Shades of grey:

a critical review of grey-number optimization

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Abstract -- A grey number is an uncertain number with fixed lower and upper bounds but unknown distribution. Grey numbers find use in optimization to systematically and proactively incorporate uncertainties expressed as intervals plus communicate resulting stable, feasible ranges for the objective function and decision variables. This paper critically reviews their use in linear and stochastic programs with recourse. It summarizes grey model formulation and solution algorithms. It advances multiple counter-examples that yield risk-prone grey solutions that perform worse than a worst-case analysis and do not span the stable feasible range of the decision space. The paper suggests reasons for the poor performance and identifies conditions for which it typically occurs. It also identifies a fundamental shortcoming of grey stochastic programming with recourse and suggests new solution algorithms that give more risk-adverse solutions. The review helps clarify the important advantages, disadvantages, and distinctions between risk-prone and risk-adverse grey-programming and best/worst case analysis.

Keywords: *interval number; linear program; stochastic program with recourse; optimization with uncertainty.*

Introduction

Over the last three decades, a variety of techniques have surfaced to optimize in the face of uncertainty. Techniques such as chance constraints, grey numbers, fuzzy numbers, probabilistic, possibilistic, flexible, and stochastic programs with recourse have been presented to systematically and proactively incorporate numerical uncertainties in optimization models (Sahinidis, 2004). Here, I review the proactive systems analysis technique of grey number optimization and suggest some modifications.

A grey number (also called an interval number) takes an unknown distribution between fixed lower and upper bounds, i.e., $w^{\pm} \in \left[w^{-}, w^{+}\right]$ or $w^{-} \leq w^{\pm} \leq w^{+}$, where w^- and w^+, are, respectively, the lower and upper bounds for *w*. In optimization, grey numbers find use to systematically and proactively incorporate uncertainties expressed as intervals plus communicate resulting stable, feasible ranges for the objective function and all decision variables. Grey number programs are decomposed into two computationally-efficient, interacting deterministic sub-models that are then solved sequentially. Decision makers

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use the resulting grey intervals for decision variables to select alternatives within proscribed bounds.

Grey numbers have been applied to a variety of linear (Ishibuchi and Tanaka, 1990, Huang, Baetz and Patry, 1992, Huang and Moore, 1993), mixed integer (Huang, Baetz and Patry, 1995, Huang, 1998), quadratic (Huang and Baetz, 1995, Li and Huang, 2007), and stochastic (Huang and Loucks, 2000, Maqsood and Huang, 2003, Maqsood, Huang and Zeng, 2004, Maqsood, Huang and Yeomans, 2005, Maqsood, Huang, Huang and Chen, 2005, Li, Huang and Nie, 2006, Li and Huang, 2006, Li, Huang and Baetz, 2006, Li, Huang, Nie and Huang, 2006, Li, Huang and Nie, 2007, Rosenberg and Lund, 2008) programs with applications including hypothetical numerical examples for solid waste management (Huang, Baetz and Patry, 1992, Huang, Baetz and Patry, 1995, Maqsood and Huang, 2003, Maqsood, Huang and Zeng, 2004, Li and Huang, 2006, Li, Huang, Nie, Nie and Maqsood, 2006), water resources allocation (Huang and Loucks, 2000, Maqsood, Huang and Yeomans, 2005, Maqsood, Huang, Huang and Chen, 2005, Li, Huang and Nie, 2006), and flood diversion planning (Li, Huang and Nie, 2007). Limited practical examples include for water quality management in China (Huang, 1998), solid waste management for the city of Regina (Li and Huang, 2006), and water system planning in Amman, Jordan (Rosenberg and Lund, 2008).

Apart from the practical examples, most grey optimization work has focused on model formulations and solution techniques for hypothetical examples. There has been little interpretation of solution results nor comparison to results from other solution approaches such as sensitivity or best/worst case analysis.

Sensitivity analysis (also called range-of-basis) is a *reactive* approach that—after solution—examines how or whether the optimal solution changes with changes in input parameter values. Sensitivity can be examined manually (changing an individual input, resolving, and noting solution changes) or by analyzing the range-of-basis output produced by most optimization solvers. Unfortunately, range-of-basis results apply only to *individual* changes in input parameters and not *combinations* of parameter changes as accommodated by grey-number approaches.

The long-standing approach of best/worst-case analysis simply solves a program twice for the combinations of parameter values that represent the most favourable (best case) and least favourable (worst case) conditions. Rosenberg and Lund (2008) compared a grey stochastic program with recourse to deterministic-equivalent, robust, and best/worst case formulations and found that the grey model performed worse than the worst-case analysis. This paper further explores reasons for the risk-prone performance, characterizes conditions under which the problem is likely to arise, and suggests alternative solution approaches that are more risk adverse.

The paper is organized as follows. Sections 2 and 3 review problem formulation and solution techniques for grey linear programs and grey stochastic programs with recourse. Each section identifies problems with existing solution techniques and characterizes situations in which these problems arise. Section 4 presents two alternative grey solution techniques that are more risk adverse. Section 5 discusses and highlights the important advantages, disadvantages, and distinctions between risk-prone and risk-adverse grey-programming and best/worst case analysis. Section 6 concludes.

Grey linear programming

Model formulation and solution

Early applications of grey linear programming incorporated grey numbers into the objective function (Ishibuchi and Tanaka, 1990), constraint matrix (Huang and Moore, 1993, Tong, 1994), right-hand sides of constraints, and all of the above (Huang, Baetz and Patry, 1992, Huang, Baetz and Patry, 1995, Huang, 1996). The process works as follows. A linear program with objective function *f*, decision variables X_i, objective function coefficients c_i, constraint matrix coefficients a_{ij}, and right-hand-side constraint coefficients b_j

$$\text{Max } f = \sum_i c_i X_i \tag{1a}$$

$$\text{s.t. } \sum_i a_{ij} X_i \le b_j, \forall j \tag{1b}$$

$$X_i \ge 0, \forall i \tag{1c}$$

is turned into a grey linear program by substituting grey numbers for each of the input coefficients $a^{\pm}$, $b^{\pm}$, and $c^{\pm}$. These substitutions turn the objective function ($f^{\pm}$) and decision variables ($X_i^{\pm}$) grey and yield the grey linear program (2):

$$\text{Max } f^{\pm} = \sum_i c_i^{\pm} X_i^{\pm} \tag{2a}$$

$$\text{s.t. } \sum_i a_{ij}^{\pm} X_i^{\pm} \le b_j^{\pm}, \forall j \text{ , and} \tag{2b}$$

$$X_i^{\pm} \ge 0, \forall i \tag{2c}$$

where $f^{\pm}$ is the uncertain grey objective function with lower- and upper bounds, respectively, f^- and f^+; similarly for the other decision variables and input coefficients.

We solve grey linear program (2) by decomposing it into two deterministic sub-models (Huang, 1996). The two sub-models correspond to the upper and lower bounds of the grey objective-function and interact.

For a maximization problem, solve the upper bound sub-model first. The upper bound sub-model corresponds to f^+ and uses input coefficients (c^+, a^-, and b^+) that maximize the objective function and allow X_i to reach their upper limits.

$$\text{Max } f^+ = \sum_i c_i^+ X_i^+ \tag{3a}$$

$$\text{s.t.} \ \sum_i a_{ij}^- X_i^+ \le b_j^+, \forall j \text{, and} \tag{3b}$$

$$X_i^+ \ge 0, \forall i \, . \tag{3c}$$

Model (3) is also the best-case formulation.

The lower bound sub-model corresponds to f^- and uses the input coefficients (c^-, a^+, and b^-) that minimize the objective function and force X_i towards their lower limits.

$$\text{Max} \ f^- = \sum_i c_i^- X_i^- \tag{4a}$$

$$\text{s.t.} \ \sum_i a_{ij}^+ X_i^- \le b_j^-, \forall j \text{, and} \tag{4b}$$

$$X_i^- \ge 0, \forall i \, . \tag{4c}$$

$$X_i^- \le X_{i\,opt}^+, \forall i \tag{4d}$$

Further, lower bound sub-model (4) also contains an interaction constraint (4d) that requires the lower bound solution (X_i^-) to be in the solution basis of the upper-bound sub-model ($X_i^+{}_{opt}$). The interaction constraint forces solution consistency across the upper- and lower-bound sub-models.

Model discussion and comparisons

Solutions to sub-models (3) and (4) span maximal, stable, feasible ranges for the objective function and decision variables. These ranges are $f^\pm{}_{opt} = [f^-, f^+]$ and $X^\pm{}_{i\,opt} = [X_i^-, X_i^+]$ where f^+ and $X^+{}_i$ are solutions to upper-bound sub-model (3) and f^- and $X^-{}_i$ are solutions to lower-bound sub-model (4). The interaction constraints allow decision makers to choose X_i between their ranges X_i^- and X_i^+ and be guaranteed an objective function value between f^- and f^+.

For a minimization problem, the solution order discussed above is reversed. First solve the lower bound sub-model (without interaction constraint (4d)). Second, solve the upper bound sub-model (with an interaction constraint $X_i^+ \ge X_{i\,opt}^-, \forall i$).

Further note that for a maximization problem the best case formulation is identical to the upper bound sub-model (3) while the worst case formulation is simply the lower bound sub-model (4) without the interaction constraint (4d). Because the Best/Worst case solutions do not limit the basis, solutions can help judge the system's capability to realize a desired goal but do not necessarily construct a set of stable solutions for generating decision alternatives.

Problems

For a simple optimization problem

$$\begin{aligned} &\text{Max}\,[3,4]X_1 + [5,6]X_2 \\ &X_1 + X_2 \le 1 \\ &X_1 \ge 0;\ X_2 \ge 0 \end{aligned} \tag{5}$$

the grey linear program formulation and solution algorithm gives the grey solution $f^{\pm}_{opt}$ = [5, 6]; $X_{1\,opt}$ = [0]; and $X_{2\,opt}$ = [1]. Here the grey solution is identical to the best/worst case solution and the solution basis for the two cases both contain X_2. However, when the lower bound of the objective function coefficient for X_2 changes from 5 to 2, the solutions diverge (Table 1). In this case the grey linear program identifies X_2 as part of the solution basis in the upper bound sub-model while seeking a maximum value for the objective function, but the interaction constraint excludes X_1 from the solution basis for the lower-bound sub-model. Consequently, the objective function value falls to 2. Absent the interaction constraint, the worst-case analysis switches the solution basis to X_1 with an improvement in the objective function value to 3. The grey linear program identifies the maximal, stable feasible range for the objective function but performs worse than the worst case. The lower-bound sub-model is more constrained than the worst-case sub-model.

Moreover, because the interaction constraint forces X_1 to stay at zero, the grey linear program fails to report the full stable, feasible range for decision variable X_1 (i.e., for unfavourable coefficient values it is preferable to implement X_1). Similar performance worse than the worse case and failure to report the full stable feasible range for the decision variables is also seen when the constraint matrix coefficient for X_2 changes from [1, 1] to [1, 4] (Table 1).

Solution mischaracterization and performance worse than the worse case identify the grey solution algorithm as risk prone. When unfavourable conditions arise, a decision maker implementing the grey-number solution could do better by adopting a worst-case or possibly other solution.

Conditions under which the problems arise

A retrospective analysis of grey linear program examples (Huang, Baetz and Patry, 1992, Huang and Moore, 1993, Huang, Baetz and Patry, 1995, Tong, 1994, Huang, 1996) shows that several of grey-number solutions perform worse than the worst case (Table 2). The range of the grey objective function is wider than objective function range for the best/worst cases. However, this analysis was complicated by the facts that several of the works are (i) infeasible as published (Huang, Baetz and Patry, 1992, Huang, Baetz and Patry, 1995, Tong, 1994), (ii) instead use a best/worst case solution algorithm (without interaction constraints) but call it a grey solution technique (Huang, Baetz and Patry, 1992, Huang and Moore, 1993, Tong, 1994), or (iii) do not present enough input data to verify the published solution (Huang and Moore, 1993, Huang, 1998, Huang, 1996, Yeh, 1996).

Reworking with feasible data and the grey solution algorithm shows that several of the examples perform no different that best/worst case analysis (Huang, Baetz and Patry, 1992, Huang and Moore, 1993, Huang, 1996) while others perform worse that the worst-case analysis (Huang, Baetz and Patry, 1995, Tong, 1994). In the former examples, the interaction constraints do *not* bind, whereas in the later cases they *do* bind and force a solution that would not otherwise be desirable.

More generally, we note that performance worse than the worst case and solution mischaracterization are seen whenever the grey-number interaction constraints bind. The grey linear program imposes risks and costs to maintain maximal stable feasible ranges for the decision variables. The cost is the shadow value (Lagrange multiplier) associated with the binding grey linear program interaction constraint and the risk is, under conditions of unfavourable parameter values, the objective function performs worse than the worst case.

Grey Stochastic Programming with Recourse

Grey number optimization has also been applied to stochastic programs with recourse, including two-stage linear programs (Huang and Loucks, 2000, Maqsood and Huang, 2003), two-stage mixed integer programs (Maqsood, Huang and Zeng, 2004, Li and Huang, 2006, Rosenberg and Lund, 2008), fuzzy two-stage programs (Maqsood, Huang and Yeomans, 2005, Li, Huang and Nie, 2007), and multi-stage programs (Li, Huang and Nie, 2006), among others. Grey number stochastic programs with recourse incorporate uncertainties expressed as probability distributions and as intervals and work as follows.

Decisions are partitioned into two types. Primary-stage decisions are taken before stochastic information is realized. After the stochastic information is realized, second-stage (recourse) decisions are then implemented to cover the shortfalls not met by primary-stage decision levels. Since shortfalls differ for different stochastic realizations, recourse decisions apply only to a particular realization. Stochastic realizations are described by a probability distribution, which, for a stochastic linear program, is approximated by a set of discrete levels and likelihoods (probabilities). Together, primary stage decisions plus sets of recourse decisions for each stochastic realization constitute the decision portfolio—mix of actions—to respond to the stochastic events.

Model formulation and solution

A two-stage stochastic program that has primary decisions of water allocation targets X_i and recourse decisions that are shortage allocations D_{ie} to each sector i for unmet targets given water availability levels q_e in events e, can be expressed as follows (Huang and Loucks, 2000):

$$\text{Max } f = \sum_i b_i X_i - \sum_{i,e} p_e c_{ie} D_{ie} \tag{6a}$$

$$\text{s.t. } q_e \geq \sum_i \left(X_i - D_{ie}\right), \forall e \tag{6b}$$

$$X_i \geq 0, \forall i;\ D_{ie} \geq 0,\ \forall i, e. \qquad (6c)$$

Here, f is the objective function, b_i are benefits from water allocation to water use sector i, c_{ie} are costs or penalties in water availability event e for delivering a volume below the target, $X_i - D_{ie}$ are actual water deliveries to sector i in event e, and q_e and p_e are, respectively, the water availabilities levels and their associated probabilities. Together, p_e and q_e describe a set of discrete water availability levels and probabilities that approximate the stochastic distribution of water availability.

Substituting grey numbers for each of the input coefficients ($b^{\pm}_i$, $c^{\pm}_{ie}$, and $q^{\pm}_e$) and decision variables ($X_i^{\pm}$ and $D_{ie}^{\pm}$) turns two-stage linear program (6) into a grey two-stage linear program (7):

$$\text{Max } f^{\pm} = \sum_i b_i^{\pm} X_i^{\pm} - \sum_{i,e} p_e c_{ie}^{\pm} D_{ie}^{\pm} \qquad (7a)$$

$$\text{s.t. } q_e^{\pm} \geq \sum_i \left(X_i^{\pm} - D_{ie}^{\pm}\right), \forall e \qquad (7b)$$

$$X_i^{\pm} \geq 0, \forall i;\ D_{ie}^{\pm} \geq 0,\ \forall i, e \qquad (7c)$$

According to Huang and Loucks (2000), we solve grey two-stage linear program (7) by decomposing it into two deterministic sub-models. The two sub-models correspond to the upper and lower bounds of the grey objective-function $f^{\pm}$ and interact. With maximization, uncertain primary-stage decisions (X_i^{+}) are identified by first solving the upper-bound sub-model. Then, the determined primary-stage water allocation targets (now called $X_i^{*}{}_{opt}$) are used to solve the lower-bound sub-model for upper limits on recourse decisions (D_{ie}^{+}). This ordering identifies the maximal and widest range of system benefits. Decomposition and solution requires three steps.

Step 1. Set up and solve the sub-model to identify the objective function upper bound, f^{+}. Use parameter values that maximize benefits and minimize the need for recourse-stage shortages ($X_i^{\pm}$ and D_{ie}^{-}) [i.e., large benefits (b_i^{+}), small penalties (c_{ie}^{-}), and large water availability levels (q_e^{+})]. The program solves for long-term decision levels ($X_i^{\pm}$) since these values influence the objective function positively or negatively depending on recourse (short-term) decisions. The upper-bound sub-model is:

$$\text{Max } f^{+} = \sum_i b_i^{+} X_i^{+} - \sum_{i,e} p_e c_{ie}^{-} D_{ie}^{-} \qquad (8a)$$

$$\text{s.t. } q_e^{+} \geq \sum_i \left(X_i^{+} - D_{ie}^{-}\right), \forall e \qquad (8b)$$

$$X_i^{+} \geq 0, \forall i;\ D_{ie}^{-} \geq 0,\ \forall i, e \qquad (8c)$$

The solution identifies optimal primary-stage water allocation targets ($X^{*}_{i\ opt}$) and recourse-decision shortage levels (D_{ie}^{-}) that maximize net benefits under favourable economic conditions. Water allocation target levels ($X^{*}_{i\ opt}$) that maximize system benefits become inputs to the lower-bound sub-model.

Step 2. Set up and solve the lower bound sub-model to identify f^-. Use objective function coefficients and constraint values that minimize net benefits and increase the need for shortages (D_{ie}^+) [i.e., small benefits (b_i^-), large penalties (c_{ie}^+), and small water availability levels (q_e^-)]. The sole decisions are recourse-decision shortage levels (D_{ie}^+) that minimize benefits with unfavourable economic conditions. The lower-bound sub-model is:

$$\text{Max } f^- = \sum_i b_i^- X_{i\,opt}^* - \sum_{i,e} p_e c_{ie}^+ D_{ie}^+ \tag{9a}$$

$$\text{s.t. } q_e^- \geq \sum_i \left(X_{i\,opt}^* - D_{ie}^+ \right), \forall e \tag{9b}$$

$$D_{ie}^+ \geq 0, \ \forall i, e \tag{9c}$$

$$D_{ie}^+ \geq D_{ie\,opt}^-, \ \forall i, e \tag{9d}$$

Here, interaction constraint (9d) enforces a stable feasible range for the recourse decisions and the model omits non-negativity constraints for the primary-stage decisions since the upper-bound sub-model fixes the water allocation targets ($X^*_{i\,opt}$).

Step 3. Solutions to sub-models (8) and (9) span maximal, stable, feasible ranges for the objective function and recourse-stage decision variables. These ranges are $f^\pm_{opt} = [f^-, f^+]$, X_i^*, and $D^\pm_{ie\,opt} = [D_{ie}^-, D_{ie}^+]$ where f^+, X_i^*, and D_{ie}^- are solutions to upper-bound sub-model (8) and f^- and D_{ie}^+ are solutions to lower-bound sub-model (9).

Model discussion and comparisons

As with the grey linear program, the best-case formulation for a stochastic maximization problem is the same as the upper bound sub-model (8). The worst-case formulation allows primary-stage decisions, does not have an interaction constraint, and is:

$$\text{Max } f^- = \sum_i b_i^- X_i^- - \sum_{i,e} p_e c_{ie}^+ D_{ie}^+ \tag{10a}$$

$$\text{s.t. } q_e^- \geq \sum_i \left(X_i^- - D_{ie}^+ \right), \forall e \tag{10b}$$

$$X_i^- \geq 0, \forall i; \ D_{ie}^+ \geq 0, \ \forall i, e \tag{10c}$$

Here X_i^- are primary-stage water allocation targets identified under pessimistic economic conditions. The primary difference between the two-stage best/worst case and grey number formulations is that the primary-stage decision variable values are fixed across the grey-number sub-models (interaction) whereas they can change between the best and worst case models. Also, solutions to the best/worst case sub-models do not necessarily construct a set of stable, feasible ranges for selecting decision alternatives.

For a grey two-stage minimization problem, the solution algorithm is essentially reversed. First, solve the lower-bound sub-model allowing primary-stage decisions and without interaction constraint (i.e., sub-model [10]). Second, solve the upper-bound sub-model using the primary-stage decision values fixed from the lower-bound sub-model solution and with an interaction constraint on recourse-stage decisions ($D_{ie}^{-} \leq D_{ie\,opt}^{+}, \forall i, e$).

Problems and conditions under which they arise

Retrospective analyses comparing grey stochastic program example solutions (Huang and Loucks, 2000, Maqsood and Huang, 2003, Maqsood, Huang and Zeng, 2004, Maqsood, Huang and Yeomans, 2005, Li, Huang and Nie, 2006, Li and Huang, 2006, Li, Huang and Nie, 2007, Rosenberg and Lund, 2008) to their best/worst case counterparts show that grey solutions always perform worse than their worst case counterparts (Table 3). Here, the grey widths for the grey objective functions (f^{+}— f^{-}) are much wider than the widths associated with the best/worst case sub-models. These results identify grey stochastic solutions as very risk prone—subject to large, undesirable consequences under unfavourable conditions that decision makers could improve upon with a different solution approach such as solutions recommended by a worst-case analysis.

Performance is significantly worse than the worst case because the grey-solution method chooses optimistic primary-stage decision values to maximize system benefits under best-case conditions. Further, the grey-solution method fixes these optimistic primary-stage decisions across the upper- and lower-bound sub-models. For unfavourable conditions, the grey-number approach must implement the same program of optimistic decision values to maintain feasible ranges for decisions across sub-models. This sub-model interaction then requires the grey-number approach to counteract the fixed and optimistic program of primary-stage decisions with many additional and more costly recourse decisions. The worst-case analysis is not similarly constrained. Under unfavourable conditions, the worst-case basis for primary-stage decisions can exclude, scale back, or identify more appropriate primary-stage decision targets.

Moreover, fixing primary-stage decisions across grey-number sub-models undermines one of the tenants of grey number programming: to identify the stable, feasible range for the decision variables. Existing grey-solution techniques (Huang and Loucks, 2000) do not identify a range for the most important primary-stage planning decisions; they only identify a grey range for the less important recourse-stage operational decisions. I now propose some promising grey-solution techniques that (i) narrow the grey width of the objective function, and (ii) also identify a stable, feasible range for primary-stage decisions.

Alternative grey-solution techniques

Herein, I develop two alternative grey-solution techniques for stochastic linear programs, provide ratiocinations for each technique, and present and discuss solution results for numerous examples. The first technique is termed risk adverse and seeks to reduce the grey-width of the objective function by identifying a single set of primary-stage decisions and stable, feasible ranges for recourse decisions. The second technique imposes interaction constraints on both the primary- and recourse-stage decisions and identifies stable, feasible ranges for both sets of decisions. This approach is termed an interacting primary-stage grey solution technique. Both solution approaches guarantee objective function values equal or better than the worst-case value and work as follows.

Risk adverse technique

The existing risk-prone grey-solution technique (Huang and Loucks, 2000) identifies primary-stage solutions by solving the best-case (upper-bound for a maximization problem) sub-model first. This approach gives a wide-ranging objective function value because significant (and costly) recourse decisions are required should unfavourable conditions (represented by worst-case parameter values) arise. Reversing the solution process to first solve the worst-case (lower-bound for a maximization problem) sub-model can identify a more appropriate set of primary-stage allocation targets and reduce the need for costly recourse decisions.

Solution process

For a maximization problem, the risk adverse solution process works as follows.

Step 1. Set up and solve worst-case sub-model (10) to identify the objective function lower bound (f^-), primary-stage allocation targets under unfavourable parameter values (X_i^-), and upper bounds on recourse decisions (D_{ie}^+). Primary-stage water allocation target levels identified for pessimistic parameter values ($X_{i\ opt}^-$) become inputs to the upper-bound sub-model.

Step 2. Set up and solve an upper-bound sub-model to identify f^+. Here, the sole decisions are recourse-decision shortage levels (D_{ie}^-) that maximize benefits under optimistic parameter values. This upper-bound sub-model is:

$$\text{Max } f^+ = \sum_i b_i^+ X_{i\,opt}^- - \sum_{i,e} p_e c_{ie}^- D_{ie}^- \qquad (11a)$$

$$\text{s.t. } q_e^+ \geq \sum_i \left(X_{i\,opt}^- - D_{ie}^- \right), \forall e \qquad (11b)$$

$$D_{ie}^- \geq 0, \ \forall i,e \qquad (11c)$$

$$D_{ie}^- \leq D_{ie\,opt}^+, \ \forall i,e \qquad (11d)$$

Interaction constraint (11d) enforces a stable feasible range for the recourse decisions and the model omits a non-negativity constraint for the primary-stage decisions since water allocation targets ($X_{i\ opt}^-$) are fixed in the lower-bound sub-model.

D_{ie}^- are the sole decision variables in linear sub-model (11), so we can formulate an analytical solution rule for D_{ie}^-. This rule is: for each event e, minimize shortages (i.e., maximize delivery increases $D_{ie}^+ - D_{ie}^-$) to sector i with the highest water shortage cost (c_{ie}^-) subject to increased water availability ($q_e^+ - q_e^-$) and D_{ie}^- within the non-negativity (11c) and interaction (11d) constraints. For sectors with lower shortage costs, maximize delivery increases subject to increased water availability minus delivery increases to sectors with higher-shortage costs. Mathematically, this recursive solution rule is:

$$D_{je}^+ - D_{je}^- = \text{Minimum}\left\{\left(q_e^+ - q_e^-\right) - \sum_{k=1}^{j-1}\left(D_{ke}^+ - D_{ke}^-\right),\ D_{je}^+\right\},\ \forall j, e. \qquad (11e)$$

Here, j and k are the water use sectors ranked by shortage costs, c_{je}^-, so that $c_{1e}^- \geq c_{2e}^- \geq \ldots \geq c_{Ie}^-$. This rule is obtained by subtracting (10b) from (11c), eliminating the common $X^-_{i\ opt}$ terms, separating shortage decisions D_{je}^+ and D_{je}^- for the jth water use sector from shortage decisions for the other water use sectors, bringing these terms to one side, and combining with constraint (11d). Further, since the solution to a constrained linear optimization problem falls on the boundary of the feasible solution space, the binding inequality constraint becomes an equality. Rearranging (11e) gives the analytical decision rule for upper-bound shortage decisions D_{ie}^- as:

$$D_{je}^- = D_{je}^+ - \text{Minimum}\left\{\left(q_e^+ - q_e^-\right) - \sum_{k=1}^{j-1}\left(D_{ke}^+ - D_{ke}^-\right),\ D_{je}^+\right\},\ \forall j, e. \qquad (11f)$$

Step 3. Solutions to sub-models (10) and (11) span stable, feasible ranges for the objective function and decision variables. These ranges are $f^{\pm}_{opt} = [f^-, f^+]$, X_i^-, and $D^{\pm}_{ie\ opt} = [D_{ie}^-, D_{ie}^+]$ where f^-, X_i^-, and D_{ie}^+ are solutions to worst-case sub-model (10) and f^+ and D_{ie}^- are solutions to the upper-bound sub-model (11).

Ratiocination

The proof that the risk-adverse technique gives a narrower objective function width with a feasible solution and objective function value equal or better than the worst-case value is straightforward. The proof involves reinterpreting a prior theorem and proof made by Huang et al. (1995) and then showing solution feasibility and ranges.

In their Theorem 2, Huang et al. (1995, p. 599-602) show that solving the upper-bound sub-model first (for a maximization problem) is necessary to identify “the two extreme bounds for given system condition variations” (p. 601). This ordering generates grey solutions of good quality. Here, “quality” refers to the grey degree or width of the grey decision variables and objective function values (differences between their upper and lower bounds)(Huang, Baetz and Patry, 1995, Definition 13, p. 597) while “good” means these widths are maximal and large. Huang et al. (1995, p. 598) also show the converse—solving the lower-bound sub-model first (for a maximization problem) is unable to generate grey solutions with good quality.

First, we note that large-ranging grey widths that comprise the extreme bounds and force the objective function to perform worse than the worse-case solution when interaction constraints bind are neither "good" nor desirable outcomes. Under unfavorable parameter conditions, decision makers may regret that they could have done better had they adopted a worst-case or possibly other solution. We therefore reinterpret Huang et al.'s (1995) definitions of "good" and desirable to allow as acceptable first solving the lower-bound sub-model to generate grey solutions of indeterminate quality.

Second, we show that feasible solutions exist for the upper-bound sub-model (11). This proof is straightforward. By definition of the grey-number parameter, $q_e^+ \geq q_e^-$, ∀e and examining solution expression (11f), we note $0 \leq D_{ie}^- \leq D_{ie}^+$, ∀i,e, which is compatible with constraints (11c) and (11d) and gives feasible solutions for D_{ie}^-. Initially, large increased water availabilities ($q_e^+ - q_e^- > D_{ie}^+$) force the second argument of the Minimum function in (11f) to dominate and set shortages to zero for sectors with high shortage costs. Subsequently, increased allocations to sectors with higher shortage costs will balance the increased water availability so that the first argument of the Minimum function will fall to a minimum of zero. This minimum only allows D_{ie}^- to reach an upper-limit of D_{ie}^+ and maintains the feasible range of solution values.

Third, we show the objective function value (f^+) for the upper-bound sub-model (11) will always be greater than or equal to the objective function value (f^-) for the lower-bound (and worst-case) sub-model (10). By definitions of the grey-number parameters $b_i^+ \geq b_i^-$, ∀i; $c_{ie}^- \leq c_{ie}^+$, ∀i,e; and from interaction constraint (11d) where $D_{ie}^- \leq D_{ie}^+$, ∀i,e; we have the objective function value ordering:
$f^- = \sum_i b_i^- X_i^- - \sum_{i,e} p_e c_{ie}^+ D_{ie}^+ \leq \sum_i b_i^+ X_i^- - \sum_{i,e} p_e c_{ie}^- D_{ie}^- = f^+$. The width of the risk adverse objective function range is $f^+ - f^- = \sum_i \left(b_i^+ - b_i^-\right) X_i^- + \sum_{i,e} p_e \left(c_{ie}^+ D_{ie}^+ - c_{ie}^- D_{ie}^-\right)$. Substituting in Equation (11f) gives

$$f^+ - f^- = \sum_i \left(b_i^+ - b_i^-\right) X_i^- + \sum_{i,e} p_e \left(c_{ie}^+ - c_{ie}^-\right) D_{ie}^+ + \sum_{je} p_e c_{je}^- \mathrm{Min}\left\{ \left(q_e^+ - q_e^-\right) - \sum_{k=1}^{j-1} \left(D_{ke}^+ - D_{ke}^-\right), D_{ke}^+ \right\}$$

which shows the risk adverse objective function width depends only on increased benefits ($b_i^+ - b_i^-$), decreased costs ($c_{ie}^+ - c_{ie}^-$), and increased water availability ($q_e^+ - q_e^-$) multiplied by lower-bound costs for sectors with the most costly shortages.

Fourth, we note (as do Huang et al. (1995, p. 601)) that solving the lower-bound (worst-case) sub-model first will generate the worst-case solution but that the associated upper-bound solution will likely not reach the best-case objective function value. However, this behavior is not required for the risk-adverse solution approach. (Such behavior will occur only when solutions to the best- and worst-case sub-models comprise the same solution basis and the upper-bound sub-model interaction constraint does not bind).

Finally, combining results from Points #3 (feasible objective function value ≥ worst-case objective function value) and #4 (upper-bound objective function value ≤ best-case objective function value) gives a risk-adverse grey objective function range ($f^+ - f^-$) that is narrower than range obtained by the existing risk-prone solution method.

Example results

Resolving each stochastic program example with the risk-adverse grey-solution technique shows that the technique gives an objective function value range that is narrower than both the risk-prone grey-number and best/worst case methods (Table 3). One bound of the objective function corresponds to the worst-case (lower-bound for a maximization problem; upper-bound for a minimization problem) while the other bound falls "inside" the best-case solution (upper-bound less than the best-case for a maximization problem; lower-bound greater than the best-case for a minimization problem).

The risk-adverse technique identifies primary-stage decisions and stable, feasible range of recourse-stage decisions that minimize deviations of the objective function value. Further, the objective function avoids risk-prone performance worse than the worst-case solution. However, the risk adverse solution approach (like the risk-prone approach) fixes primary-stage decision values across the sub-models; we correct this failing with an interacting primary-stage grey solution approach.

Interacting primary-stage technique

The risk-prone and risk-adverse grey solution techniques fix primary-stage decision values across the upper- and lower-bound sub-models and fail to identify a stable, feasible range for all decision variables. Here, we introduce interaction constraints for primary-stage decisions to identify the stable, feasible ranges for these variables.

Solution process

For a maximization problem, first solve the worst-case sub-model (10) to identify the objective function lower bound (f^-), lower bound on primary-stage allocation targets under unfavourable parameter values (X_i^-), and upper bounds on recourse decisions (D_{ie}^+). Second, solve an upper-bound sub-model to identify the objective function upper bound (f^+), upper bound on primary-stage allocation targets under favourable parameter values (X_i^+), and lower bounds on recourse decisions (D_{ie}^-).

$$\text{Max } f^+ = \sum_i b_i^+ X_i^+ - \sum_{i,e} p_e c_{ie}^- D_{ie}^- \tag{12a}$$

$$\text{s.t. } q_e^+ \geq \sum_i \left(X_i^+ - D_{ie}^- \right), \forall e \tag{12b}$$

$$X_i^+ \geq 0, \ \forall i; \ D_{ie}^- \geq 0, \ \forall i,e \tag{12c}$$

$$D_{ie}^- \leq D_{ie\ opt}^+, \ \forall i,e \tag{12d}$$

$$X_i^+ \geq X_{i\ opt}^-, \ \forall i \tag{12e}$$

Here, decision variables include both primary- and recourse-stages (X_i^+ and D_{ie}^-) with an interaction constraint (12e) requiring primary-stage decisions under favorable conditions to be above the levels identified in the worst-case sub-model ($X_{i\ opt}^-$). We can derive an analytical solution for the recourse-stage shortage decisions D_{ie}^- as was done for the risk-adverse approach. Except, here, primary-stage decisions from the two sub-models are not necessarily identical and may not cancel. Thus,

$$D_{je}^- = D_{je}^+ \text{ - Minimum}\left\{\left(q_e^+ - q_e^-\right) - \sum_i \left(X_i^+ - X_i^-\right) - \sum_{k=1}^{j-1}\left(D_{ke}^+ - D_{ke}^-\right),\ D_{je}^+\right\},\ \forall j, e. \qquad (12f)$$

Together, solutions to sub-models (10) and (12) span stable, feasible ranges for the objective function and both primary- and recourse-stage decision variables. These ranges are $f^{\pm}_{opt} = [f^-, f^+]$, $X^{\pm}_{i\ opt} = [X_i^-, X_i^+]$, and $D^{\pm}_{ie\ opt} = [D_{ie}^-, D_{ie}^+]$ where f^-, X_i^-, and D_{ie}^+ are solutions to worst-case sub-model (10) and f^+, X_i^+, and D_{ie}^- are solutions to the upper-bound sub-model (12).

Ratiocination

The mathematical proof that the interacting primary stage solution technique gives a feasible solution, objective function value equal or better than the worst-case value, and objective function width that is equal or wider than the risk-adverse technique follows the ratiocination provided for the risk-adverse technique. Here, we simply add and account for interaction constraints on the primary-stage variables.

First, we again reinterpret the Theorem 2 and proof made by Huang et al. (1995) to allow that first solving the lower-bound sub-model will generate grey solutions of indeterminate but acceptable quality. Second, we show that feasible solutions exist for upper-bound sub-model (12). This proof is straightforward. Subtracting (10b) from (12b), combining and separating terms, gives $q_e^+ - q_e^- \geq \sum_i \left(X_i^+ - X_i^-\right) + \sum_i \left(D_{ie}^+ - D_{ie}^-\right) \geq 0, \forall e$.

This expression is compatible with the prior grey-number parameter definition for q_e and interaction constraints (12d) and (12e), and gives feasible solutions for upper-bound allocation targets (X_i^+) and lower-bound shortages (D_{ie}^-). Together, increases in X_i^+ and decreases in D_{ie}^- cannot exceed increased water availabilities seen when moving from unfavorable to favorable parameter conditions. But the expression still allows for a wide range of X_i^+ and D_{ie}^-. At worst, $X_i^+ = X_i^-$ and (12f) reduces to (11f). In this case feasibility conditions shown in the ratiocination for the risk-averse technique similarly apply.

Third, we show $f^+ \geq f^-$ from upper-bound sub-model (12) and lower-bound (and worst-case) sub-model (10). This proof is also straightforward. By prior definitions of the grey-number parameters and interaction constraints on recourse decisions ($D_{ie}^- \leq D_{ie}^+$, ~~∀~~i,e, [Eq. 12d]) and on primary-stage decisions ($X_i^+ \geq X_i^-$, ~~∀~~i, [Eq. 12e]), we simply have:
$$f^- = \sum_i b_i^- X_i^- - \sum_{i,e} p_e c_{ie}^+ D_{ie}^+ \leq \sum_i b_i^+ X_i^+ - \sum_{i,e} p_e c_{ie}^- D_{ie}^- = f^+.$$

Fourth, we show that the objective function width for the interacting primary stage solution technique ($f^+_{ips} - f^-_{ips}$) is greater than or equal to the width for the risk-adverse technique ($f^+_{ra} - f^-_{ra}$). Since both solution techniques use the same lower-bound (worst case) sub-model (10), we need only examine the upper-bound objective function values and show $f^+_{ips} \geq f^+_{ra}$. Here, note that the upper-bound risk-adverse sub-model solution (11) is part of the solution space to the upper-bound interacting primary-stage sub-model (12) ($[X^*_{i\,opt}, D_{ie}^-]_{ra} \in [X^+_i, D_{ie}^-]_{ips}$) by virtue of interaction constraint (12e). Further, the upper-bound risk-adverse sub-model is more constrained than the interacting primary-stage model (the later has $X_i^+ \geq X_i^-$, $\forall$i, [Eq. 12e] and the former has $X_i^+ = X^*_{i\,opt}$, $\forall$i). Therefore, $f^+_{ips} \geq f^+_{ra}$. The increase is $f^+_{ips} - f^+_{ra} = \sum_i b_i^+ \left(X_i^+ - X^-_{i\,opt}\right) - \sum_{i,e} p_e c^-_{ie} \left(D^-_{ie\,ips} - D^-_{ie\,ra}\right)$.

Substituting in (11f) and (12f) and noting that the sole difference between $D^-_{ie\,ips}$ and $D^-_{ie\,ra}$ is the term $-\sum_i \left(X_i^+ - X^-_{i\,opt}\right)$ (which represents decreased water availability from increased primary-stage water allocation targets), gives

$$f^+_{ips} - f^+_{ra} \leq \sum_i \left(b_i^+ - \sum_e p_e c^-_{ie}\right)\left(X_i^+ - X^-_{i\,opt}\right).$$

We can also obtain the same expression by formulating the Lagrangian for sub-model (12), specifying the Kuhn-Tucker conditions, and substituting to eliminate the Lagrange multiplier associated with constraint (12b). This expression says that the interacting primary-stage objective function value will increase above the risk adverse value whenever upper-bound benefits exceed expected lower-bound shortage costs.. Should benefits not exceed expected shortage costs, constraint (12e) will bind so that $X_i^+ - X^-_{i\,opt} = 0$ with no increase.

Finally, we note again that the associated upper-bound objective function value f^+_{ips} will likely not reach the best-case objective function value. However, this behavior is not required of the interacting primary-stage solution approach.

Example results

Resolving each of the stochastic program examples using the interacting primary-stage grey-solution method shows that the approach generates solutions whose objective function widths are wider than the risk-adverse solutions but narrower than the risk-prone or best/worst case solutions (Table 3). Figure 1 illustrates and compares these objective function widths for the water resources allocation problem posed by Huang and Loucks (2000). The primary-stage interaction solution performs no worse than the risk-adverse approach (both methods use worst-case sub-model (10) to solve for the lower bound of the objective function), but shows an improvement over the risk-adverse approach for favourable parameter values. This improvement nearly approaches the large, optimistic upper-bound objective function value seen for the best-case and risk-prone solution methods. The interacting primary-stage method avoids the pitfall of the risk-prone approach (performance worse than the worst-case), allows flexibility to choose primary-stage decision values within the identified range, and improves objective function performance for favourable parameter values compared to the risk-adverse solution approach.

Discussion

The Best/Worst-case formulations solve a linear program twice using the most favourable (best case) and least-favourable (worst case) parameter values. Solutions from the two sub-models can help judge the system's capability to realize a desired goal but do not necessarily construct a set of stable ranges for generating decision alternatives. When the solution basis for the best case differs from the solution basis for the worst case, there can be confusion about how to operate the system in the face of uncertain parameter inputs.

Grey linear programs identify maximal, stable, feasible ranges for decision variables by first solving the best-case (upper-bound for a maximization problem) sub-model. They then solve the lower-bound sub-model and introduce interaction constraints to require lower-bound solutions be less than or equal to upper-bound solutions. This interaction identifies stable, feasible ranges for decision variables and simultaneously communicates that decision variables can be chosen within the proscribed ranges while assuring that the objective function value will vary only within the associated specified range.

When the range of uncertainty for input parameters is small and the interaction constraints do not bind, the grey linear program and best/worst-case formulation solutions are identical. In this case, the solution bases for the best and worst cases are also the same. However, when the range of uncertainty for input parameters is significant and the interaction constraints bind, the grey linear program objective function value will be worse than the worst case. The grey linear program will also fail to identify part of the solution basis that is preferable under unfavourable parameter values. There are risks and costs to impose a maximal, stable, feasible range of solutions. The cost is the shadow value (Lagrange multiplier) associated with the binding interaction constraint and the risk is, under unfavourable parameter values, performance worse than the worse case. In these situations, decision makers will likely prefer to adopt a worst-case or other more risk-adverse solution. Two simple numerical problems and retrospective analysis of grey linear program examples from the literature demonstrate these problems.

These problems are magnified for grey stochastic programs that have primary- and recourse-stage decisions and incorporate uncertainties expressed as probability distributions and as intervals. Existing grey-solution methods which we term risk-prone identify maximal, stable, feasible ranges for the objective function and recourse-stage decision variables by solving the best-case sub-model first. They then use identified primary-stage decision values as inputs to the lower-bound sub-model. Fixing the primary-stage decision values across the sub-models, risk-prone grey-solution methods fail to identify stable, feasible ranges for primary-stage decisions and often require significant and costly recourse-stage decisions for unfavourable parameter values. This requirement results in wide-ranging and risk-prone objective function values that perform worse than the worst case. Again, under unfavorable parameter conditions, decision makers could do better by adopting a worst-case or other more risk-adverse solution.

To narrow the width of objective function deviations and guarantee performance at or better than the worst case, a risk-adverse grey-solution method solves the worst-case sub-model first, then uses the identified primary-stage decision values to solve the upper-bound sub-model. Identifying primary-stage decision levels first for unfavourable parameter values minimizes the cost of and need for recourse-stage decisions, but also reduces potential benefits under favourable parameter conditions. Like the risk-prone approach, the risk-adverse method also fixes primary-stage decision variable values across sub-models and fails to identify a stable, feasible range for these decision variables.

A third solution approach uses interaction to identify stable, feasible ranges for the objective function, primary-stage, and recourse-stage decision variables. The interacting primary-stage grey solution method solves the worst-case sub-model first to identify lower-bounds on the objective function and primary-stage decision variables. Then it solves the upper-bound sub-model and uses an interaction constraint on primary-stage decisions to identify the upper bounds on the objective function and primary-stage decision variables. Together, solutions form stable, feasible ranges for selecting decision alternatives. Because interaction identifies a range for primary-stage decision values, the interacting primary-stage grey solution method is better able to adapt to favourable parameter conditions and typically gives an objective function range that is wider than the risk-adverse approach and nearly approaches the best-case solution value.

Table 4 summarizes and compares the four methods to solve stochastic programs with recourse that incorporate uncertainties expressed as intervals. The choice of solution method depends on the modeler’s aims, particularly his/her tolerance for objective function deviations. If large deviations and performance worse than the worse case are acceptable should unfavourable conditions arise, then use the existing risk-prone grey solution approach. First solve the best-case (upper-bound for a maximization problem) sub-model and use primary-stage decision values identified for optimistic conditions. However, if objective function value deviations are to be reduced and a solution guaranteed to be at or better than the worst-case, instead use the risk-adverse or interacting primary-stage grey solution approaches. In this case, first solve the worst-case (lower-bound for a maximization problem) sub-model and use the primary-stage decision values identified for pessimistic conditions. Algorithmically, risk tolerance boils down to a choice of first solving the best- or worst-case sub-model.

Conclusions

A grey number expresses uncertainty as an interval between fixed lower and upper bounds. Grey numbers find use in optimization to proactively incorporate uncertainties expressed as intervals and identify maximal, stable, feasible ranges for the objective function and decision variables. These ranges are identified by introducing interaction constraints to limit decision variable values for unfavourable conditions based on decision variable levels first identified for favourable conditions. Ranges for decision variables can then be used to select decision alternatives within proscribed bounds.

Grey number programs represent an improvement over best/worst case analysis because the latter approach, lacking interaction constraints, often offers solutions with different bases for favourable and unfavourable parameter values. However, the interaction constraints also limit grey solutions and grey programs often fail to identify part of the feasible solution space, particularly in the face of unfavourable parameter values. Moreover, interaction constraints often lead the grey-number objective function value to perform worse than the worst-case analysis. This solution mischaracterization and risk-prone performance worse that the worse case occurs whenever the interaction constraints bind. The paper shows this mischaracterization and risk-prone performance for numerous linear and stochastic programming examples. Further, the existing grey-solution approach for stochastic programs with recourse fixes primary-stage decision variable values across sub-models and fails to identify a stable, feasible range for these important planning decision variables.

Two alternative grey-solution algorithms are presented to overcome these problems. A risk-adverse grey-solution technique solves the worst-case sub-model first, reduces deviations in the objective function value, and guarantees an objective function value no worse than the worst case. An interacting primary-stage technique introduces interaction constraints on primary-stage decisions, identifies a stable, feasible range for these decision variables, guarantees an objective function value no worse than the worst case, yet offers a range that is wider and an improvement over the risk-adverse technique. These solution behaviors are ratiocinated, demonstrated, and verified for numerous stochastic programming examples.

Ultimately, a modeler's or decision maker's choice of solution method to include uncertainties expressed as intervals depends on their risk preferences—particularly their tolerance for objective function deviations. If wide deviations are acceptable with performance worse than the worst case possible under unfavourable parameter values, then use existing grey-solution techniques. However, if wide deviations are to be avoided such as in risk-adverse decision-making, then the alternative solution approaches may be preferable. Should the goal be only to characterize system performance across favourable and unfavourable conditions without need to enforce solution stability across these different environments, then Best/Worst case analysis may be used. These tradeoffs and distinctions highlight the important advantages, disadvantages, and differences between risk-prone and risk-adverse grey-number programming and best/worst case analysis.

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Table 1. Comparison of grey linear program and best/worst case solutions for simple optimization programs

Solution Element	**Max [3, 4]X_1 + [5, 6]X_2 X_1 + X_2 <= 1**		**Max [3, 4]X_1 + [2, 6]X_2 X_1 + X_2 <= 1**		**Max [3, 4]X_1 + [5, 6]X_2 X_1 + [1, 4]X_2 <= 1**	
	Grey Linear Program	Best/Worst Case	Grey Linear Program	Best/Worst Case	Grey Linear Program	Best/Worst Case
f	[5, 6]	<5, 6>	[2, 6]	<3, 6>	[1.25, 6]	<3, 6>
X_1	[0]	<0, 0>	[0]	<1, 0>	[0]	<1, 0>
X_2	[1]	<1, 1>	[1]	<0, 1>	[0.25, 1]	<0, 1>

Table 2. Comparison of grey number and worst-case solutions for linear program examples

Author	Year	Programming Approach	Solution Method	Application	Direction	Objective Function		Verification / Problems
						Grey Solution	Worst Case	
Huang, Baetz, and Patry	1992	Grey linear program	Best/Worst case	Numerical example SWM example	Max Min	[765, 1931] [238, 517]*	765 517	Infeasible as published. Landfill capacity too small for optimal solution published. Instead, use [2.36, 3.24]10^6 tons.
Huang and Moore	1993	Grey linear program	Best/Worst case	Numerical example WR example	Max Min	[764, 1930]* [8.13, 27.6]	764	Infeasible as published for upper bound of numerical example. Correct as published in Huang et al (1992). Not enough input data published to verify WR solution.
Tong	1994	Interval linear program	Best/Worst case	Chicken feed numerical example	Min	[242.2, 420]*	415	Infeasible as published. Total forage constraint misinterpreted. Should be X_1+X_2 <= 1000; X_1+X_2 <= 1130. Upper bound submodel performs worse than worst case.
Huang, Baetz, and Patry	1995	Grey integer program	Interacting Best/Worst case	SWM example	Min	[385, 708]*	702	Error in specification of waste generation rates for cities 2 & 3. Upper bound submodel performs worse than worst case.
Yeh	1996	Grey linear program	Interacting Best/Worst case; Full and partial greylization	Reservoir capacity example	Min	[4, 12]	NA	Not enough input data published to verify solution or calculate worst case.
Huang	1996	Grey linear program	Interacting Best/Worst case	Numerical example WQ example	Max Max	[8.2, 15.4] [15.4, 20.0]	8.2	Lower bound is same as worst case. Not enough input data to verify WQ example.
Huang	1998	Grey linear chance constraint program	Interacting Best/Worst case	WQ ex., p=0.10 WQ ex., p=0.05 WQ ex., p=0.01	Max	[20.1, 22.8] [17.9, 21.2] [15.4, 20.0]	NA	Not enough input data published to verify solution or calculate worst case.

* denotes author's calculation does not verify against published grey-number solution for reasons described in verification column

Table 3. Comparison of grey number and best/worst case solutions for stochastic linear program examples

					Objective Function Ranges			
						Grey Number Solution Methods		
Author	**Year**	**Programming Approach**	**Application**	**Direction**	**Best / Worst Cases**	**Existing** (risk prone)	**Risk Adverse**	**Interacting Primary Stage**
Huang and Loucks	2000	Two-stage stochastic program	WR numerical example	Max	<346, 592>	[260, 592]	[346, 462]	[346, 560]
Maqsood and Huang	2003	Two-stage stochastic program	SWM example	Min	<0.147, 0.255>	[0.147, 0.260]*	[0.15, 0.26]	[0.149, 0.255]
Maqsood, Huang, and Zeng	2004	Two-stage mixed integer program	SWM example	Min	<249, 432>	[249, 478]*	[302, 432]	[272, 432]
Maqsood, Huang, and Yeomans	2005	Fuzzy two-stage stochastic program	WR numerical example	Max	<203, 571>	[154, 571]*	[203, 462]	[203, 538]
Li, Huang, Nie, Nie, and Maqsood	2006	Two-stage mixed integer program	SWM example	Min	<119, 278>	[119, 283]*	[124, 278]	NA
Li, Huang, and Nie	2006	Multi-stage stochastic program	WR numerical example	Max	<1435, 2605>	[1240, 2605]*	[1435, 2404]	[1435, 2606]
Li, Huang, and Nie	2007	Fuzzy two-stage mixed integer program	Flood diversion example	Min	<1899, 2215>	[1899, 2634]*	[2083, 2215]	NA
Rosenberg and Lund	2008	Two-stage mixed integer program	Water supply planning	Min	<-15, 112>	[-15, 281]	[4, 112]	[4.8, 112]

* denotes author's calculation does not verify against published risk prone solution for reasons described in the text

Table 4. Comparison of solution methods to stochastic programs with recourse

	Solution Method	**Sub-models**	**Decision Variables** Primary Stage	**Decision Variables** Recourse Stage	**Notes**
Best/Worst cases		(8) and (10)	$<X_i^-, X_i^+>$	$<D_{ie}^-, D_{ie}^+>$	Solutions do not necessarily construct stable, feasible ranges for selecting decision alternatives.
Grey-number	Existing risk prone	(8) then (9)	X_i^*	$[D_{ie}^-, D_{ie}^+]$	Wide-ranging objective function performs worse than worst-case. Primary-stage decisions fixed across sub-models.
Grey-number	Risk adverse	(10) then (11)	X_i^-	$[D_{ie}^-, D_{ie}^+]$	Minimizes objective function deviations. Objective function performs no worse than worst-case. Primary-stage decisions fixed across sub-models.
Grey-number	Interacting primary stage	(10) then (12)	$[X_i^-, X_i^+]$	$[D_{ie}^-, D_{ie}^+]$	Interaction constraints identify range of primary-stage decisions. Objective function performs no worse than worst-case and better than risk-adverse technique.

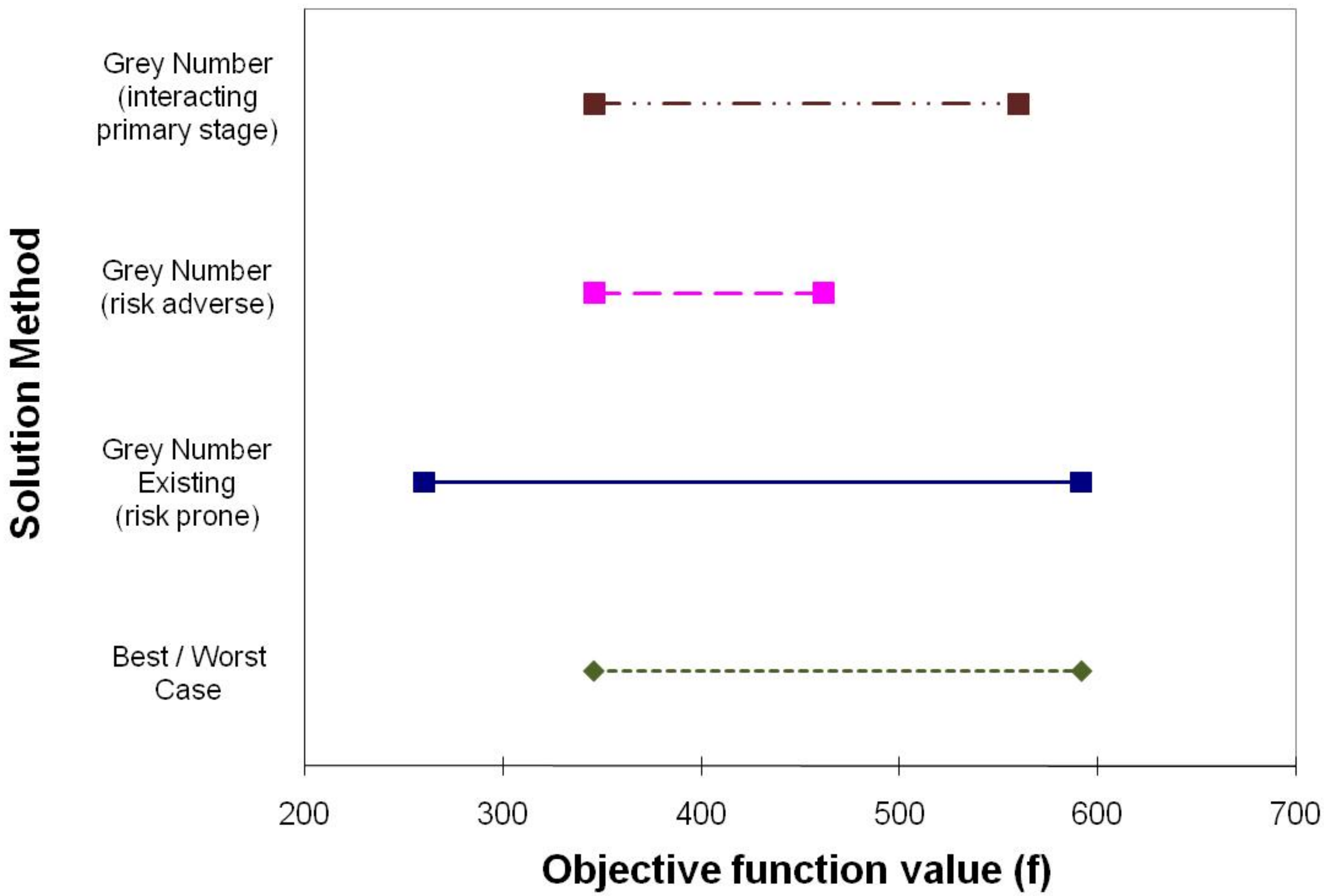

Figure 1. Comparison of objective function values for best / worst case and grey-number solution methods for the water allocation problem posed by Huang and Loucks (2000)