fig. 11. During each iteration through the regular grid network the values of $r$ and $z$ at these grid points represented by $i = 1,2,3, ..., NH$, $j = NS-2$ and $i = 1,2,3, ..., NH$, $j = NS-1$ (where $NH$ and NS represent the values of $i$ and $j$ for the final $\Phi_t =$ constant and $\psi =$ constant lines respectively) are used to determine the values of $r$ and $z$ for the grid.

![Subdivided grid network at one-half and one-quarter of regular spacing along the upper boundary of the $\Phi_t \psi$ plane.](image)

Fig. 11. Subdivided grid network at one-half and one-quarter of regular spacing along the upper boundary of the $\Phi_t \psi$ plane.
points in the subdivided grid network denoted by \( i_1 = 1, 2, 3, \ldots, NP \), \( j_1 = 1 \) (where \( NP = 2NH - 1 \)) with half of the regular spacing. A special operator is required to relate the variables \( r \) and \( z \) at the point \( 0 \) shown in Fig. 11 with half of the regular spacing to the variables at regular grid points 1, 2, 3, and 4 and grid points 5 and 6 on the subdivided grid network with half of the regular spacing (see Fig. 11). Suppose a function \( f = f(x, \psi) \) and a rectangular mesh is subdivided into halves of the original spacing as shown in Fig. 12, the values of \( f \) at any two points denoted by \((x_1, \psi)\) and \((x_2, \psi + m)\) in which \( d = \frac{1}{2} \Delta x \) and \( m = \frac{1}{2} \Delta \psi \) are related by the Taylor's series

\[
\begin{align*}
\Phi(x + d, \psi + m) &= f(x_1, \psi) + \left( \frac{\partial}{\partial x} + m \frac{\partial}{\partial \psi} \right) f(x_1, \psi) + \frac{1}{2!} \left( \frac{\partial^2}{\partial x^2} + m \frac{\partial^2}{\partial \psi^2} \right) f(x_1, \psi) \\
&+ \cdots \\
&+ \frac{1}{(n-1)!} \left( \frac{\partial^n}{\partial x^n} + m \frac{\partial^n}{\partial \psi^n} \right) f(x_1, \psi) + R_n. \quad (95)
\end{align*}
\]

where \( R_n \) is the remainder term and is given by

\[
R_n = \frac{1}{n!} \left( \frac{\partial}{\partial x} + m \frac{\partial}{\partial \psi} \right) f(x, \psi) + \cdots. \quad (96)
\]

where \( 0 < \xi < 1 \).

Equation 96 can be rewritten as

\[
R_n = 0 \left( \frac{d}{d} + m \right)^n. \quad (97)
\]

Taylor's series for \( f_5 \) and \( f_6 \) about \( f_0 \) at the central point 0 can also be written to obtain a finite difference expression for the second partial derivative of \( f \) with respect to \( \psi \) at the central point.

\[
\left( \frac{\partial^2 f}{\partial \psi^2} \right) f_0 = \frac{1}{2} \left( f_1 + f_2 + f_3 + f_4 - 4 f_0 \right). \quad (99)
\]

Taylor's series for \( f_5 \) and \( f_6 \) about \( f_0 \) at the central point 0 can also be written to obtain a finite difference expression for the second partial derivative of \( f \) with respect to \( \phi \) at the central point.
By multiplying Eqs. 99 and 100 by \( K r^2 f_0^2 \) and \((1 - K r^2 f_0^2 \mathcal{D}^2)\), respectively, and combining the resulting equations, a finite difference expression for the terms enclosed by the first parenthesis in the first governing equation Eq. 26 at the central point is obtained.

\[
\frac{\partial^2 f}{\partial t^2} + \frac{1}{m} \left[ \frac{1}{2} \left( f_1 + f_2 + f_3 + f_4 \right) - f_5 - f_6 \right] \quad (101)
\]

Finite difference expressions for the terms \( \frac{\partial f}{\partial t} \) and \( \frac{\partial^2 f}{\partial \psi^2} \) at the central point can also be obtained from Eqs. 98 a through 98d

\[
\left( \frac{\partial f}{\partial t} \right)_o = \frac{1}{4d} \left[ (f_2 + f_3) - (f_1 + f_4) \right] \quad (102)
\]

and

\[
\left( \frac{\partial^2 f}{\partial \psi^2} \right)_o = \frac{1}{4m} \left[ (f_1 + f_2) - (f_3 + f_4) \right] \quad (103)
\]

Substituting Eqs. 100, 101, 102, and 103 into Eq. 26 with proper substitution of \( r \) and \( z \) for \( f \), a finite difference operator for Eq. 26 at the central point is obtained. The operator will be evaluated in terms of the values of \( r \) and \( z \) at the surrounding grid points 1, 2, 3, 4, 5, and 6 with half of the regular spacing. (See Fig. 12.)

\[
F_1 \left( r_o, z_o \right) = 16 \left( K r_o^2 \right)^2 \left[ \frac{1}{2} \left( r_1 + r_2 + r_3 + r_4 \right) - (r_5 + r_6) \right] \left( r_o^2 \right)^3 + 32 \left( K r_o^2 \right)^2 \left( r_1 + r_2 + r_3 + r_4 \right)^2 r_o^2 + 16 R \left( r_5 + r_6 \right)
- 4 R \left( K r_o^2 \right)^2 \left[ \frac{m}{K_o^2} \right] \left( r_2 + r_3 - r_1 - r_4 \right) = 0 \quad (104)
\]

By making the same substitutions, a similar operator for Eq. 27 is also obtained.

\[
F_2 \left( r_o, z_o \right) = 16 \left( K r_o^2 \right)^2 r_o^2 \left[ \frac{1}{2} \left( z_1 + z_2 + z_3 + z_4 \right) - (z_5 + z_6) \right] + 16 R \left( z_5 + z_6 - 2z_o \right)
- \left( K r_o \right)^2 \left( \frac{m}{K_o^2} \right) \left( z_o - \phi_o \right)^{n-1} \left[ (z_1 + z_2 - z_3 - z_4)^2 \right] + 4 R \left( K r_o^2 \right)^2 \left( \frac{m}{K_o^2} \right) \left( z_o - \phi_o \right)^{n-1} \left[ \frac{1}{4} R \left( z_2 + z_3 - z_1 - z_4 \right) \right]
- \left( \frac{m}{K_o^2} \right) \left[ (z_2 + z_3 - z_1 - z_4) \right]^2
+ 2 R \left( K r_o^2 \right)^2 \left( z_1 + z_2 - z_3 - z_4 \right)^2 \left( z_o + z_3 - z_1 - z_4 \right) = 0 \quad (105)
\]

Because the values of \( r \) and \( z \) at point 7 of the rectangular mesh shown in Fig. 12 are not available in this transition zone from the regular grid network to the subdivided grid network with one-half of the regular spacing, the regular five-point operators Eqs. 48 and 49 cannot be used. Instead, the new values of \( r \) and \( z \) at point 0 are obtained by solving the new seven-point operators Eqs. 104 and 105 simultaneously. The Newton-Raphson iterative method is used in this inner iteration shown by Eq. 52, and Eq. 54 is then applied for outer iteration. The expressions of the derivatives to be used in Eq. 52 are

\[
\frac{\partial F}{\partial r} = 48 \left( K r_o^3 \right)^2 \left( r_o^2 \right)^2 \left( r_1 + r_2 + r_3 + r_4 \right)^2 \left( r_5 + r_6 \right) = 0 \quad (106)
\]

\[
\frac{\partial F}{\partial z} = 2 \left( K r_o^3 \right)^2 \left( r_o^2 \right)^2 \left( z_o - \phi_o \right)^{n-1} \left[ \frac{m}{K_o^2} \right] \left( r_2 + r_3 - r_1 - r_4 \right) \quad (107)
\]
In order to iterate over the transition zone from the regular grid network to the subdivided grid network, the values of \( r \) and \( z \) on the boundary of the subdivided grid network (i.e., the line denoted by \( (j_1 = 1) \) in Fig. 11) are obtained from either the seven-point operators or the five-point operators depending on whether the grid point in question is inside or on the boundary of regular rectangular mesh. Along the boundary denoted by \( j_1 = 1 \), the values for \( r \) and \( z \) at grid points which are located at the centers of the regular rectangular meshes (i.e., the grid points denoted by \( (i_1 = 2, 4, \ldots, NH-1, j_1 = 1) \) in Fig. 11) are obtained from the seven-point operators, Eqs. 104 and 105. Values at the remaining grid points on the boundary \( G_1 = 1 \) are obtained by solving the original five-point operators with points 1, 0, 4, and 7 as surrounding points with the point 5 as the point in question. The order of the iteration over this boundary \( G_1 = 1 \) starts with \( i_1 = 2 \) and proceeds to \( i_1 = MP-1 \). The seven-point and five-point operators are used in this last pass alternatively.

The values of \( r \) and \( z \) at the soil surface and water table have to be approximated separately to satisfy their respective boundary conditions. The operator for the point at the soil surface denoted by \( (i_1 = NP, j_1 = 1) \) is obtained by noting that the value of \( z \) is constant at this boundary being equal to the depth of the water table. Since \( r \) changes by only a small amount along the surface, the value of \( r \) at \( (i_1 = MP, j_1 = 1) \) can be taken as the average of the values of \( r \) of the two regular grid points on the surface denoted by \( (i=NH, j=NS-1) \) and \( (i=NH, j=NS-2) \) on the regular grid network system. To evaluate the value of \( r \) at the point denoted by \( (i_1 = 1, j_1 = 1) \) on the water table, the condition that the streamlines enter the water table at right angle (i.e., \( \theta = 0 \)) must be satisfied. The value of \( r \) can be determined by substituting this normal derivative condition into the first governing equation, Eq. 48.

After computing values of \( r \) and \( z \) on the grid network with one-half of the regular spacing values of \( r \) and \( z \) on the grid network with one-quarter of the regular spacing denoted by \( i_2 = 1, 2, 3, \ldots, MP, j_2 = 1 \), where \( MP = 4NH-3 \) are computed in a similar manner.

After solving for the values of \( r \) and \( z \) on the grid network with one-quarter of the regular spacing by the iterative procedure described in a previous section, values of \( r \) and \( z \) at the regular grid network covering the subdivided region are equated to the values of \( r \) and \( z \) at the corresponding grid points with one-quarter of the regular spacing.

Adjustment of the number of equipotential lines which intersect the vertical outer boundary

The number of equipotential lines which intersect the vertical outer boundary is denoted by \( M_4 \). If \( M_4 \) is equal to or greater than two, one of these equipotential lines is assumed to pass through the intersection of the vertical outer boundary and the ground surface. The criterion used to determine the value of \( M_4 \) is based upon the orthogonality of streamline with equipotential line. Orthogonality can be checked at each grid point by computing the angle between equipotential line and streamline at the point. If the angle is \( 90^\circ \), the condition is satisfied; otherwise, \( M_4 \) needs to be adjusted. For example, the angle \( \beta \) is checked at the subdivided grid point where the third equipotential line from the water table and the third streamline from the vertical outer boundary intersect. If the angle is \( 90^\circ \pm 1^\circ \), the orthogonality condition is assumed to be satisfied, and the value of \( M_4 \) is not changed. If the angle is greater than \( 91^\circ \), \( M_4 \) is decreased by one, and if the angle is less than \( 89^\circ \), \( M_4 \) is decreased by one. The adjustments made for the corner point and the surrounding grid points are shown in Fig. 13.
SOLUTION AND DISCUSSION OF RESULTS

Nature of solution and results obtained from solution

Several solutions have been obtained for the steady-state flow system resulting from moisture applied over a circular area at the ground surface. By varying the values of the parameters in Eq. 16, the effect of each parameter on the resulting flow pattern is studied. The solution, which is obtained through the inverse formulation described earlier, consists of values of \( r \) and \( z \) coordinates in the radial and vertical directions respectively at each grid point in the finite difference solution. Consequently, the solution output obtained can be easily plotted to form a flownet to give a general idea about the flow pattern. Floawnets have been obtained from the Gerber plotter at the University of Utah computer center which essentially draws a straight line between adjacent coordinates. A parabolic interpolation formula was included in the FORTRAN plotting subroutine to plot smooth curves in the region near the water table where the spacings of equipotential lines are too large to obtain a good flownet by simply drawing a straight line between two adjacent grid points.

In this section a number of flownets (Figs. 14 through 23) are given which represent the solutions obtained for varying soil parameters. All of the solutions have specified that \( p_c/\gamma = 1.0 \) feet over the source circle. These solutions can be divided into three groups. The solutions in the first group (Figs. 14 through 18) have common \( p_b \) and \( \eta \) values (\( p_b/\gamma = 3.5 \) feet and \( \eta = 2.50 \)) but varying \( b \) values ranging from 1.0 to 3.0. In the second group, (Figs. 18 through 22) the solutions have common values for \( b \) and \( \eta \) (\( b = 1.0 \) and \( \eta = 2.50 \)) and different bubbling pressure heads ranging from 1.5 feet and 3.5 feet. In the last group, (Figs. 18 and 23) the solutions have common \( b \) and \( p_b \) values (\( b = 1.0 \) and \( p_b/\gamma = 3.5 \) feet) and the pore size distribution index \( \eta \) ranging from 2.5 to 3.5 were specified to obtain solutions.

The portion of the flownets with dashed lines are the regions where the spacings on the \( \Phi \psi \) plane have been reduced to one-quarter of the regular spacing to increase the accuracy of the numerical approximation in these regions. Consequently one-quarter as much flux moves between dashed lines as between adjacent solid lines.

For each flownet obtained, the orthogonality condition must be satisfied at each grid point. For the axisymmetric isotropic homogeneous porous medium, the orthogonality is obtained in the following manner. Along \( \Phi = \) constant lines

\[
\frac{\partial \Phi}{\partial r} \, dr + \frac{\partial \Phi}{\partial z} \, dz = 0
\]

or

\[
\frac{d\Phi}{dz} = -\frac{\partial \Phi / \partial z}{\partial \Phi / \partial r}
\]

Alone \( \psi = \) constant lines

\[
\frac{\partial \psi}{\partial r} \, dr + \frac{\partial \psi}{\partial z} \, dz = 0
\]

or

\[
\frac{d\psi}{dz} = -\frac{\partial \psi / \partial z}{\partial \psi / \partial r}
\]

After substituting Eqs. 5 and 6 into Eqs. 110 and 111 and multiplying two equations together, the following orthogonality condition which is independent of \( K = K(\rho) \) can be obtained for saturated and partially saturated axisymmetric isotropic homogeneous media.

\[
\frac{d\psi}{dz} \frac{d\phi}{\Phi} = -1
\]

Coping with multiple roots of finite difference operators

The finite difference operators, Eqs. 48 and 49, are solved simultaneously by the Newton-Raphson method at each grid point. At most of the interior grid points, rough initial estimates for \( r \) and \( z \) are adequate to insure that the Newton-Raphson method will select the physically correct roots from the operators. At some grid points, particularly those close to the vertical outer boundary and the water table, this is not the case. The multiple possible roots resulting from simultaneously solving Eqs. 48 and 49 are extremely difficult to analyze. For instance, in obtaining solution for \( \eta = 2.50 \), \( p_b/\gamma = 1.50^1 \), and \( b = 1.0 \), it was discovered that a slight variation of the initial values of \( z \) at the grid points in the vicinity of the vertical outer boundary and the water table lead to an alternative solution. Tables 1 and 2 demonstrate the sensitivity of the solution to the initialization. In Tables 1 and 2, \( f_1 \) is the operator Eq. 48, \( f_2 \) is the operator Eq. 49, \( n \) is the number of the Newton-Raphson iteration, and \( r_o \) and \( z_o \) are the initial values of \( r \) and \( z \) used in obtaining the solution. In this case, starting with \( r_o = 25.906 \) and \( z_o = 5.0 \) leads to a final \( r = 25.906 \) and \( z = 5.295 \) whereas starting with \( r_o = 25.906 \) and \( z_o = 4.5 \) results in the solution \( r = 25.894 \) and \( z = 4.115 \). When two roots are relatively close to each other as shown in the example, it is difficult to pick up the correct solution as required by the physical system. Only after the iterative solution has been obtained, and the orthogonality condition is not satisfied, is it apparent that an incorrect root was selected at some stage during the solution process. The best approach seems to be to modify the initialization in another attempt at a solution in hopes that the correct roots to the functional operators will be supplied by the Newton-Raphson iteration throughout the entire solution process.
Fig. 14. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 3.50$, $b = 3.00$, $\eta = 2.50$, and $p_c/\gamma = 1.0$ over the source circle.

Fig. 15. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 3.50$, $b = 2.50$, $\eta = 2.50$, and $p_c/\gamma = 1.0$ over the source circle.

Fig. 16. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 3.50$, $b = 2.00$, $\eta = 2.50$, and $p_c/\gamma = 1.0$ over the source circle.
Fig. 17. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 3.50', b = 1.50$, $\eta = 2.50$, and $p_c/\gamma = 1.0'$ over the source circle.

Fig. 18. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 3.00', b = 1.00$, $\eta = 2.50$, and $p_c/\gamma = 1.0'$ over the source circle.

Fig. 19. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 3.00', b = 1.00$, $\eta = 2.50$, and $p_c/\gamma = 1.0'$ over the source circle.
Fig. 20. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 2.50$, $b = 1.00$, $\eta = 2.50$, and $p_c/\gamma = 1.0$ over the source circle.

Fig. 21. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 2.00$, $b = 1.00$, $\eta = 2.50$, and $p_c/\gamma = 1.0$ over the source circle.

Fig. 22. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 1.50$, $b = 1.00$, $\eta = 2.50$, and $p_c/\gamma = 1.0$ over the source circle.

Fig. 23. Flownet for axisymmetric infiltration from source circle to water table with $p_b/\gamma = 3.50$, $b = 1.00$, $\eta = 3.00$, and $p_c/\gamma = 1.0$ over the source circle.
Table 1. Newton-Raphson solution for $r_0 = 25.906$ and $z_0 = 5.000$.

<table>
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<th>n</th>
<th>r</th>
<th>z</th>
<th>$\frac{\partial f_1}{\partial r}$</th>
<th>$\frac{\partial f_1}{\partial z}$</th>
<th>$\frac{\partial f_2}{\partial r}$</th>
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Table 2. Newton-Raphson solution for $r_0 = 25.906$ and $z_0 = 4.500$.

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<th>$\frac{\partial f_1}{\partial z}$</th>
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</table>

Determination of final radius of boundary value problem

The total number of the streamlines of the $\Phi$ plane, NS, is specified as an input parameter in defining the problem. The value $\text{NS} = 19$ has been used to obtain all solutions given herein. The $\text{NS} = 19$ stream surface consists of the ground surface outward from the edge of the source circle and the vertical outer cylindrical boundary of the flow system. The second stream surface from the vertical outer boundary in the regular grid network system is the stream surface enclosing 95 percent of the total flux.

The maximum radius of the 95 percent total flux stream surface of the first group of solutions has been plotted as a function of the value of the parameter $b$ in Fig. 24. The plot shows that the radius increases linearly with $b$ ($p_b/\gamma$ and $\eta$ are held constant at 3.50 feet and 2.50 respectively). This general relationship might have been anticipated from Fig. 4, since for a constant capillary pressure head, the height of the plateau or the maximum value of $\Phi$ increases as the value of $b$ decreases. That is, if the two parameters $p_b/\gamma$ and $\eta$ on the source circle are held constant, the soil will become more saturated and the flow system will become nearly one-dimensional with decreasing value of $b$.

![Fig. 24. The radius of 95 percent flux stream surface at the water table as a function of $b$ values with $p_b/\gamma = 3.50$, and $\eta = 2.50$.](image-url)
The relationship is also linear between the radius of the 95 percent flux stream surface at the water table and the bubbling pressure head as shown in Fig. 25. The maximum 95 percent radius increases as the bubbling pressure head increases. Since the bubbling pressure increases with decreasing average pore size of the soil, the soil throughout the region becomes more saturated as \( \frac{p_b}{\gamma} \) decreases. Thus, the flow system of the problem becomes more nearly one-dimensional as the bubbling pressure decreases and the remaining soil parameters are held constant. That is the radius of the 95 percent flux stream surface at the water table decreases as the bubbling pressure decreases as shown in Fig. 25.

The region within the 95 percent flux stream surface obtained from the solutions for varying \( \eta \) values with \( \frac{p_b}{\gamma} = 1.0 \) specified on the source surface has the relative permeability close to unity. Thus, within this region, the saturation increases as the values of \( \eta \) increases (Fig. 5), and the radius of 95 percent flux stream surface at the water table decreases a relatively small amount as the value of \( \eta \) increases (Fig. 26). The indicated tendency may not be true when a high constant capillary pressure is specified on the source surface.

**Determination of seepage velocity**

Expressions for computing the seepage velocity, \( V \), at any point can be obtained by defining the Jacobian for the transformation of variables. Eq. 30 can be rewritten as

\[
J = \begin{vmatrix}
\frac{\partial \phi}{\partial r} & \frac{\partial \phi}{\partial z} \\
\frac{\partial \psi}{\partial r} & \frac{\partial \psi}{\partial z}
\end{vmatrix}
\]

Substituting Eqs. 5 and 6 into Eq. 113 gives

\[
J = \begin{vmatrix}
\frac{V}{K} - \frac{V}{K} \\
rV & rV
\end{vmatrix} = \frac{rV^2}{K}
\]

The derivatives of \( \phi \) and \( \psi \) with respect to \( r \) and \( z \) are related to the derivatives of \( r \) and \( z \) with respect to \( \phi \) and \( \psi \) by J. Substituting Eq. 114 into these derivatives, the following expressions are obtained.

\[
\frac{\partial r}{\partial \phi} = \frac{K \cos \theta}{V}
\]

\[
\frac{\partial r}{\partial \psi} = \frac{\sin \theta}{rV}
\]
\[ \frac{\partial z}{\partial \psi} = -\frac{K \sin \theta}{V} \] .......................... (117)

and

\[ \frac{\partial z}{\partial \psi} = \frac{\cos \theta}{r V} \] .......................... (118)

in which \( \theta \) is the angle between the direction of flow and the horizontal \( r \)-axis. By eliminating the angle, any of the following four equations results to obtain the velocity:

\[ V = \frac{1}{r \sqrt{\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2}} \] .......................... (119)

\[ V = \frac{K}{\sqrt{\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2}} \] .......................... (120)

\[ V = \frac{1}{\sqrt{\frac{1}{K^2} \left(\frac{\partial z}{\partial \psi}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \psi}\right)^2}} \] .......................... (121)

\[ V = \frac{1}{\sqrt{\frac{1}{K^2} \left(\frac{\partial z}{\partial \psi}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \psi}\right)^2}} \] .......................... (122)

By numerically approximating the derivatives in Eqs. 119, 120, 121, or 122, the seepage velocity at any grid point can be obtained. An example of the seepage velocity distribution is shown in Fig. 27(a). The time required by the water to travel from the source surface at a distance \( r/R_0 \) from the axis of symmetry to the water table for \( \rho_b/\gamma = 3.50 \), \( \eta = 2.50 \), \( b = 1.0 \) and \( D = 16' \) is shown in Figure 27(b).

Hydrostatic pressure distribution and pressure gradient

The hydrostatic pressure or pore pressure at any point can be obtained from the solution output of \( z(\Phi, \psi) \) with the equation

\[ p = \rho g (\Phi - z) = \gamma [(I - 1) \Delta \Phi - z] \] .......................... (123)

Since a constant capillary pressure head is specified on the surface of the circular rainfall simulator and there is no other source of moisture into the system, the entire region will be partially saturated, and the pressure at each point will be negative. By definition this negative pressure is a positive capillary pressure, \( p_c \). Thus, Eq. 123 can be rewritten as

\[ p_c = \gamma [z - (I - 1) \Delta \Phi] = -p \] .......................... (124)

Fig. 28 gives the capillary pressure distribution obtained from a solution for \( \rho_b/\gamma = 3.50 \), \( b = 1.00 \), and \( \eta = 2.50 \) with a unit capillary pressure head on the surface of the source circle. The capillary pressure distribution of the solution is similar to the distribution of permeability. Analysis of the distribution of permeability for different solutions will be made in a subsequent section of this study.

Fig. 27(a). The seepage velocity distribution for \( \rho_b/\gamma = 3.50 \), \( b = 1.0 \), and \( \eta = 2.50 \).
Fig. 27(b). Time (T in hour) needed by water to travel from the source surface at a distance $r/R_0$ from the axis of symmetry to the water table. ($p_b/\gamma = 3.5', \eta = 2.50, b = 1.0, n =$ porosity, and $K_s =$ permeability at saturation in fps.)

Fig. 28. Capillary pressure distribution for $p_b/\gamma = 3.50'$, $b = 1.0$, and $\eta = 2.50$. 

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The pressure gradients can be obtained by dividing the pressure difference between adjacent grid points by the grid spacing.

**Distribution of the relative infiltration rate** (I_r) **over source circle**

Infiltration will be expressed in dimensionless terms by dividing by the permeability K_o. This dimensionless quantity I_r = I/K_o (where I is the infiltration rate per unit area) will be referred to as the relative infiltration rate. Distribution of relative infiltration rate over the source circle for several values of bubbling pressure and \( \eta = 2.50 \), \( b = 1.0 \), and \( p_c/\gamma = 1.0' \) over the source circle are shown in Fig. 29. These distributions were obtained from the mean relative infiltration rate between each pair of consecutive stream surfaces (the infiltration flux between stream surfaces, \( 2\pi \Delta \psi/K_o \) divided by its area). The infiltration rate is nearly constant except near the edge of the source circle where rapid increase of I_r occurs.

The distributions of relative infiltration rate over the source circle for different \( b \) values with \( p_b/\gamma = 3.50' \), \( \eta = 2.50 \), and \( p_c/\gamma = 1.0' \) are given in Fig. 30. The influence of edge effect becomes less as the value of \( b \) decreases.

The distribution of the relative infiltration rate over the source circle due to different pore-size distribution indexes is shown in Fig. 31. From the range of \( \eta \)-values for which solutions have been obtained it appears that the distribution of the relative infiltration rate over the source circle is not very sensitive to changes in \( \eta \). This lack of sensitivity might be expected because the functional relationship between the permeability and the capillary pressure head is insensitive to changes in \( \eta \) values at \( p_c/\gamma = 1.0' \) as shown in Fig. 5.

**Distribution of permeability parameter (K_r) on the ground surface beyond the source circle**

In partially saturated moisture movement it is useful to have information regarding saturation or effective permeability throughout the flow region. In this and subsequent sections the relationships of K_r with the soil parameters, as obtained from analyses of the solution results, are presented. Since the value of K_r at any grid point of the problem can be obtained directly from the computer solution results, and it would require considerable computer execution time to solve Eq. 44 for S_e from a known K_r or K_s at each grid point, the distributions of S_e on the ground surface beyond the source circle as well as any other surface or plane of

![Fig. 29. Distribution of relative infiltration rate, I_r, over source circle as a function of distance from the center for different p_b/\gamma values with \( \eta = 2.50 \), \( b = 1.0 \), \( K_o = 0.001 \) fps, and \( p_c/\gamma = 1.0' \).](image-url)
Fig. 30. Distribution of relative infiltration rate, $I_r$, over source circle as a function of distance from the center for different $b$ values with $\eta = 2.50$, $p/b/\gamma = 3.50$, $K_o = 0.001$ fps, and $p_c/\gamma = 1.0$.

Fig. 31. Distribution of relative infiltration rate ($I_r$) over source circle as a function of distance ($r/R_o$) from the center for different $\eta$ values, with $p/b/\gamma = 3.50$, $b = 1.0$, $K_o = 0.001$ fps, and $p_c/\gamma = 1.0$. 
interest is not given directly in this report. Instead, the distributions of \( K_r \) are given and discussed in this and subsequent sections. The distributions of \( S_e \) can be obtained from the \( K_r \) vs. \( S_e \) relationships, Fig. 6 or 7, as discussed earlier. One need only enter these figures with \( K_r \) and the appropriate soil parameters \( b, \eta \) and \( p_b/\gamma \) and obtain from the figures the corresponding value of \( S_e \).

The distribution of permeability on the ground surface beyond the circle of application is quite different from that within the source circle. The effects of each of the parameters in Eq. 1 on the distributions of \( K_r \) beyond the source circle are given in Figs. 32, 33, and 34.

Fig. 32 shows the series of lines result from solutions based on different values of \( p_b/\gamma \) but with other parameters in Eq. 1 held constant \( (\eta = 2.50, b = 1.0) \). Since in all of these solutions the capillary pressure, specified on the source circle, was less than the bubbling pressure each of the curves begins at \( K_r = 1/[(\gamma/p_b)^{2.5} + 1] \) and decreases with the distance beyond the source circle. For smaller values of \( p_b/\gamma \) the permeability parameter decreases more rapidly at small radial distances than for larger values of \( p_b/\gamma \). At larger radial distances the percentage decrease in \( K_r \) is less for small values of \( p_b/\gamma \).

The variation of the permeability parameter \( (K_r) \) with radial distance from the solutions with different values of \( b \) specified but with \( \eta \) and \( p_b/\gamma \) held constant have been plotted in Fig. 33. In contrast with the reduction in the permeability parameter \( (K_r) \) with decreasing values of \( p_b/\gamma \) shown in Fig. 32, Fig. 33 indicates that the permeability parameter \( (K_r) \) decreases as the value of the

![Fig. 32. Distribution of permeability parameter, \( K_r \), on the surface as a function of distance from the edge of the source circle for different \( p_b/\gamma \) values, with \( \eta = 2.50 \), and \( b = 1.0 \).](image_url)
parameter \( b \) is increased. The percentage reduction in \( K_r \) with large values of \( b \) is greater at smaller radial distances that at larger radial distances.

The manner in which the permeability parameter \( (K_r) \) varies with radial distance beyond the source circle for those solutions with different values of \( \eta \) specified is shown in Fig. 34. At small radial distances \( (r/R_o \approx 1.1) \) the permeability parameter \( (K_r) \) shows a slight increase with larger specified values of \( \eta \). At large radial distances, however, the permeability parameter \( (K_r) \) decreases considerably with increasing values of \( \eta \). The latter relationship occurs in regions where the capillary pressure is large, and the former where the capillary pressure is small. The variation of \( K_r \) with \( \eta \) is closely related to the relationship shown in Fig. 5 in which the separate curves all passed through the point \( p_c = p_b \) and \( K_r = 1/(1+b) \).

**Distributions of permeability parameter \((K_r)\) along the axis of symmetry**

The distributions of permeability along the axis of symmetry resulting from the solution obtained for different values of \( p_b/\gamma \) with \( \eta = 2.50 \) and \( b = 1.0 \) are shown in Fig. 35. Since all of these solutions specified \( p_c/\gamma = 1.0 \) on the surface within the source circle, the permeability \( (K_r) \) at the surface is given by \( K_r = 1/(\gamma/p_b) + b \). At the water table \( p_c = 0 \) and therefore \( K_r = 1/b \) for any value of \( p_b \). The results plotted in Fig. 35 show that the variation of the permeability is large just above the water table for small values of \( p_b/\gamma \).
The distribution of the permeability along the axis of symmetry for different values of $b$ with $\eta = 2.50$, and $p_b/\gamma = 3.50^1$ is given in Fig. 36. At the water table the permeability changes with $b$ since $K_r = 1/b$, but this is only because of the definition of $K_r$. The permeability at the surface is given by $K_r = 1/(\gamma/p_b)^{1/2} + b$. The quantity $(\gamma/p_b)$ (where $p_b$ and $\gamma$ are given constants) in the denominator becomes less significant as $b$ increases. Thus, the permeability becomes more nearly constant along the axis for larger values of $b$.

The distributions of the permeability along the axis of symmetry as given by solutions for different values of $\eta$ with $p_b/\gamma$ and $b$ constant are given in Fig. 37. The relative permeability at the surface is given by $K_r = 1/(\gamma/p_b)^{1/2} + b$. As $\eta$ increases, $K_r$ increases approaching a constant value equal to $1/b$. An increase of the pore-size distribution index indicates that the porous media is becoming coarser and the resulting flow system becoming more nearly one-dimensional.

**Distribution of equi-permeability curves**

The variation of the effective saturation, $S_e$, within the flow region under study is much less sensitive than the permeability parameter, $K_r$. The permeability parameter $(K_r)$ varies in a wider range from 0.1 to 1.0 while $S_e$ varies about from 0.6 to 1.0. In order to show a clearer picture of the flow system, the distribution of equi-permeability lines (which are lines of constant, $K_r$) is presented in this section. Actually these distributions may also be converted to distribution of equieffective saturation lines from the solutions of Eq. 45, as shown in Figs. 6 and 7.
The distribution of equi-permeability lines throughout the seepage region can be obtained by substituting the solution of \( z \) into Eq. 123 to compute the pressure, which in turn is used in Eq. 1 or Eq. 16 to compute \( K_r \). The equi-permeability lines are obtained by interpolations between the values for \( K_r \) on the Figs. 14 through 23. The distribution of equi-permeability lines for the cases studied are shown in Figs. 38 through 48. How these distributions are related to the soil parameters in Eq. 1 is presented in the following section.

The influence of soil parameters on the distribution of permeability

Changes in the \( K_r = 0.30 \) permeability curve with different values of \( p_b/\gamma \) and \( \eta = 2.50 \), and \( b = 1.0 \) are given in Fig. 49. The curves are shifted outward from the axis of symmetry as the value of \( p_b/\gamma \) increases, and the portion of the curves closer to the water table, where the capillary tension is high, has been shifted more, since the variation of the permeability parameter, \( K_r \), is greater at larger bubbling pressure heads as shown in Fig. 3.

The variation of \( K_r = 0.30 \) curve for different values of \( b \) with \( \eta = 2.50 \), and \( p_b/\gamma = 3.50\gamma \) is given in Fig. 50. The curves are shifted toward the axis of symmetry as the value of \( b \) increases. The shifting is more significant in the area of low capillary tension underneath the source surface where the variation of \( K_r \) is more significant for varying value of \( b \) as shown in Fig. 4.

The equi-permeability curves for \( K_r = 0.20, 0.30, 0.50, \) and \( 0.80 \) for different values of \( \eta \) are given in Fig. 51. The permeability \( (K_r) \) varies only slightly on the source surface where \( p_c/\gamma = 1.0 \) is specified (Fig. 5), and it appears that the \( K_r = 0.80 \) curve or any other equi-permeability curve with \( K_r \) greater than \( 1/(1 + b) = 0.50 \) is shifted outward as the value of \( \eta \) increases and the equi-permeability curves with \( K_r \) equal or less than 0.50 are shifted inward as the value of \( \eta \) increases and the shifting is more significant in areas of higher capillary tension such as the vertical outer boundary.

These same variations with \( \eta \) are revealed in Fig. 4 which shows the variation of the permeability parameter, \( K_r \), with the capillary pressure for different pore-size distribution indexes.

Effects of soil parameters on the "spreading effect"

The lateral movement of flow in soil from an infiltrometer or a source circle will be affected by the size of the infiltrometer or the source circle, the surface head, and the soil properties defined by Eq. 1. Marshall and Stirk (1950) studied the effect of lateral movement of water in soil from flooded plots and found that the relative lateral movement is greatest in the smallest plot. They introduced a correction factor to compensate for lateral movement and reduce the influence of plot size. Schiff (1953) studied the effect of surface head on final infiltration rates based on the
The relative infiltration rate for one-dimensional steady downward flow to a water table can be obtained from the capillary pressure distribution equation during steady downward flow toward a water table. Arbhabhirama and Kridakorn (1968) used an equation similar to Eq. 1 proposed by King (1965) except $b = 1.0$ to obtain integral form of the Scott-Corey (1961) equation,

$$z = \int_0^{\frac{p_c}{\gamma}} \left( \frac{dp_c}{\gamma} \right) \left[ 1 - \left( \frac{q}{K_r} \right) \right] . \quad (125)$$

where $z$ is the elevation above the water table, $q$ is the volume flow per unit area in direction $z$, and obtained a simple equation to represent the capillary pressure distribution during steady one-dimensional downward flow. If Eq. 1 is substituted into Eq. 125 and the integration is performed by using the convergent series suggested by Arbhabhirama and Kridakorn (1968), the following expression for the capillary pressure distribution during steady downward flow can be obtained (see Appendix A).

$$z = \frac{p_o}{1 - bq_o} \left[ 1 - \frac{1}{\eta + 1} \ln \left( \frac{q_o p_o \eta}{1 - bq_o} \right) \right] . \quad (126)$$
Fig. 39. Distribution of equi-permeability curves for $\eta = 3.00$, $p_b/\gamma = 3.50'$, and $b = 1.0$.

Fig. 40. Distribution of equi-permeability curves for $\eta = 2.50$, $p_b/\gamma = 3.50'$, and $b = 1.0$.

Fig. 41. Distribution of equi-permeability curves for $\eta = 2.50$, $p_b/\gamma = 3.00'$, and $b = 1.0$.

Fig. 42. Distribution of equi-permeability curves for $\eta = 2.50$, $p_b/\gamma = 2.50'$, and $b = 1.0$. 
Fig. 43. Distribution of equi-permeability curves for \( \eta = 2.50, p_b / \gamma = 2.00', \) and \( b = 1.0. \)

Fig. 44. Distribution of equi-permeability curves for \( \eta = 2.50, p_b / \gamma = 1.50', \) and \( b = 1.0. \)

Fig. 45. Distribution of equi-permeability curves for \( \eta = 2.50, p_b / \gamma = 3.50', \) and \( b = 1.50. \)

Fig. 46. Distribution of equi-permeability curves for \( \eta = 2.50, p_b / \gamma = 3.50', \) and \( b = 2.00. \)
Fig. 47. Distribution of equi-permeability curves for $\eta = 2.50$, $p_b/\gamma = 3.50'$, and $b = 2.50$.

Fig. 48. Distribution of equi-permeability curves for $\eta = 2.50$, $p_b/\gamma = 3.50'$, and $b = 3.00$.

Fig. 49. The $K_r = 0.30$ equi-permeability curves obtained from solutions for different $p_b/\gamma$ values, with $\eta = 2.50$, and $b = 1.0$.

Fig. 50. The $K_r = 0.30$ equi-permeability curves obtained from solutions for different $b$ values, with $p_b/\gamma = 3.50'$, and $\eta = 2.50$. 
Fig. 51. The $K_r = 0.20, 0.30, 0.50$, and $0.80$ equi-permeability curves obtained from solutions for different $\eta$ values, with $p_b/\gamma = 3.50'$ and $b = 1.0$.

$P_c/\gamma = 1.0'$

$16'$

Water table

where $\eta > 2$, $z_o = z/(p_b/\gamma)$, $p_o = p_c/p_b$, and $q_o = q/K_o$. For the region where $p_o < 1$ and the values of $p_o^3$ are small, Eq. 126 can be approximated as the straight line.

$$z_o = \frac{p_o}{1 - b q_o} \quad . . . . . . . . . . . . . . . . \quad (127)$$

If the soil parameters, the elevation above the water table, and capillary pressure defined over the surface are known, the relative infiltration rate per unit area $q_o$ can be obtained from Eq. 126, and the total intake of water $I_1$ from the one-dimensional case which corresponds to the flux $I_3$ in the axisymmetric case is obtained by multiplying $q_o$ by the area of the source circle. The relative infiltration rate for the axisymmetric case, $I_3$, has been obtained from the computer solution results.

Values for $I_3/I_1$ are plotted in Fig. 52 as a function of the scaled bubbling pressure head, $p_b/(\gamma D)$, with $\eta = 2.50$, $b = 1.0$, and $D = 16.0'$ (the depth of the water table below the source circle). This plot shows that $I_3/I_1$ is directly proportional to the scaled bubbling pressure head. $I_3/I_1$ increases as the dimensionless bubbling pressure head $p_b/(\gamma D)$ increases. Fig. 53 shows $I_3/I_1$ plotted against $b$ with $p_b/\gamma = 3.50'$, and $\eta = 2.50$. This relationship is a straight line on the semi-log plot, with $I_3/I_1$ increasing as $b$ increases. Fig. 54 contains a plot of $I_3/I_1$ against the pore-size distribution index $\eta$. A straight line on the semi-log paper also defines this relation.

In order to define the total relationship between the soil parameters and the lateral flow of moisture, a normal regression analysis which minimizes the sum of squares of deviations in the normal direction (Jeppson and Huber, 1969) was performed to arrive at a simple expression relates $I_3/I_1$ to the soil parameters in Eq. 1. The resulting regression equation is

$$\frac{I_3}{I_1} = 0.66096 - 3.54250 \frac{p_b}{\gamma D} + 0.91072 \log b + 0.46874 \log \eta \quad . . . . . . . . . . . . . . \quad (128)$$

Fig. 52. Scaled relative infiltration rate ($I_3/I_1$) as a function of scaled bubbling head ($p_b/(\gamma D)$), with $\eta = 2.50$, and $b = 1.0$. 
in which the correlation coefficient is 0.9995. A nomogram was constructed from Eq. 128 and provides a relationship for \( I_3/I_1 \) as a function of \( \eta, b, \) and \( P_b/\gamma D \). This nomogram is given as Fig. 55.

Values of \( I_3/I_1 \) can be considered a correction factor in obtaining one-dimensional infiltration rates from circular infiltrometer measurements. Let

\[
C_1 = \frac{I_3}{I_1}
\]

(129)

then

\[
I_1 = \frac{I_3}{C_1}
\]

(130)

For any known soil (i.e. the soil parameters in Eq. 1 are known) \( C_1 \) can be obtained from the nomogram Fig. 55, and one-dimensional relative infiltration rate, \( I_1 \), can be obtained from Eq. 130.

Fig. 53. Scaled relative infiltration rate \( (I_3/I_1) \) as a function of \( b, \) with \( P_b/\gamma = 3.50 \), and \( \eta = 2.50 \).

Fig. 54. Scaled relative infiltration rate \( (I_3/I_1) \) as a function of \( \eta, \) with \( P_b/\gamma = 3.50 \), and \( b = 1.0 \).

Fig. 55. Nomogram which relates the ratio of the axisymmetric infiltration rate divided by the one-dimensional rate to the soil parameters.
SUMMARY AND CONCLUSIONS

Solutions for the problem of steady-state partially saturated axisymmetric infiltration resulting from moisture applied over a horizontal source surface which moves toward a water table have been obtained. A commonly accepted relationship between permeability and capillary pressure proposed by King (1965) has been utilized in conjunction with Darcy’s law to formulate the mathematical model. The method of solution of the model has utilized an inverse formulation and finite difference techniques. The inverse formulation considers the cylindrical coordinates z and y as the dependent variables and the potential function \( \phi \) and the stream function \( \psi \) as the independent variables (i.e. the problem is solved for \( \phi \) and \( \psi \) on the \( \phi \psi \) plane). Because of the nonlinear nature of the equations, numerical solutions are obtained by combining a Newton-Raphson inner iteration with the usual successive over relaxation outer iteration. To increase the accuracy of the numerical solution, a subdivided grid network has been set up for the region between the last two streamlines where the moisture content is expected to be extremely low.

The approach used for solving the problem of partially saturated axisymmetric infiltration is practical with modern digital computers. It requires approximately 3 minutes of execution time on a UNIVAC 1108 digital computer to obtain a solution to a problem such as those presented herein. The computer output gives the z coordinates at each finite difference grid point on the \( \psi \psi \phi \phi \) plane, which can readily be plotted in a flownet form to show the characteristics of the flow pattern at a glance. From the computer solutions, the distribution of capillary pressure, permeability, or effective saturation over any surface or plane of interest can be obtained. Such quantities have been obtained from the solutions and analyzed to define their relationship with soil properties. The results from these analyses are given in a number of graphs.

The solutions indicate that significant radial movement of moisture occurs. This “spreading effect” is found to be closely related to the hydraulic properties of soils as characterized in the parametric relationship used to describe these properties. The amount of radial movement increases as the bubbling pressures \( P_b \) increases. It also increases as the pore-size distribution index \( \eta \) increases and as the other soil parameter \( \beta \) increases. The maximum radius of the stream surface at the water table, which contains 95 percent of the total flux, increases as the value of \( P_b \) or \( \beta \) increases, and decreases slightly as \( \eta \) increases.

While the results are obtained based on flow in porous media concepts, they give insight into such physical flow occurrences in nature as the distribution of the permeability or effective saturation on the surface, along the axis of symmetry, or on any plane including the axis of symmetry and how these distributions are related to the soil parameters defining the hydraulic properties of the soil. The results show how the moisture content decreases on the surface beyond the area of application, and how radial movement into this region causes a high infiltration rate at the edge of the source circle.

The following conclusions have been drawn directly from the presentation and discussion of results sections. (The common specifications of the problem are \( R_o = 10^1 \), \( D = 16^1 \), and \( P_b \gamma = 1.0 \) on the source surface.)

1. The “spreading effect” due to the radial movement of moisture has been shown to cause a higher infiltration rate to occur for the axisymmetric case than for the one-dimensional case. Within the range of the bubbling pressure heads tested, i.e. \( P_b \gamma \) varies from 1.5' to 3.5', with \( \eta = 2.50 \) and \( \beta = 1.0 \), the radial movement of moisture causes an excess of infiltration rate over the one-dimensional infiltration rate from 19 percent to 62 percent; the 19 percent increase being associated with \( P_b \gamma = 1.5' \) and the 60 percent with \( P_b \gamma = 3.5' \). When values of \( \beta \) vary from 1.0 to 3.0, with \( \eta = 2.50 \) and \( P_b \gamma = 3.5' \), the excess infiltration rate increases linearly from 62 percent to as high as 105 percent. The excess infiltration rate is also related linearly to \( \eta \) varying from 62 percent to 69 percent as \( \eta \) increases from 2.50 to 3.50, with \( \beta = 1.0 \) and \( P_b \gamma = 3.5' \). The results indicate that a higher percentage increase in infiltration rate can be expected for soils with larger values of bubbling pressure, pore-size distribution index and \( \beta \).

2. The radial movement of moisture has been shown to cause the streamlines to spread radially from the source circle toward the water table. The maximum radius of the stream surface at the water table which contains 95 percent of the total flux expressed by \( r/R_o \) increases from 2.1 to 3.0 as \( P_b \gamma \) increases from 1.5 to 3.5, with \( \beta = 1.0 \) and \( \eta = 2.50 \). It decreases slightly from 2.6 to 2.4 as \( \eta \) increases from 2.50 to 3.50, with \( \beta = 1.0 \) and \( P_b \gamma = 3.5' \), and increases from 2.56 to 2.92 as \( \beta \) increases from 1.0 to 3.0, with \( \eta = 2.50 \) and \( P_b \gamma = 3.5' \). Therefore, the results show that the radial spreading of stream surfaces from the source circle toward the water table can be significant if large values of \( P_b \) and \( \beta \) are specified for the soil.

3. The solutions indicate that the moisture content on the ground surface beyond the source circle decreases as the distance from the source circle increases. For instance, the moisture content decreases from approximately 0.8 - 0.99 at the edge of the source circle to about 0.3 - 0.7 at \( r/R_o = 3.0 \).

4. The solutions indicate that a definite edge effect causing a high infiltration rate at the edge of the source circle exists. The edge effect decreases as the values of \( P_b \) or \( \beta \) decrease and is not sensitive to variations in \( \eta \).
LITERATURE CITED


Schiff, L. 1953. The effect of surface head on infiltration rates based on the performance of ring infiltrometers and ponds. Transactions, American Geophysical Union, Vol. 34, No. 2, April, pp. 257-266.


APPENDICES

Appendix A

One-dimensional Steady Downward Flow
To a Water Table

In an earlier study, Scott and Corey (1961) used the long-column technique of Childs and Collins-George (1948) and the controlled pressure method of Richards (1931) to determine the distribution of capillary pressure in columns of both homogeneous and stratified sands. An equation was derived in terms of dimensionless parameters to describe the distribution of capillary pressure during steady downward flow and was referred as the Scott-Corey equation. The equation can be integrated to obtain an integral expression for the elevation above the water table, \( z \).

\[
\frac{K}{K_0} \left( \frac{P_c}{P_b} \right)^\eta + b = K \quad \text{(A-2)}
\]

in which \( K \) is the permeability for partially saturated flow, \( P_c \) is the capillary pressure, and \( q \) is the volume flux rate. Arbhabhirama and Kridakorn (1968) used the dimensionless form of Gardner's equation proposed by King (1965) to obtain.

\[
z = \int_0^{P_c/\gamma} \frac{1}{1 - (q/\lambda)} \frac{dP_c}{\gamma} \quad \text{............. (A-1)}
\]

Letting \( \lambda = \frac{p_o(q_0/(1-bq_o))^{1/\eta} \), Eq. A-4 can be written as

\[
z_o = \left( \frac{1}{1 - bq_o} \right)^{1/\eta} \int_0^{x \over 1 - x^\eta} \frac{dx}{1 - x^\eta} \quad \text{(A-5)}
\]

To evaluate \( \int_0^x \frac{dx}{(1-x^\eta)} \), the the term \( 1/(1-x^\eta) \) is expressed by a convergent series for \( x \leq 1 \) (Arbhabhirama and Kridakorn's procedure)

\[
\frac{1}{1 - x^\eta} = 1 + x^\eta + x^{2\eta} + x^{3\eta} + \ldots \quad \text{(A-6)}
\]

Integrating the above equation term by term gives

\[
\int_0^x \frac{dx}{1 - x^\eta} = x + x^\eta + \left( x^\frac{2\eta}{2} + x^\frac{3\eta}{3} + \ldots \right)
\]

\[
+ \left[ \frac{x^{2\eta+1}}{(2\eta+1)(2\eta+2)} \right] + \left[ \frac{x^{3\eta+1}}{(3\eta+1)(3\eta+2)} + \ldots \right] + \ldots
\]

\quad \text{............. (A-7)}

the values of the terms in the last two parentheses are very small when the values of \( \eta \) are greater than 2. Accordingly, these terms can be neglected. Nearly all media have values of \( \eta \) greater than 2. Then, the integration can be approximated by

\[
\int_0^x \frac{dx}{1 - x^\eta} = x + \left( x^\frac{2\eta}{2} + x^\frac{3\eta}{3} + \ldots \right)
\]

\[
= x + \frac{x^\eta}{\eta+1} \ln (1 - x^\eta) \quad \text{............. (A-8)}
\]

Substituting Eq. A-8 and the expression for \( x \) into Eq. A-5, it becomes

\[
z_o = \frac{P_o}{1 - bq_o} \left[ 1 - \frac{1}{\eta+1} \ln \left( 1 - \frac{q_o p_o^{\eta}}{1 - bq_o} \right) \right] \quad \text{(A-9)}
\]
Eq. A-9 is a general equation describing the capillary pressure distribution during steady downward flow.

For the homogeneous soil, at the region close to the surface of the soil column where the capillary pressure is uniform, and

\[ 1 - \frac{(q_o p_o)}{q_o (1 - b q_o)} = 0, \text{ or } p_o = \left(\frac{1 - b q_o}{q_o}\right) \]

or, in general form

\[ K_o = \frac{1}{\left(\frac{q_c}{p_b} + b\right)^{1/n}} \]  

(Eq. A-10)

Eq. A-10 is used to estimate the infiltration rate of the one-dimensional steady downward infiltration.
FORMAT (16F5.3)
Q(I) = 0.5*(R(I,NS2) - R(I,NS1))
S(I) = 0
DO 130 I = 2, N
II = 2*I - 1
S(II) = 0.5*(Z(II,NS1) - Z(II,NS2))
Q(II) = 0.5*(R(II,NS1) - R(II,NS2))
S(II-1) = 0.25*(Z(II-1,NS1) + Z(II-1,NS2))
C(II) = 0.25*(Z(II,NS1) + Z(II,NS2)) + 0.7*(Z(II,NS1) + Z(II,NS2))
IF (NRUN.EQ.0) GO TO 1920
1910 READ (5,10) NS, NH, NN4, NRD, W1, ERR, DO, GAMA, 8, PR, ENTA, AKO
IF (NS.EQ.99) GO TO 999
READ (5,6) W2, RNS, G, H, NIT, NWT, NUB, NRUN
PBD = B
BREAD = B
GB = GAMA/PB
BI = I / B
BI2 = BI * BI
GBN = GB * ENTA
ET = ENTA-1.
GBNN = GBN * ENTA
WRITE (6,19)
WRITE (6,12)
WRITE (6,11)
WRITE (6,13)
1920 CONTINUE
68 WRITE (6,13)
DLB = READ - BLAST
DB = DELB / 5.0
B = BLAST / DB
DBP = DB - BLAST
DBBP = DBP / 10.0
PB = DBP - DPLST
GB = GAMA / PB
GBN = GB * ENTA
GBNN = GBN * ENTA
39 FORMAT (11H1)
19 FORMAT (1DH)
801 FORMAT (I8)
C ITERATION BEGINS
NCOUNT = 1
500 NN = NCOUNT
NWT = NUB
GO TO 231
232 U(M4P+4) = U(M4P+6)
T(M4+6) = 0.5*(T(M4+6) + T(MAN+6))
U(M4+5) = 0.5*(U(M4+5) + U(MAN+5))
U(M4+4) = 0.5*(U(M4+4) + U(MAN+4))
M4 = M4 + 1
M4PP = M4P + 1
M4PPP = M4PP + 1
M4N = M4 - 1
231 IF (NCOUNT .GT. 80) GO TO 221
220 IF ((RAM .EQ. 90) .OR. N = 1) GO TO 221
226 IF (M4 .EQ. 2) GO TO 135
U(M4+6) = 0.5*(U(M4+6) + U(M4P+6))
T(MAN+6) = T(M4+6)
U(MAN+5) = 0.5*(U(MAN+5) + U(M4P+5))
U(MAN+4) = 0.5*(U(MAN+4) + U(M4P+4))
M4 = M4 - 1
M4PP = M4P + 1
M4PPP = M4PP + 1
M4N = M4 - 1
GO TO 221
135 M4 = 10
221 CONTINUE
171 CONTINUE
N = NN
NN = N + 1
IF (NCOUNT .NE. NN) GO TO 72:
C COMPUTE RNS
DO 100 I = 1, 10 + RNS
RRN = 1.1*10 + RNS
RNS = 1.1*10 + RNS
U = U(1) - U(1 - 1)
U = U(1) - U(1 + 1)
DO = DX - DX1
IF (DOX .LT. 0.0) DOX = 0.0
DO = DX + DX1
DX = 2.5*DX2
IF (DX .LT. DX2) DX = DX2
DX = DX + DX1
DX = 2.5*DX2
IF (DX .LT. DX2) DX = DX2
DX = DX + DX1
RNS = U(1) + DX3
IF (RNS .GT. RRN) RNS = RRN
IF (RNS .LT. U(1 + 5)) RNS = U(1 + 5) + 1.05
IF (RNS .GT. 50.) RNS = 50.
W = (M4-11)/n
DO 930 I = 1, n
930 CONTINUE
IF (M4 .EQ. 10) I = M4 + 1
IF (M4 .EQ. 10) I = M4R + 1
DO 801 I = 1, M4R
801 CONTINUE

C ITERATION ALONG THE WATER TABLE
DO 180 J = 2 + NS2
Z1 = Z(1 + J)
Z2 = Z(2 + J)
R1 = R(1 + J)
R2 = R(2 + J)
Z1 = Z1 - Z1
Z2 = Z2 - Z2
AA = (Z1 - Z2)/((R1 - R2))
BB = Z1 - R1 + AA
CF = 1.005
IF (J .LT. 6) CF = 1.005
180 CONTINUE
C INTEGRATION ALONG (1) - (5), SOIL SURFACE, (R)
PHAI =
CALL KVAR( AK1 + GRN + ENTA + B + PHAI + ZPNT)
DO = 1.33333333*Z(NH+1) - 3.0*Z(NH+1) + 1.5*Z(NH+2)
- 1.3333333*Z(NH+2)
DR = 1.3333333*Z(NH+2) - 3.0*Z(NH+1) + 1.5*Z(NH+2)
- 1.3333333*Z(NH+2)
R = R(NH+2) + SORT(DR) + DR1 + GA/AA1
DO 101 J = 5
K = J + 1
DR = 1.3333333*Z(NH+K) - 3.0*Z(NH+K) + 1.5*Z(NH+K)
- 1.3333333*Z(NH+K)
DELZ = 1.0*Z(NH+K) + (DR1 + DR2) - 0.8*Z(NH+K) + (DR + DR2)
R(NH+J) = SORT(R(NH+J) - R(NH+J-1) + DELZ/AA1)
DO 102 DR = DR1
102 CONTINUE
DR = DR1
101 CONTINUE
RR = Z(NH+J) + DIFR
DR = DR1
DIFR = 1 - DRB/RBB
DO 99 J = 2 + NS1
99 CONTINUE
R(NH+J) = Z(NH+J) + DIFR
72 NPT = NCOUNT + 1
NPT = NPT/10
NPT = NPT + 1
NPT = NPT - 1
IF (NPT .NE. NCOUNT) GO TO 960
WRITE(6,19)
WRITE(6,143) NCOUNT
143 FORMAT(4X*7HNCOUNT=I3)
C ITERATION ALONG THE AXIS OF SYMMETRY (2)
960 SUM2=0.D0
NAS=NCOUNT
NAS=NAS/2
NAS=NAS+2
IF (NAS.EQ.0) GO TO 660
CALL AXISZ(SUMZ,DP5)
106 FORMAT(4X*4HMP5=F10.5,4X*2HE=F8.3)
C ITERATION OF THE INTERIOR REGION
660 SUMR=0.D0
CALL R2ITR(SUMR,SUMZ,DP5)
IF (INPT.NE.NCOUNT) GO TO 120
WRITE(6,110) SUMZ,SUMR,DP5,G(U(2,5),T(2,5),U(2,4),T(2,4))
110 FORMAT(4X*5SHSUMZ=F10.5,4X*5SHDP5=F8.3)
WRITE(6,801) (R(I,J),J=1,NS)
WRITE(6,801) (P(RH,J),J=1,NS),R8B
WRITE(6,801) (T(I,J),J=1,NS)
MP15=MP15-15
MP15=M-15
WRITE(6,801) (UI(I),I=1,MP15)
WRITE(6,801) (UI(I),I=1,MP15)
C INTEGRATION ALONG THE UPPER BOUNDARY OF THE SUBDIVIDED
C NETWORK (4) - (5), (1).
120 NN=NCOUNT
NN=NN/NUB
NN=NN/NUB
IF (NCOUNT.NE.NN) GO TO 451
UI(MP,6)=R(NH,NS)
CCH=H*FLOAT(MP1)
M4B=M4P
IF (M4A.EQ.10) M4B=M4
IF (M4A.EQ.0) U(M4,6)=U(M4,6)+1
M4B=M4B+1
Y1=U(M4B,5)
Y2=U(M4B,4)
X1=U(M4B,5)
X2=U(M4B,4)
AA=(Y2-Y1)*Y1/(X2-X1)
BB=U*(Y1-A1)
UI(M4B,6)=BB/AA+0.975
IF (U(M4B,6)*LT.U(M4B,6)+1) U(M4,6)=1.05*U(M4B,6)
IF (U(M4B,6)*GT.RNS) UI(M4B,6)=RNS
T(M4B,6)=0
DO 770 I=M4B1,MP1

ZT=0
PHI=CCCC*FLOAT(I-1)
CALL KVARI(AKK+GBN+NTA+8,ZT,PHI,ZPNT)
AK2=AKK+AKK
ZT=ZT-PHI
R1=U(I-1)+6
R3=U(I-1)+6
RN=U(I-5)
R13=R1-R3
R3P=R1+R3
F1=U+AK2
F2=U+R4
F3=U+62
GBNN=GBNN+ZPNT/ZTP
FN=(R13*G-5*G*R13*AKK+GBNN*DDPS)
F5=25*G+R13*R13
NCT=0
RT(U(I-6))
771 RT2=RT+RT
RT=RT2
RT4=RT3-RT
FR=F1+RT4+F2*RT3+F3*RT2+R4*RTF5
DFDF4=F1*RT3+F2*RT2+R4*RTF4
DR=FR/DFDF
RT=RT-DR
NCT=NCT+1
IF (ABS(DR).GT.0.00001.AND.NCT.LT.10) GO TO 771
770 UI(I-6)=RT
DO 251 I=1,NN+R
251 UI(I-6)=U(I-6)
C SOLVE Z ALONG RNS
PHI=H/FNH
DDPS=6*PHI
DDPS=25*DDPS
CC=H*FLOAT(MP1)
RT=NNR
RT=RT-RT
IF (M4A.EQ.2) GO TO 451
M4NN=M4N-1
X1=RNRS-U(M4N+5)
X2=RNRS-U(M4N+4)
Y1=U(M4N+5)
Y2=U(M4N+4)
AA=(X2-X1)*X1/(Y2-Y1)
BB=U*X1-A1
T(M4N)+BB/AA+1.01
DO 740 I=2+M4NN
PHI=CCC*FLOAT(I-1)
ZT=U(I-6)
ZT=U(I-5)
IF INCOUNT.GE.NIT GO TO 757
IF (SUMRZ.GT.5000.) GO TO 400
IF (INCOUNT.LT.110) GO TO 7001
NWE=50
NUBY=50
7001 CONTINUE
INCOUNT=INCOUNT+1
C TO COMPUTE DSAI/DPHAI
GG=0.0
CCCC=H/FNH
DN1=R(1.6)-R(1.5)
AK1=81
AK2=81
DO 660 I=1,NH1
III=1
RM=0.25*(R(1.5)+R(1.6)+R(II.5)+R(II.6))
DS1=SORT(R(II.5)-R(1.5))*2*(Z(II.5)-Z(1.5))*2
DS2=SORT(R(II.6)-R(1.6))*2*(Z(II.6)-Z(1.6))*2
DN2=SORT(R(II.6)-R(II.5))*2*(Z(II.6)-Z(II.5))*2
PHI=CCC*FLOAT(I)
CALL KVAR(AK1,GBK,ENTA,B,Z(II.5),PHI,ZTP)
CALL KVAR(AK1,GBK,ENTA,B,Z(II.5),PHI,ZTP)
AK1=0.25*(AK1+AK2+AK3+AK4)
DO 401 J=1,NH
DO 403 K=1.NH
DO 404 P=1,NH
DN1=DN1+DN2
AK1=AK4
660 AK2=AK4
G=G*FH/NH
G2=G*G
DPS=G*H/FNH
DPS=DPS*0.25*DPS
IF (INCOUNT.EQ.15) B=B+DB
IF (INCOUNT.EQ.25) B=B+DB
IF (INCOUNT.EQ.35) B=B+DB
IF (INCOUNT.EQ.45) B=B+DB
NN=INCOUNT
NN=NN/4
NN=NN/4
IF INCOUNT.EQ.NH PB=PB+DPB
IF (PN*GT.5) PB=PB*PB
GB=GB*GB
GBNN=GBK*ENTA
GBNN=GBK*ENTA
GBNN=GBK*ENTA
IF INCOUNT.EQ.95 GO TO 434
-473 WRITE(6,391)
DO 430 I=1,NH
430 WRITE(6,801)R(I,J),J=1,NH
WRITE(6,391)
DO 431 I=1,NH
431 WRITE(6,801)Z(I,J),J=1,NH
WRITE(6,39)
DO 432 I=1,NH
432 WRITE(6,801)U(I,J),T(I,J),J=1,NH
434 CONTINUE
WRITE(6,110)SUMZ,SUMP,DPS,G,(U(2,5),T(2,5),U(2,4),T(2,4)
GO TO 5000
757 WRITE(6,-19)
WRITE(6,106)DPS,G
C STORE THE SOLUTION ON THE STORAGE TAPE
DO 290 J=1,NH
CALL INOUT(0.1*R(1.5),NH)
290 CALL INOUT(0.1*Z(1.5),NH)
DO 291 J=1,NH
CALL INOUT(0.1*U(1.5),MP)
291 CALL INOUT(0.1*T(1.5),MP)
CALL ENFILE(1)
NSKP=NSKP+1
GO TO 5000
5000 CONTINUE
GO TO 500
400 WRITE(6,39)
DO 401 I=1,NH
401 WRITE(6,801)R(I,J),J=1,NH
WRITE(6,39)
DO 403 I=1,NH
403 WRITE(6,801)Z(I,J),J=1,NH
WRITE(6,39)
DO 404 I=1,NH
404 WRITE(6,801)U(I,J),T(I,J),J=1,NH
404 WRITE(6,801)
DO 405 J=1,NH
DO 406 MP
DO 407 J=1,NH
DO 408 MP
DO 409 J=1,NH
DO 409 J=1,NH
IF (SUMRZ.GT.5000.) GO TO 555
IF (INRUN.EQ.5) GO TO 350
IF (INRUN.EQ.10) GO TO 350
GO TO 350
350 CONTINUE
CALL RZPLT(R,Z,U,T,NH,MP,NHC,NHC,NS,H,M,P,0.35,3.5)
351 CONTINUE
WRITE(6,550)
555 FORMAT(4X,25HTHE SOLUTION IS DIVERING/)
IF (INRUN.EQ.5) GO TO 560
CALL REWIND(1,0)
CALL SKPL5(1,NSKP+1)
DO 570 J=1,NH
570 CALL INOUT(1.1*R(1.5),NH)
DO 571 J=1,NH
571 CALL INOUT(1.1*U(1.5),MP)
571 CALL INOUT(1.1*T(1.5),MP)
CALL SKPL5(1,4,0)
560 CONTINUE
PBIST=PBRD
BLAST=READ
IF(NRUN.NE.2)GO TO 1910
CALL UNLOAD(I1)
999 STOP
END
I1
SUBROUTINE RZWFTN(RT+ZT+RD+R2+R3+R4+Z0+Z1+Z2+Z3+Z4+
SPH+1+DP3)
COMMON R(50+50),Z(50+50),U(200+6),T(200+6),Q(100+51),
S(100+51),P(100+2)+W(100+7),E,R+1,NS+NH+NS1+NS2+NS3,
NH+NH2+NH3+NH4+NH4P+NH4PP+NH5+NH6+FSZ+FHZ+FHS,
NHF+F5+N+FN+M+NP1+NP2+NP3+MP+M4+M4P+M4N+M4PP+M6+PB,
S5+GR+6+BNN+6BNN1+AKN+ENTA+ETA+NRD+0G+G2+BI+BI2+H
RT=RD
ZT=ZD
R1=R1-R3
R1P=R1+R3
R2=R2-R4
R2P=R2+R4
R124=R13+R24
Z13=Z13-Z23
Z13P=Z13+Z23
Z24P=Z24+Z4
Z24P=Z24+Z24
E=-25*R13+R13+G3
CC=-5*Z13+Z4+G
NCT=0
40 CALL CVARB(AK1+GBN+ENTA+ZT+PHA1+ZPN1)
AK17=AK1+AK1
AK13=AK12+AK1
A=2.+AK12
B1=AK12+2NP
RT2=RT+RT
RT3=RT2+RT
RT4=RT3+RT
Z24P=Z24P+2.*ZT
ZPT=ZPT+PHA1
ZPT1=ZPT+ZTP
ZPT2=ZPT1+ZTP
GNMZ=GBN+AKN+ZTP2
GBZ=1+GBN+ZTP1
C=-.5*AK1+GBZ*R132+G*+.25*AK1+R242-2.*G2
D1=1313P+G2-5*AK1+DP5+GBZ+113+G6
F1=AK14+R1+RT3+C+RT2+O1+RT+4
DF=AK1+Z24P-2.5*AK12+GBZ+2Z24
XX=SCF*X(I,J)
YY=SCF*Y(I,J)
20 CALL PLOT(XX,YY+2)
IF(J.EQ.1) GO TO 40
J=J-1
K=1
IF(J.GE.KL) K=2
KK=K+1
I=NH
XX=SCF*X(I,J)
YY=SCF*Y(I,J)
CALL PLOT(XX,YY+3)
I=I-1
25 XX=SCF*X(I,J)
YY=SCF*Y(I,J)
CALL PLOT(XX,YY+2)
IF(I.EQ.K) GO TO 26
I=I-1
GO TO 25
26 IF(J.EQ.1) GO TO 40
J=J-1
GO TO 30
40 J=2
46 K=1
IF(J.GE.KL) K=2
KK=K+1
I=NH
XX=SCF*X(I,J)
YY=SCF*Y(I,J)
CALL PLOT(XX,YY+3)
I=I-1
41 XX=SCF*X(I,J)
YY=SCF*Y(I,J)
CALL PLOT(XX,YY+2)
IF(I.EQ.K) GO TO 42
I=I-1
GO TO 41
42 IF(J.EQ.NS2) GO TO 37
J=J+1
IF(J.GE.KL) K=2
KK=K+1
XX=SCF*X(K,J)
YY=SCF*Y(K,J)
CALL PLOT(XX,YY+3)
DO 45 I=KK,NH
XX=SCF*X(I,J)
YY=SCF*Y(I,J)
45 CALL PLOT(XX,YY+2)
IF(J.EQ.NS2) GO TO 37
J=J+1
K=1
IF(J.GE.KL) K=2
KK=K+1
GO TO 46
37 XX=SCF*U(I,2)
YY=SCF*T(I,2)
CALL PLOT(XX,YY+3)
DO 38 I=3,NSP
XX=SCF*U(I,2)
YY=SCF*T(I,2)
CALL PLOT(XX,YY+3)
XX=SCF*X(NH,NS)
YY=SCF*Y(NH,NS)
CALL PLOT(XX,YY+3)
XX=SCF*U(I,6)
YY=SCF*T(MP,6)
CALL PLOT(XX,YY+2)
50 I=NH
60 NSP=NS1
IF(I.EQ.1) NSP=NS
61 J=NSP
XX=SCF*X(I,J)
YY=SCF*Y(I,J)
CALL PLOT(XX,YY+3)
J=J-1
51 XX=SCF*X(I,J)
YY=SCF*Y(I,J)
CALL PLOT(XX,YY+2)
IF(J.EQ.1) GO TO 52
J=J-1
GO TO 51
52 DO 55 J=2,NSP
XX=SCF*X(I,J)
YY=SCF*Y(I,J)
55 CALL PLOT(XX,YY+2)
63 IF(I.EQ.1) GO TO 77
I=I-1
IF(I.EQ.1) NSP=NS
67 J=NSP
XX=SCF*X(I,J)
YY=SCF*Y(I,J)
CALL PLOT(XX,YY+3)
J=J-1
56 XX=SCF*X(I,J)
YY=SCF*Y(I,J)
CALL PLOT(XX,YY+2)
IF(J.EQ.1) GO TO 57
J=J-1
GO TO 56
57 DO 58 J=2,NSP
XX=SCF*X(I,J)
YY = SCF * Y(I,J)
58 CALL PLOT(XX,YY,2)
59 IF (I.EQ.1) GO TO 77
   P = I-1
60 GO TO 60
77 KK = 0
   AKK = 1.0
62 = 5
65 XX = SCF * U(2,J) * AKK
   YY = SCF * T(2,J)
   CALL PLOT(XX,YY,3)
   X1 = XX
   Y1 = YY
   MPP = MPP
   IF (J.EQ.5) MPP = MP
   DO 70 I = 3, MPP
   X2 = SCF * U(I,J) * AKK
   Y2 = SCF * T(I,J)
   CALL C PLOT(X1,X2,Y1,Y2,DS)
   X1 = X2
70 Y1 = Y2
   I = MPP
   IF (J.EQ.3) GO TO 75
   J = J-1
   XX = SCF * U(I,J) * AKK
   YY = SCF * T(I,J)
   CALL PLOT(XX,YY,3)
   X1 = XX
   Y1 = YY
   I = I-1
62 X2 = SCF * U(I,J) * AKK
   Y2 = SCF * T(I,J)
   CALL C PLOT(X1,X2,Y1,Y2,DS)
   X1 = X2
   Y1 = Y2
   IF (J.EQ.2) GO TO 84
   J = J-1
   GO TO 70
73 IF (J.EQ.3) GO TO 75
   J = J-1
   GO TO 65
75 I = 2
80 XX = SCF * U(I+2) * AKK
   YY = SCF * T(I+2)
   CALL PLOT(XX,YY,3)
   X1 = XX
   Y1 = YY
   M = 5
62 IF (I.LT.21) GO TO 86
   M = 4
   I = I-1
IK = IK+4
IF (IK.EQ.II) MOD = 6
86 CONTINUE
   DO 82 J = 3, MOD
   X2 = SCF * U(J,J) * AKK
   Y2 = SCF * T(J,J)
   CALL C PLOT(X1,X2,Y1,Y2,DS)
   X1 = X2
82 Y1 = Y2
   IF (I.EQ.3) GO TO 79
   I = I-1
   MOD = 5
   IF (I.LT.21) GO TO 87
   I = I-1
   IK = IK+4
   IF (IK.EQ.II) MOD = 6
87 CONTINUE
   XX = SCF * U(I+MD) * AKK
   YY = SCF * T(I+MD)
   CALL PLOT(XX,YY,3)
   X1 = XX
   Y1 = YY
   J = MOD-1
83 X2 = SCF * U(I,J) * AKK
   Y2 = SCF * T(I,J)
   CALL C PLOT(X1,X2,Y1,Y2,DS)
   X1 = X2
   Y1 = Y2
   IF (J.EQ.2) GO TO 84
   J = J-1
   GO TO 83
84 IF (I.EQ.3) GO TO 79
   I = I-1
   GO TO 85
79 IF (KK.EQ.1) GO TO 88
   KK = 1
   AKK = 1.0
   GO TO 80
88 KK = 1
   AKK = FLOAT(KK)
89 DO 15 J = 10, N52
   Z1 = (I+J)*SCF
   Z2 = (I+J)*SCF
   RT = X(I,J)+SCF
   CALL PLOT(RT,ZT,3)
   Z1 = (J+2)*SCF
   Z2 = (J+2)*SCF
   R1 = X(I+2,J)+SCF
   R2 = X(I+2,J)+SCF
   AA = (Z2*Z2-Z1*Z1)/(R2-R1)

BB = Z1*Z1 - AA*R1
AXIS = BB/AA
DELY = RT - AXIS
DDX = DELX/10.
DZ = Z1/10.
NT = 1
16 Y = DX*FLOAT(NT)
   DX = DDX*FLOAT(10-N)
   XX = (YY*YY - BB)/(AA + DX)
   CALL PLOT(XX,YY)
   IF (NT .EQ. 10) GO TO 15
   NT = NT + 1
   GO TO 16
15 CONTINUE
   XX = U(1,2) + SCF*AKK
   YY = T(1,2) + SCF
   BB = Y1 - YY
   X1 = BB/AA
   AXIS = BB/AA
   CALL PLOT(XX)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
CALL R2SIX(IOT, ZT, RD, R1, R2, R3, R4, S, Z6, Z0, Z1, Z2, Z3, Z4, Z5)
A FOR I SUB7
SUBROUTINE AXISZ(SUMZ, DPS)
COMMON R(150,50), Z(150,50), U(250,6), T(200,6), O(100,5),
S(100,5), P(100,3), W(100,2), ERF, W1, NS, NH, NS1, NS2, NS3,
NH1, NH2, NH3, N4, NH4, NH5, NH6, FS2, FH, NH5,
NHF, FN, HMP, M1, MP2, MP1, MP3, MPN, MP1, M4, MP, M4, MPN, M4, MP, M4, MPN, M4, MP
SGB, GBN, GBNH, GBNN, AKO, ENTA, ETA, NRO, O, G, G2, BI, B12, H
BB=0, O=0, D=12, 16
DO 150 I = 2, NH1
IP = I + 1
IN = I - 1
RC = 0.1*R(I, 2)*X2
ZS = Z(I, 2)
ZC = Z(I, 1)
NUM = 0
ZCP = Z(1, IP, 1)
ZCN = Z(IN, 1)
PHI = H/FHN*FLOAT(IN)
150 CONTINUE
RETURN
END

A FOR I SUB8
SUBROUTINE ZTTR(SUMR, SUMZ, DPS)
COMMON R(50,51), Z(50,50), U(200,6), T(200,6), O(100,5),
S(50,5), P(100,2), W(100,2), ERF, W1, NS, NH, NS1, NS2, NS3,
NH1, NH2, NH3, N4, NH4, NH5, NH6, FS2, FH, NH5,
NHF, FN, HMP, M1, MP2, MP1, MP3, MPN, MP1, M4, MP, M4, MPN, M4, MP, M4, MPN, M4, MP
SGB, GBN, GBNH, GBNN, AKO, ENTA, ETA, NRO, O, G, G2, BI, B12, H
BB=0, O=0, D=12, 16
DO 150 J = 2, NS2
JP = J + 1
JN = J - 1
AIP = IP
AIN = IN
PHI = H/FHN*AIN
DO 200 J = 2, NS2
JP = J + 1
JN = J - 1
150 CONTINUE
RETURN
END
INITIALIZATION OF SUBDIVIDED GRID NETWORK (Z)

DO 25 J=1,6
   T(JP,J)=D
25 CONTINUE

II=1
IK=1
DO 27 I=5,M+4
   IF(IK=I) THEN
      II=II+1
      TI(I+6)=Z(I+II)
      SI(K+1)=Z(I+II)
      TI(-2+6)=TI(-4+6)+5*(TI(-6)-TI(-4+6))
      TI(-3+6)=TI(-4+6)+5*(TI(-2+6)-TI(-4+6))
      TI(-1+6)=TI(-2+6)+5*(TI(-6)-TI(-2+6))
27 CONTINUE

II=II+1
DO 28 I=M+4,M+5
   TI(I+6)=D
   II=II+1
28 CONTINUE

300 DO 29 I=1,NH
   WI(I)=Z(I+NS2)
29 CONTINUE

II=II+1
CALL PATCH(X+1,N+1,N+1)
DO 35 I=1,NH
   IM=2*(II-1)
   III=III+1
   SI(II,1)=XI(II)
   SI(II,2)=XI(II-1)
   IF(I.EQ.0,0) SI(II+1,2)=0.5*(Z(I+NH)+Z(I+NH)+Z(I+1+NS2))
   SI(II+1,2)=0.5*(Z(I+NS1)+Z(I+1+NS1))
   SI(II+2,1)=Z(I+1+NS1)
   SI(II,1)=5*(SI(II+1,2))+SI(II+1,2)
   SI(II+1,3)=25*(Z(I+NS1)+Z(I+NS1)+Z(I+1+NS1)+Z(I+1+NS1))
   SI(II,1)=5*(SI(II+1,2))+SI(II+1,2)
   TI(I+1,1)=SI(II,1)+SI(II+1,2)
   TI(I+1,1)=5*(SI(II+1,2)+SI(II+1,2))
   TI(I+1,1)=5*(SI(II+1,2)+SI(II+1,2))
   TI(I+1,1)=5*(SI(II+1,2)+SI(II+1,2))
   IF(LIU.EQ.600) GO TO 35
   XI(II)=XI(II)+1
35 CONTINUE

II=II+1
CALL INITIATE R VALUES
DO 49 I=1,NH
   RI(I+1)=0.0
   R2=RI+RO*FNS
   RI(NH+1)=SQRT(R22)
   DO 51 J=3,NS1
      AN=J-1
      S(NP+J)=D
51 CONTINUE

S(NP+J)=D