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Centrality Measures of Graphs utilizing Continuous Walks in Hilbert Space

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Centrality Measures of Graphs utilizing Continuous Walks in Hilbert Space

Physics 4900 Research Project

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Abstract

Centrality is most commonly thought of as a measure in which we assign a ranking of the vertices from most important to least important. The importance of a vertex is relative to the underlying process being carried out on the network. This is why there is a diverse amount of centrality measures addressing many such processes. We propose a measure that assigns a ranking in which interference is a property of the underlying process being carried out on the network.
Abstract

Centrality is most commonly thought of as a measure in which we assign a ranking of the vertices from most important to least important. The importance of a vertex is relative to the underlying process being carried out on the network. This is why there is a diverse amount of centrality measures addressing many such processes. We propose a measure that assigns a ranking in which interference is a property of the underlying process being carried out on the network.

Introduction

Networks are perhaps one of the most ubiquitous structures in nature. They arise for example in cellular biology connecting genes and proteins [5, 1, 6], in neuroscience connecting neurological regions of the brain [1, 3, 7], in sociology connecting the interactions of people, and recently in quantum computing. The analysis of the underlying topology of these discrete structures has thus gained widespread attention. Likewise, there has been a significant focus on designing measures to assess certain topological features of a network by assigning quantitative values to the nodes. These quantitative values have a subtle interpretation insofar as there are implicit assumptions of the underlying process being carried out on the network.

Borgatti has identified a typology of flow processes with specific trajectories that use trails, geodesics, paths, or walks. In this framework the flow has a specific type of transmission corresponding to some concrete application. Borgatti gives examples such as used goods, currency, infections, and gossip. Suppose we want to model a flow process in which the flow may interfere with itself. This interference may be the result of collisions in the network where oppositely oriented flows may annihilate. How then can we model such a flow? Our proposition is to model continuous walks on the network insofar as interference becomes an emergent property.

Definition: Centrality is a measure in which the nodes of a network are assigned a ranking with respect to an implicit assumption of the flow characteristics of the network.
The adjacency matrix of a simple graph is defined as follows,

\[ a_{ij} = \begin{cases} 
1, & \text{if } i \text{ is adjacent to } j \\
0, & \text{otherwise} 
\end{cases} \]

where both \( i \) and \( j \) are nodes in the graph. One of the earliest and simplest centrality measures, Degree Centrality, was developed by Freeman [11]. It is defined as,

\[ \text{deg}(i) = \sum_{j=1}^{N} a_{ij} = (Ae)_i \]

which simply counts all nodes adjacent to \( i \). Here, \( e \) is the vector of all one’s. In this manner degree centrality measures the local connectivity of the graph. To this end degree centrality can be intuitively thought of as the immediate risk of coming into contact with what is flowing through the network.

The generalization of degree centrality is Katz Centrality [4] where both the immediate neighbors and their connections are counted under the influence of an attenuation parameter \( \alpha \). Katz centrality is then defined as,

\[ \kappa(i) = \sum_{j=1}^{N} \sum_{k=0}^{\infty} \alpha^k (A^k)_{ij} = ((I - \alpha A)^{-1} - I)e)_i \]

where \( I \) is the \( N \)-dimensional identity matrix and where \( \alpha \) is an arbitrary constant which must be smaller than the reciprocal of the largest eigenvalue of \( A \). Katz centrality describes the long term behavior of random walks on the network where every additional step is attenuated by \( \alpha \). Katz centrality can then be thought of as the long-term risk of coming into contact with what is flowing through the network. Another important centrality measure is closeness centrality, [4] defined as,

\[ C(i) = \left[ \sum_{j=1}^{n} d(i,j) \right]^{-1} \]

where \( d(i,j) \) is the graph-theoretic distance from nodes \( i \) to \( j \). Closeness centrality is then the sum of the inverse of the shortest path between nodes \( i \) and \( j \). This measure is intuitively thought of as the expected time it takes for something to flow through the network [9].
All centrality measures may be classified as either being radial or medial. Radial measures count specific walks between nodes whereas medial measures count specific walks that pass through some given node. A walk of length \( n \) is an ordered list of nodes \( i, u_1, u_2, \ldots, u_{n-1}, j \) such that all nodes in the list are connected. Furthermore the nodes need not be distinct and may include both \( i \) and \( j \). A closed walk is a walk that both begins and ends at the same node.

We may enumerate all walks of length \( n \) by taking successive powers of the adjacency matrix,

\[
(A^n)_{ij} = \sum_{u_1=1}^{N} \sum_{u_2=1}^{N} \cdots \sum_{u_{n-2}=1}^{N} \sum_{u_{n-1}=1}^{N} a_{i,u_1} a_{u_1,u_2} \cdots a_{u_{n-2},u_{n-1}} a_{u_{n-1},j}
\]

The rest of this paper focuses on lifting the restriction of \( n \) strictly being an integer. Of course then the above equation is useless so we introduce the matrix function \( f(a) = a^x \) where \( x \in \mathbb{R} \) and apply it to the adjacency matrix.

**Results**

**Lemma 1.1:** The power-law function \( f(a) = a^x, \ x \in \mathbb{R} \) of the adjacency matrix produces complex functions as the entries.

**Proof:** Let \( A \) be the adjacency matrix of a simple nonempty graph. \( A \) is a traceless symmetric matrix, \( \text{Tr}[A] = [A, A^T] = 0 \). Since it is symmetric it is always diagonalizable, we then have \( A = PDP^{-1} \) where \( P \) are the eigenvectors collected as a matrix and \( D \) is the diagonal matrix consisting of the eigenvalues of \( A \). We then have \( \text{Tr}[A] = \text{Tr}[PDP^{-1}] = \text{Tr}[D(PP^{-1})] = \text{Tr}[D] = \sum \lambda_i = 0 \). Since the graph is nonempty and the sum of the eigenvalues is zero we are therefore guaranteed to have at least one negative eigenvalue. The function of a matrix can be expressed as \( f(A) = Pf(D)P^{-1} \), where the spectral decomposition is,

\[
A^x = PD^xP^{-1} = \sum_i \lambda_i^x u_i u_i^T = \sum_i \lambda_i^x u_i u_i^T
\]

and where \((-\lambda)^x = \lambda^x e^{i\pi x} \). We then can express every entry of \( A^x \) as,

\[
\varphi_{jk}(\lambda, x) = \sum_{i \in \Lambda \setminus \{0\}} \lambda_i^x u_{ik} + e^{i\pi x} \sum_{i \in \Lambda} \lambda_i^x u_{ik}
\]

4
where $\Lambda^+ \setminus \{0\}$ is the multiset of all positive eigenvalues not including zero and $\Lambda^-$ is the multiset of all negative eigenvalues. Since we are guaranteed at least one negative eigenvalue $\varphi_{jk}$ is complex always.

**Definition:** Hilbert Space 1 is given by the set of functions defined on the interval $(0 \leq x \leq L)$ with finite norm and Hilbert Space 2 is given by the set of functions defined on the whole $x$ interval.

$$\|\varphi\|^2 = \int_0^L \varphi^* \varphi \, dx < \infty \, \mathcal{H}_1$$

$$\|\varphi\|^2 = \int_{-\infty}^{\infty} \varphi^* \varphi \, dx < \infty \, \mathcal{H}_2$$

**Properties:**

1) Hilbert Space is linear. A function space is linear when the two following conditions hold: 1) If $\alpha$ is a constant and $\varphi$ is an element of the space then $\alpha \varphi$ is also an element of the space and 2) if $\psi$ and $\varphi$ are any two elements of the space then $\psi + \varphi$ is also an elements of the space.  

2) There exists an inner product, $\langle \psi | \varphi \rangle$, for any two elements of the space defined over the appropriate interval.  

3) Any element of the space has a norm (length) defined as $\|\varphi\|^2 = \langle \varphi | \varphi \rangle$.  

4) $\mathcal{H}$ is complete. Every Cauchy sequence of functions in $\mathcal{H}$ converges to an element of $\mathcal{H}$. A Cauchy sequence $\{\varphi_n\}$ is such that $\|\varphi_n - \varphi_l\| \to 0$ as $n$ and $l$ approaches infinity [12].

**Theorem 1.1:** $\varphi_{jk}$ is an element of Hilbert Space.

**Proof:** Taking the complex conjugate we get,

$$\varphi_{jk}^* \varphi_{jk} = \left( \sum_{l \in \Lambda^+ \setminus \{0\}} \lambda_l^x u_{ilk} + e^{-i\pi x} \sum_{l \in \Lambda^-} \lambda_l^x u_{ilk} \right) \left( \sum_{l \in \Lambda^+ \setminus \{0\}} \lambda_l^x u_{ilk} + e^{i\pi x} \sum_{l \in \Lambda^-} \lambda_l^x u_{ilk} \right)$$

$$= \left[ \sum_{l \in \Lambda^+ \setminus \{0\}} \lambda_l^x u_{ilk} \right]^2 + 2 \left[ \sum_{l \in \Lambda^-} \lambda_l^x u_{ilk} \right]^2 + 2 \text{Cos}(\pi x) \left[ \sum_{l \in \Lambda^+ \setminus \{0\}} \lambda_l^x u_{ilk} \right] \left[ \sum_{l \in \Lambda^-} \lambda_l^x u_{ilk} \right]$$

$$= \sum_{n \in \Lambda} \lambda_n^x u_{nk}^2 + 2 \text{Cos}(\pi x) \sum_{l \in \Lambda^-} \lambda_l^x u_{ilk} \left[ \sum_{l \in \Lambda^-} \lambda_l^x u_{ilk} \right] \text{ where } \Lambda = \Lambda^+ \setminus \{0\} \cup \Lambda^-$$

$$= \sum_{n \in \Lambda} \lambda_n^x u_{nk}^2 + 2 \sum_{m < n} \lambda_m \lambda_n x u_{nm} + 2 \text{Cos}(\pi x) \sum_{l \in \Lambda^-} \sum_{l \in \Lambda^-} \lambda_l^x u_{ilk}$$

where $u_{nm} = u_{nk} u_{mk}$ and $u_{ll'} = u_{lk} u_{lk}$ respectively. Finally upon integration we have,
\[ \int_0^L \varphi_{jk} \varphi_{jk} \, dx = \sum_{n \in \Lambda} u_{nk}^2 \int_0^L (\lambda_n)^{2x} \, dx + 2 \sum_{m < n} u_{nm} \int_0^L (\lambda_n \lambda_m)^x \, dx + 2 \sum_{i \in \Lambda \setminus \{0\}} \sum_{\nu \in \Lambda} u_{\nu i} \int_0^L \cos(\pi x) (\lambda_i \lambda_i')^x \, dx \]

\[ = \sum_{n \in \Lambda} u_{nk}^2 \frac{\lambda_n^{2L} - 1}{2 \ln(\lambda_n)} + 2 \sum_{m < n} u_{nm} \frac{(\lambda_n \lambda_m)^L - 1}{\ln(\lambda_n \lambda_m)} + 2 \sum_{i \in \Lambda \setminus \{0\}} \sum_{\nu \in \Lambda} \frac{\ln(\lambda_i \lambda_i') - (\lambda_i \lambda_i')^L (\cos(\pi L) \ln(\lambda_i \lambda_i') + \pi \sin(\pi L))}{\pi^2 + (\ln(\lambda_i \lambda_i'))^2} \]

On the right-hand side of the integral we have two indeterminates of the form \( \frac{0}{0} \) when \( \lambda_n \to 1 \) and when \( \lambda_n \lambda_m \to 1 \). Upon a change of variable the limit is,

\[ \lim_{x \to 1} \frac{x^L - 1}{\ln(x)} = \lim_{x \to 1} \frac{x^{2L} - 1}{2 \ln(x)} = L \]

The integral then converges over the interval and we have the desired result, \( \varphi_{jk} \in \mathcal{H}_1 \). This is an interesting mathematical result that relates the combinatorial structure of graphs to elements of Hilbert Space. Essentially, what we have done is allowed continuous walks on a discrete structure which has the equivalence of taking a vector space and completing it. Recall that a walk is an ordered list of nodes where the number of nodes is the length of the walk. By allowing walks to be any real length we lose the ability of listing the nodes in any order. By labeling the nodes of a graph they can be considered discrete locations within the graph. When asking how many ways there are to go from node’s \( i \) to \( j \) in \( n \) steps \( (n \in \mathbb{Z}) \) we can explicitly list every location involved from \( i \) to \( j \). However, when we ask how many ways are there to go from node \( i \) to \( j \) in \( x \) steps \( (x \in \mathbb{R}) \) we cannot explicitly list any location involved from \( i \) to \( j \). Below we plot a few \( \varphi_{jk} \)’s for several graphs.

Figure 1A. Real and Imaginary parts of \( \varphi \) for \( P_2 \) (Path Graph on two nodes)
A natural question to ask is what structural features of the graph are expressed in $\varphi(x)$ and more importantly which of these features can be obtained from operations on $\varphi(x)$? First and foremost let’s recover the adjacency relation. Since $\varphi(x)$ counts all walks of length $x$ and the adjacency relation is given by all walks of length 1 then $\varphi_{jk}(1) = a_{jk}$, written in terms of an inner product we have $\langle \delta(x - 1) | \varphi_{jk}(x) \rangle = a_{jk}$. This begs the question, does there exist functions for which the inner product produces specific graph parameters?
Figure 1D. Real and Imaginary parts of $\varphi$ for $C_4$ (Cycle Graph on 4 nodes)

To assign quantitative values to the nodes we introduce the ensemble vector,

$$\zeta_j = \sum_{k=1}^{N} \varphi_{jk}$$

which counts all walks of length $x$ from node $j$ to all other nodes in the network. Below we plot several ensemble vectors from the previous graphs.

Figure 2A. $\zeta$ for $P_2$

Figure 2B. $\zeta$ for $K_3$ and $C_4$
We may now define a particular degree measure in which we have a superposition of interference terms from nodes $j$ with respect to all other nodes. We may call this Degree-Interference and is formally defined as,

$$\mathcal{D}_I j \equiv \| \mathcal{J}_j \| = \sqrt{\langle \mathcal{J}_j | \mathcal{J}_j \rangle} = \sqrt{\int_0^1 \mathcal{J}_j^* \mathcal{J}_j dx} = \sqrt{\int_0^1 \left( \sum_{k=1}^N \varphi_{jk}^* \right) \left( \sum_{k' = 1}^N \varphi_{jk'} \right) dx}$$

This measure is simply the length or magnitude of the ensemble vector defined over the $(0,1)$ interval. It is an analogue of the degree of a node since the magnitude of the $\mathcal{J}_j$ is defined over an interval which contains the adjacency relation. Degree centrality counts all walks of length one whereas Degree-Interference counts all walks that constructively interfere from zero to one. The implicit assumption of the flow characteristics on the network is a flow that can interfere with itself. Recall that this is entirely a topological phenomenon by considering continuous walks on the network.

The generalization of degree-interference is Katz-Interference which is the magnitude of the weighted ensemble vector defined over the $(0, \infty)$ interval. To assure convergence we must weigh the adjacency matrix with an attenuation coefficient $\alpha$ such that $\alpha^{-1} > \max \Lambda$. Katz-Interference is then given by,

$$\mathcal{K}_I j \equiv \| \mathcal{J}_j \| = \sqrt{\langle \mathcal{J}_j | \mathcal{J}_j \rangle} = \sqrt{\int_0^\infty \mathcal{J}_j^* \mathcal{J}_j dx} = \sqrt{\int_0^\infty \left( \sum_{k=1}^N \varphi_{jk}^* \right) \left( \sum_{k' = 1}^N \varphi_{jk'} \right) dx}$$

As Katz centrality is a generalization of degree centrality Katz-Interference is the generalization of Degree-Interference, where we count all walks of all possible lengths that constructively interfere. Below we give several examples of both measures.
<table>
<thead>
<tr>
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<th>$D_{Ij}$</th>
<th>Katz</th>
<th>$K_{Ij}$</th>
</tr>
</thead>
<tbody>
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<td>1.17</td>
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<td>1.05</td>
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<tr>
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<td>2</td>
<td>1.17</td>
<td>1.25</td>
<td>1.05</td>
</tr>
<tr>
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<td>2</td>
<td>1.17</td>
<td>1.25</td>
<td>1.05</td>
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<th>Degree</th>
<th>$D_{Ij}$</th>
<th>Katz</th>
<th>$K_{Ij}$</th>
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<td>1.28</td>
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<td>3</td>
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<td>1.05</td>
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<td>3</td>
<td>1.28</td>
<td>1.42</td>
<td>1.05</td>
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<tr>
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<th>Katz</th>
<th>$K_{Ij}$</th>
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<td>1.25</td>
<td>1.06</td>
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<td>1.04</td>
<td>1.25</td>
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<td>1.25</td>
<td>1.06</td>
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<th>Degree</th>
<th>$D_{Ij}$</th>
<th>Katz</th>
<th>$K_{Ij}$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>0.96</td>
<td>1.12</td>
<td>0.89</td>
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<td>2</td>
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<td>1.18</td>
<td>1.23</td>
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<td>4</td>
<td>1</td>
<td>0.96</td>
<td>1.12</td>
<td>0.89</td>
</tr>
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</table>
Proposition 1: \[ \| \zeta_i - \zeta_j \| = 0 \ \forall \ i, j \equiv G \text{ is k-regular} \]

Proof: A graph is k-regular iff the principle eigenvalue is \( \lambda_0 = k \) with eigenvector \( u_0 = (1,1,\ldots,1) \) and where \( \sum_{i=1}^{n} u_i = 0 \). Recall that the spectral decomposition can be written as \( a_{ik}x = \sum_{i=0}^{n} \lambda_i x u_i j v_{jk} \) where \( u_{ij} \) are the eigenvectors collected as a matrix and \( v_{jk} \) is the inverse of that matrix. We then have \( v_0 = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \) where \( \sum_{k=1}^{n} u_k = 0 \). We then express the matrix product as a sum of outer-products,

\[
a_{ik}x = \lambda_0 x \left[ \begin{array}{c} 1 \\ 1/n \\ 1/n \\ \vdots \\ 1/n \\ 1/n \\ 1/n \end{array} \right] + \lambda_1 x v_1 u_1 + \cdots + \lambda_i x v_i u_k
\]

and then sum over the columns,

\[
\sum_k a_{ik}x = \sum_k \lambda_0 x \left[ \begin{array}{c} 1 \\ 1/n \\ 1/n \\ \vdots \\ 1/n \\ 1/n \\ 1/n \end{array} \right] + \lambda_1 x v_1 u_1 + \cdots + \lambda_i x v_i u_k
\]

\[
= \lambda_0 x + 0 + \cdots + 0
\]

\[
= k^x
\]

and therefore \( \| k_i^x - k_j^x \| = 0 \ \forall \ i, j \).
We may now define a particular distance measure which is the analogue of closeness centrality. However, from Proposition 1 we see this measure is ill-suited for regular graphs. Closeness-Interference is defined as,

\[ CL_i = \left[ \sum_{j=1}^{n} d(\xi_i, \xi_j) \right]^{-1} = \left[ \sum_{j=1}^{n} ||\xi_i - \xi_j|| \right]^{-1} = \left[ \sum_{j=1}^{n} \sqrt{\langle \xi_i - \xi_j | \xi_i - \xi_j \rangle} \right]^{-1} \]

which is the distance between all pairs of ensemble vectors. Again this is a measure of how close two nodes are when considering continuous walks between all pairs of nodes. How close any given pair of nodes are depends on how the walks interfere with each other. Below we give two examples of closeness-interference.

<table>
<thead>
<tr>
<th>Node</th>
<th>Closeness</th>
<th>(CL_i)</th>
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</thead>
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<tr>
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<td>0.75</td>
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<td>0.15</td>
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<tr>
<td>4</td>
<td>1</td>
<td>0.14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Node</th>
<th>Closeness</th>
<th>(CL_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.66</td>
<td>0.77</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.61</td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>0.77</td>
</tr>
</tbody>
</table>

**Discussion**

The concepts of a continuous walk seem counterintuitive at face value. What does it mean to have a continuous walk on a graph? Continuous walks on graphs live in Hilbert Space which is a complete metric space whereas the discrete walk does not. As a consequence, the phenomenon of interference manifests itself in the topology of the graph. The trajectory of a random walker on the graph is influenced by the global structure of the graph rather than its local structure when restricting the distance the walker may travel. What we mean by this is that the total number of ways from moving from one vertex to the next depends on how all
pairs of vertices interfere with each other regardless of how far the walker is allowed to travel. We may restate this by only allowing a random walker to move in an induced subgraph of $G$. The behavior of the random walk depends only on the topology of the induced subgraph. The behavior of a continuous walk however would depend on the topology of the entire graph since there are nonzero interference terms with respect to vertices inside and outside of the induced subgraph. Stated in terms of the quantum tunneling phenomenon if we allow the random walker to move only a length of 5 steps we may find the walker on a vertex that requires a length of 6 steps to reach.

The above can be relaxed by simply thinking of a special type of flow on a network that can interfere with itself. This flow captures the mathematical features of quantum particles without explicitly using the operators of quantum mechanics. To get an idea of the behavior of this flow we defined three centrality measures which are the analogues of degree, katz, and closeness centrality. From the comparison of degree to degree-interference and katz to katz-interference the rankings were almost identical with one exception in the kite graph. The comparison between closeness and closeness-interference however had an inverse relationship where the peripheral vertices are more central. Here vertices that are more central under closeness centrality destructively interfere under closeness-interference allowing the peripheral vertices to be the most central. We have shown that by allowing continuous walks on a graph we get infinite-dimensional vectors in Hilbert Space that can be used to construct centrality measures influenced by interference among all pairs of vertices. Using such measures we have only scratched the surface of the topological consequences of continuous walks on discrete structures. Can we use these complex functions to count specific substructures of graphs by a clever manipulation of destructive and constructive interference of walks? Is there a deeper connection between graph theory and quantum mechanics? There are many more questions to be asked and to be answered which merits further investigation.

References


Physics 4900 Summary

Jarod P. Benowitz

The 4900 project followed a meandering path starting with the investigation of cellular automata, moving into the investigation of the Graph Isomorphism problem, dabbling with possible cryptographic applications, and ultimately ended with centrality measures utilizing Hilbert Space. At the time I became fascinated with Graph Theory primarily because of its ubiquity and flexibility in the natural sciences. The broad range of applicability of graphs in the sciences was attractive and left room for new and novel applications. In the beginning I was interested in using topologically dynamic graphs to model cellular automata, however due to the computationally exhaustive nature of the approach it was abandoned. At this point I decided I was going to choose an unsolved problem in graph theory and attempt to solve it. I chose the Graph Isomorphism problem because of its status and wanted a challenge. All previous attempts for the past 40 or so years have failed. Because of this I wanted to approach the problem from a completely different perspective.

The Graph Isomorphism (GI) problem has remained one of the most important unsolved problems in discrete mathematics. Distinguishing whether or not two graphs are structurally identical in polynomial time has its applications in diverse fields of study. In bioinformatics identifying which conformations of a protein have the same function remains a crucial task, likewise in chemoinformatics identifying which conformations of a molecule have the same
Chemical Hamiltonian is crucial, in computational complexity which time complexity class GI is classified in is of widespread interest, and in cryptography using GI as a trapdoor function is also of interest. Solving GI would thus be incredibly beneficial to many scientific communities. Perhaps most importantly is GI’s relationship with the millennium prize problem, P vs NP. Significant progress in solving GI has a direct correlation with P vs NP and may help pave the road to its solution.

I wanted a unique way of looking at the problem. It seemed apparent that the standard mathematical arsenal was insufficient. Either novel algorithms needed to be invented or there was hidden structure that could be exploited. I took the latter approach assuming there was additional structure of the graph that can be exploited to solve GI. Many experts would vehemently disagree urging that there simply doesn’t exist graph parameters that are both necessary and sufficient for an isomorphism. If additional structure indeed does exist how could we find it? From here we reformulated the question to read: Does there exist a physical system in which the solution to GI can be encoded, and if so which physical system? The hypothesis assumes the existence of this ambiguous physical system where the goal is to derive its physical properties without any assumptions of what they may be. The prediction was that this physical system would most likely be quantum mechanical in nature. This draws from a commonly shared intuition among the quantum computing community where quantum algorithms are expected to have an exponential speed up with respect to their classical counterparts. Finding
the exact nature of this physical system has thus remained the primary challenge. Although as
the primary theorem of the paper demonstrates, Hilbert Space is a key indicator of a quantum
system.

We first began by taking a closer look at how graphs are traversed. Motion in a graph is
classified by walks, paths, trails, and geodesics. Since a graph is a discrete structure all motion
on the structure has thus been constrained to be discrete. Since we are treating graphs as
ambiguous physical systems we thought it would be fruitful to allow continuous motion on the
graph. A natural question to ask is: what is the physical significance of having continuous
motion on a discrete structure? The reconciliation of continuous processes on underlying
discrete structures has remained the primary concept of the paper and as we will see
interference emerges from such a treatment. By raising the adjacency matrix to some integer
value \( n \) we enumerate all walks of that length from vertex \( i \) to \( j \). If we extend this to the
continuous case we simply allow \( n \) to be any real number. We use the power-law function,
\[
 f(a_{ij}) = a_{ij}^x
\]
to enumerate all walks of length \( x \) among all pairs of vertices. Consequentially, the number of possible walks of some continuous length is a complex number, see lemma 1.1
for further reference. This was interesting since on conceptual grounds it was very similar to
the Feynman Path Integral. Likewise, this result was consistent with the prediction that our
physical system of interest is quantum mechanical.
At this point we had derived functions that assign complex numbers to continuous walks in the graph. In this framework the classical graph-theoretic metric no longer applies which merited the development of an appropriate metric. The construction of this metric space was crucial to the additional structure of the graph we were aiming to exploit. Furthermore, the nature of the space should also be indicative of the physical system we seek. Keeping these concepts in mind we turned to information geometry as a potential reservoir of mathematical analysis. It seemed rather promising to treat these functions as points on a complex statistical manifold where the metric is a measure of how much information is lost when using one function to approximate another. In this manner the statistical manifold would be the canonical ensemble of our physical system where the graph represents a particular state of the system. GI then relaxes to identifying whether or not two physical systems are in the same state. The identification scheme then follows from computing the distance between all pairs of vertices of the respective graphs using the Hellinger Metric. We conclude that the respective graphs are isomorphic if and only if there is a one-to-one correspondence of zero distances. This was the reasoning at the time and as we shall see this method did not succeed.

As we discovered for certain pathological cases, i.e. regular graphs every pairwise distance was zero. In proposition 1 we demonstrated that GI under this framework was metric independent. In other words regardless of any metric one may choose an isomorphism cannot be distinguished. Regular graphs admitted symmetry that more or less cloaks structural
information of the graph. Perhaps there really does not exist any graph parameter that is both necessary and sufficient for an isomorphism. This phase of the project became rather discouraging as I began to abandon GI all together. However, lemma 1.1 still had its utility so I began looking for possible applications. It occurred to me that lemma 1.1 could be used to generalize the graph power which may be used in a cryptographic scheme.

The graph power is defined as,

\[ \text{Adj}(G^k) \equiv \sum_{i=1}^{k} A^i \]

where the resulting graph has the same set of vertices, but two vertices are adjacent if and only if their distance is at most \( k \). Generalizing this to continuous distances we get an inner product,

\[ \text{Adj}(G^x) \equiv \int_0^x \varphi^* \varphi dx = \langle \varphi | \varphi \rangle = \partial_{ij} \]

where two vertices are adjacent if and only if \( d(\zeta_i, \zeta_j) \leq \partial_{ij} \), where \( \zeta_i \) is the ensemble vector defined in the paper. In this manner the interference terms govern the topology of the resulting graph. Constructive interference allows edges between vertices whereas destructive interference does not. The cryptographic scheme follows by using the exponent as a private key and the underlying graph as a public key where we would have \( (G^{x_{alice}})^{x_{bob}} = (G^{x_{bob}})^{x_{alice}} \). It is necessary and sufficient to prove that the above equality holds for all public and private keys.
and that the encryption-decryption scheme can be done in a reasonable amount of time. However, it can be easily shown that solving for $x_{alice}$ given $G^{x_{alice}}$ is extraordinarily difficult since $x_{alice}$ is not the only solution. While I found this application interesting I failed at gaining significant ground with the attempted proof techniques. At this time the 4900 project was scattered and lacked a clear end goal but fortunately I had one last idea that would become the primary theorem of the paper.

It occurred to me one day that perhaps $\varphi$ is an element of Hilbert Space. As the primary theorem of the papers proves this is indeed the case. This is fascinating because to my knowledge it is the first explicit relationship between graphs and Hilbert Space. Furthermore, this is indirect evidence that the ambiguous physical system that may encode GI is indeed quantum mechanical. Moreover we may view graphs as lower-dimensional discrete projections of Hilbert Space. This was the additional structure I was so desperately trying to find. So how can we exploit Hilbert Space from the perspective of graph theory? The body of the paper attempts to qualitatively understand the nature of this relationship by generalizing several centrality measures and applying them to simple graphs.

I was curious as to how the structure of Hilbert Space would affect the importance of nodes in a network. The importance of nodes in a network is captured by centrality measures which assigns a ranking of the nodes with respect to implicit assumptions of the flow characteristics of the network. In this respect the flow may interfere with itself both
constructively and destructively. This was exciting since to my knowledge there has never been a serious attempt at developing centrality measures for networks in which interference is a flow characteristic. Furthermore, a tantalizing aspect of this work is that there should exist linear hermitian operators for every graph parameter just as there exists linear hermitian operators for every physical observable.

In summary the 4900 Project was successful in that a discovery was made although the utility of the discovery is not completely understood. There are many brilliant minds in this world and I hope someone one day will take what I have laid down and use it in new and novel ways. Mathematical research is at times agonizing for the mere fact that what is being done may not see its utility for some time to come. I hope that one day someone will pick up where I left off and discover something truly remarkable.