Schrödinger representation for the polarized Gowdy model

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Abstract. The polarized $\mathbf{T}^3$ Gowdy model is, in a standard gauge, characterized by a point particle degree of freedom and a scalar field degree of freedom obeying a linear field equation on $\mathbf{R} \times \mathbf{S}^1$. The Fock representation of the scalar field has been well-studied. Here we construct the Schrödinger representation for the scalar field at a fixed value of the Gowdy time in terms of square-integrable functions on a space of distributional fields with a Gaussian probability measure. We show that "typical" field configurations are slightly more singular than square-integrable functions on the circle. For each time the corresponding Schrödinger representation is unitarily equivalent to the Fock representation, and hence all the Schrödinger representations are equivalent. However, the failure of unitary implementability of time evolution in this model manifests itself in the mutual singularity of the Gaussian measures at different times.

1. Introduction

The Gowdy class of vacuum spacetimes [1] has for many years been an attractive toy model for investigations in quantum gravity; see *e.g.*, references [2–9]. These spacetimes arise from a symmetry reduction of the Einstein theory by an Abelian 2-parameter group and define a large class of inhomogeneous cosmological models featuring a big bang. From the point of view of quantum gravity the "polarized" sub-class of models define exactly soluble quantum field theories owing to the infinite number of degrees of freedom remaining after the symmetry reduction. Thus these models can be used to explore intrinsically field-theoretic effects which may feature in a complete quantum theory of gravity.

The traditional quantization of the polarized $\mathbf{T}^3$ Gowdy model uses the fact that in an appropriate gauge the phase space of the theory is characterized by a point particle degree of freedom and a scalar field degree of freedom, the latter satisfying a linear field equation. The point particle degree of freedom is treated in the usual quantum mechanical way, while the scalar field is quantized via a Fock representation of the canonical commutation relations deriving from a natural notion of positive frequency solutions to the field equation. Here we investigate the corresponding Schrödinger representation of the scalar field, *i.e.*, we construct a representation in terms of functions of the value of the scalar field at a fixed time. There are several reasons why this representation is of interest. Perhaps most importantly, the Schrödinger representation is the setting for the original (and still relatively poorly understood) approach to canonical quantum gravity based upon functions of 3-metrics

[10, 11]. It is also akin to the representation currently proposed for the full theory using the better understood "loop quantization", based upon functions of connections [12]. In loop quantum gravity the dynamical evolution of the theory is still not well-understood, while in the polarized Gowdy model the dynamical evolution is quite tractable and, as we shall see, has an important interplay with the Schrödinger representation. Moreover, the polarized Gowdy model sits as an exactly soluble model within the larger class of unpolarized Gowdy models. The latter are self-interacting field theories and could be studied using perturbative methods starting from the polarized model. Functional integral approaches based upon the Schrödinger representation have been one of the principal methods of rigorously constructing interacting quantum field theories [13]. Finally, the Schrödinger representation gives immediate physical insight into fundamental properties of the quantum field. In particular, the support of the measure used to define the representation reveals the mathematical behavior of "typical" field configurations, *i.e.*, typical quantum metrics. This behavior is far more intricate for field theories than for quantum mechanical systems.

The Schrödinger representation for the polarized $\mathbf{T}^3$ Gowdy model is defined in terms of functions on a space of distributional field configurations representing the scalar field at a fixed value of the Gowdy time coordinate. This space of field configurations is equipped with a Gaussian probability measure determined by requiring equivalence with the usual Fock representation. We will show that the support of the measure is within a Sobolev space which is slightly more "singular" than that of square-integrable field values. This representation can be constructed for any value of the Gowdy time and the resulting representations are all unitarily equivalent. One of the novel, intrinsically field-theoretic features of this model is the failure of unitary implementability of time evolution [8, 9]. This causes no difficulties in the mathematical construction and/or physical interpretation of the Schrödinger representation. However, this feature does prevent the mutual continuity of the Gaussian measures at different times. Indeed, we show that the measures at different times are mutually singular.

2. The classical Gowdy model

The Gowdy model we will study arises by assuming spacetime is not flat, that its manifold is $\mathcal{M} = \mathbf{R}^+ \times \mathbf{T}^3$ with spacelike $\mathbf{T}^3$, and that there is an Abelian 2-parameter isometry group, $G = \mathbf{T}^2$, with spacelike orbits $\mathbf{T}^2 \subset \mathbf{T}^3$ generated by a pair of commuting, hypersurface-orthogonal Killing vector fields. We will use coordinates (t, x, y, z) on $\mathcal{M}$, where $t > 0$, and $(x, y, z) \in (0, 2\pi)$. These coordinates are chosen such that the $t = const.$ surfaces foliate $\mathcal{M}$ with spacelike $\mathbf{T}^3$ hypersurfaces (coordinates (x, y, z)), and such that the Killing vector fields are $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. In such coordinates the metric g on $\mathcal{M}$ can be put into the form [14]

$$g = [- (N^\perp)^2 + e^{\gamma - \psi}(N^x)^2]dt^2 + 2e^{\gamma - \psi}N^x dt dx + e^{\gamma - \psi}dx^2$$
$$+ \tau^2 e^{-\psi}dy^2 + e^{\psi}dz^2. \tag{2.1}$$

The functions $N^\perp(t, x)$ and $N^x(t, x)$ determine the lapse function and shift vector for G-invariant spacelike foliations of $\mathcal{M}$. The function $\tau(t, x) > 0$ defines the area of the orbits of G and will be used as a time coordinate. The gradient of τ is assumed to be everywhere timelike. The functions $\gamma(t, x)$ and $\psi(t, x)$ are unrestricted.

A Hamiltonian formulation of the symmetry-reduced Einstein equations is given in [14], and we shall use this formulation here. Using a prime and dot to denote differentiation with respect to x and t respectively, the canonical action functional for the theory defined on $\mathcal{M}/G \approx \mathbf{R}^+ \times \mathbf{S}^1$ is given by

$$
\begin{aligned}
S[N^\perp, N^x, \gamma, \tau, \psi, P_\gamma, P_\tau, P_\psi] = \int_{t_1}^{t_2} dt \int_0^{2\pi} dx \Bigg(& P_\gamma \dot{\gamma} + P_\tau \dot{\tau} + P_\psi \dot{\psi} \\
& - N^\perp e^{(\psi-\gamma)/2} \Big[-P_\gamma P_\tau + 2\tau'' - \tau' \gamma' + \frac{1}{2} \left(\tau^{-1} P_\psi^2 + \tau \psi'^2 \right) \Big] \\
& - N^x \Big[-2P_\gamma' + P_\gamma \gamma' + P_\tau \tau' + P_\psi \psi' \Big] \Bigg),
\end{aligned}
\tag{2.2}
$$

which displays the canonical coordinates $(\gamma, \tau, \psi, P_\gamma, P_\tau, P_\psi)$ for the phase space Γ. The symplectic 2-form on Γ is

$$
\Omega = \int_0^{2\pi} dx \, (\delta P_\tau \wedge \delta\tau + \delta P_\gamma \wedge \delta\gamma + \delta P_\psi \wedge \delta\psi).
$$

Variation of the action with respect to the lapse and shift functions $(N^\perp, N^x)$ yields the Hamiltonian and momentum constraints

$$
H_\perp = 0 = H_x,
\tag{2.3}
$$

where the Hamiltonian and momentum constraint functions are given by:

$$
H_\perp = e^{(\psi-\gamma)/2} \Big[-P_\gamma P_\tau + 2\tau'' - \tau' \gamma' + \frac{1}{2} \left(\tau^{-1} P_\psi^2 + \tau \psi'^2 \right) \Big],
\tag{2.4}
$$

$$
H_x = -2P_\gamma' + P_\gamma \gamma' + P_\tau \tau' + P_\psi \psi'.
\tag{2.5}
$$

The restriction that the spacetime gradient of τ is timelike leads to a corresponding restriction on the phase space Γ:

$$
P_\gamma < -|\tau'|.
\tag{2.6}
$$

We will work in the gauge defined by

$$
\tau/\mathcal{P} = t, \quad P_\gamma = -\mathcal{P},
\tag{2.7}
$$

where‡

$$
\mathcal{P} = -\frac{1}{2\pi} \int_0^{2\pi} dx P_\gamma > 0.
\tag{2.8}
$$

Preserving the gauge fixing conditions in time gives (modulo the constraints)

$$
N^\perp = e^{(\gamma-\psi)/2}, \quad (N^x)' = 0.
\tag{2.9}
$$

The constraints become

$$
H_\perp = e^{(\psi-\gamma)/2} \left[\mathcal{P} P_\tau + \frac{1}{2} (\frac{1}{t\mathcal{P}} P_\psi^2 + t\mathcal{P}\psi'^2) \right] \approx 0,
\tag{2.10}
$$

$$
-\mathcal{P}\gamma' + P_\psi \psi' \approx 0,
\tag{2.11}
$$

with solution

$$
P_\tau = -\frac{1}{2}\frac{1}{\mathcal{P}}(\frac{1}{t\mathcal{P}} P_\psi^2 + t\mathcal{P}\psi'^2),
\tag{2.12}
$$

‡ We note that $\mathcal{P}$ has vanishing Poisson brackets with the constraint functions.

$$\gamma(x) = \gamma_0 + \int_0^x dx' \frac{1}{\mathcal{P}} P_\psi(x') \psi'(x'),$$

(2.13)

and the remaining constraint

$$K = \int_0^{2\pi} dx \, P_\psi \psi' \approx 0.$$

(2.14)

This is the only first-class constraint remaining after gauge fixing. K generates a 1-parameter group of canonical transformations corresponding to a rigid translation in x, which is the coordinate freedom remaining after imposing (2.7).

The conditions (2.7), (2.12) and (2.13) define a subspace $i: \tilde{\Gamma} \to \Gamma$; the induced symplectic structure is

$$i^* \Omega = -dh \wedge dt + d\mathcal{P} \wedge d\mathcal{Q} + \int_0^{2\pi} dx \, \delta P_\psi \wedge \delta \psi$$

(2.15)

where

$$\mathcal{Q} = -2\pi\gamma_0 - \frac{t}{\mathcal{P}} h - \int_0^{2\pi} dx \int_0^x dx' \frac{1}{\mathcal{P}} P_\psi(x') \psi'(x').$$

(2.16)

and

$$h = \int_0^{2\pi} dx \, \frac{1}{2} (\frac{1}{t\mathcal{P}} P_\psi^2 + t\mathcal{P}\psi'^2).$$

(2.17)

The quantity h is therefore the Hamiltonian for the dynamics of the gauge-reduced system on $\tilde{\Gamma}$ with canonical chart $(\mathcal{Q}, \mathcal{P}; \psi, P_\psi)$, still subject to the constraint

$$\int_0^{2\pi} dx \, P_\psi \psi' \approx 0.$$

(2.18)

We now make a convenient change of variables on $\tilde{\Gamma}$. We define

$$\phi := \sqrt{\mathcal{P}}\psi, \quad P_\phi := \frac{1}{\sqrt{\mathcal{P}}} P_\psi.$$

(2.19)

and

$$Q = \mathcal{P}\mathcal{Q} + \frac{1}{2} \int_0^{2\pi} dx \, P_\psi \psi, \quad P = \ln \mathcal{P}.$$

(2.20)

Because

$$d\mathcal{P} \wedge d\mathcal{Q} + \int_0^{2\pi} dx \, \delta P_\psi \wedge \delta \psi = dP \wedge dQ + \int_0^{2\pi} dx \, \delta P_\phi \wedge \delta \phi$$

(2.21)

it is clear this is a canonical transformation.

We are thus led to the following description of the dynamics after deparametrization. The phase space $\tilde{\Gamma}$ includes a point particle degree of freedom described by canonical variables (Q, P) and a field degree of freedom described by canonical variables (ϕ, P_ϕ). These variables are evolved in the time t by the Hamiltonian(s)

$$H = \int_0^{2\pi} dx \, \frac{1}{2} (\frac{1}{t} P_\phi^2 + t\phi'^2).$$

(2.22)

The field degrees of freedom are subject to a first-class constraint

$$K = \int_0^{2\pi} dx \, P_\phi \phi' \approx 0.$$

(2.23)

The Hamilton equations of motion (in the gauge $N^x = 0$) are

$$\dot{Q} = \dot{P} = 0, \tag{2.24}$$

and

$$\dot{\phi} = \frac{1}{t} P_\phi, \quad \dot{P}_\phi = t\phi'', \tag{2.25}$$

which imply

$$\ddot{\phi} + \frac{1}{t}\dot{\phi} - \phi'' = 0. \tag{2.26}$$

The solutions of the equation of motion are

$$Q(t) = Q_0, \quad P(t) = P_0, \tag{2.27}$$

$$\phi(t) = q + p\ln t \tag{2.28}$$

$$+ \frac{1}{2\sqrt{2}} \sum_{n\neq 0} \left(a_n H_0(|n|t)e^{-inx} + a_n^* H_0^*(|n|t)e^{inx} \right), \tag{2.29}$$

$$P_\phi(t) = t\dot{\phi}(t), \tag{2.30}$$

Here H_0 is the zeroth order Hankel function of the second kind. The symplectic structure (2.21) implies the following non-vanishing Poisson brackets:

$$[q, p] = 1 = [Q, P], \quad [a_n, a_m^*] = -i\delta_{nm}. \tag{2.31}$$

The constraint $K = 0$ is equivalent to

$$\sum_{n\neq 0} n|a_n|^2 = 0. \tag{2.32}$$

3. Quantization: the Fock representation

Here we briefly review the by-now standard Fock space quantization of the model. Define a Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = L^2(\mathbf{R}^2) \otimes \mathcal{F}, \tag{3.1}$$

where $\mathcal{F}$ is the symmetric Fock space built from the Hilbert space of square-summable complex sequences, $\zeta_n, n = \pm 1, \pm 2, \ldots,$

$$\sum_{n\neq 0} |\zeta_n|^2 < \infty. \tag{3.2}$$

Any $\Psi \in \mathcal{F}$ can be represented as an infinite sequence of complex sequences§

$$\Psi = (\psi_0, \psi_{m_1}, \psi_{m_1 m_2}, \ldots, \psi_{m_1 \cdots m_k}, \ldots), \tag{3.3}$$

where $\psi_0 \in \mathbf{C}$,

$$\psi_{m_1 \cdots m_k} = \psi_{(m_1 \cdots m_k)}, \tag{3.4}$$

and

$$|\psi_0|^2 + \sum_{k=1}^{\infty} \sum_{m_1 \cdots m_k \neq 0} |\psi_{m_1 \cdots m_k}|^2 < \infty. \tag{3.5}$$

§ Here ψ_0 is the "vacuum amplitude", ψ_{m_1} is the amplitude for "1-particle with momentum m_1", etc. For convenience, we represent the entire sequence $\{\psi_k, \ k = \pm 1, \pm 2, \ldots\}$ simply by the symbol ψ_k.

We now define operators $(\hat{Q}, \hat{P}, \hat{\phi}, \hat{P}_\phi)$ corresponding to the canonical variables we constructed above. The canonical pairs $(\hat{Q}, \hat{P})$ and $(\hat{q}, \hat{p})$ are represented as identity operators on $\mathcal{F}$ and are represented on (a dense domain in) $L^2(\mathbf{R}^2)$ exactly as one would canonical coordinates and momenta for a particle moving in two dimensions, *e.g.*,

$$\psi = \psi(x, y) \in L^2(\mathbf{R}^2), \tag{3.6}$$

$$\hat{q}\psi = x\psi, \quad \hat{p}\psi = \frac{1}{i}\partial_x\psi, \quad \hat{Q}\psi = y\psi, \quad \hat{P}\psi = \frac{1}{i}\partial_y\psi. \tag{3.7}$$

(Other equivalent representations are, of course, possible.) The remaining degrees of freedom in the field $\hat{\phi}$ are represented as identity operators on $L^2(\mathbf{R}^2)$ and are represented on the Fock space $\mathcal{F}$ as follows. Using the representation (3.3) for $\Psi \in \mathcal{F}$, we define annihilation and creation operators for each $l \neq 0$ by

$$\hat{a}_l\Psi = (\psi_l, \sqrt{2}\psi_{lm_1}, \sqrt{3}\psi_{lm_1m_2}, \ldots) \tag{3.8}$$

$$\hat{a}_l^*\Psi = (0, \psi_0\delta_{m_1l}, \sqrt{2}\delta_{l(m_1}\psi_{m_2)}, \sqrt{3}\delta_{l(m_1}\psi_{m_2m_3)}, \ldots). \tag{3.9}$$

These operators (on their common domain) satisfy

$$\hat{a}_n^* = (\hat{a}_n)^\dagger, \quad [\hat{a}_n, \hat{a}_m^*] = \delta_{nm}\hat{I}, \tag{3.10}$$

where $\hat{I}$ is the identity operator on $\mathcal{H}$. The quantum field $\hat{\phi}$ is defined as an operator-valued distribution on $\mathbf{R}^+ \times \mathbf{S}^1$ using the operator representation of $(\hat{q}, \hat{p}, \hat{a}_n, \hat{a}_n^*)$ in the expansion (2.29).

This quantization just described satisfies the prescription

$$\{\text{Poisson Bracket}\} \leftrightarrow \frac{1}{i}\,[\text{commutator}] \tag{3.11}$$

for the canonical coordinates and momenta. As shown *e.g.*, in [16], quantizations of ϕ of this type can be placed in correspondence with a choice of scalar product (equivalently, a choice of complex structure) on the space of solutions to (2.26). This is tantamount to making a choice of "positive frequency solutions" to the field equation. The choice of positive frequency being made here can be understood as follows. As shown in [14] one can view the scalar field and its field equation as arising from a symmetry reduction of a massless scalar field restricted to a submanifold of $2 + 1$ dimensional Minkowski space. It is straightforward to show that the notion of positive frequency (*i.e.*, choice of scalar product, complex structure) being used to define the Fock representation above is the same as the usual notion of positive frequency in this $2 + 1$ dimensional Minkowski space.

It has been shown that, within this Fock representation, it is not possible to represent the Heisenberg picture time evolution of the field operators using a unitary transformation. In other words, there is no operator $\hat{U}$ on the Fock space such that

$$\hat{\phi}(t_2) = \hat{U}^{-1}\hat{\phi}(t_1)\hat{U}. \tag{3.12}$$

Roughly speaking, time evolution of the field variables can be viewed as a deformation of the scalar product (complex structure, notion of positive frequency) used to define the Fock representation. This deformation is too "large" to be represented as a unitary transformation. While this is a technical inconvenience (*e.g.*, the Schrödinger *picture* of time evolution is not available), it is not clear what — if any — problem this really causes for the physical viability of the quantization. In the context of the present investigation, we shall see that while this field-theoretic feature of the model has implications for the support of the measure in the Schrödinger representations associated with different times, it does not cause any difficulties with the construction/interpretation of the Schrödinger representation.

4. The Schrödinger representation

The Fock representation for the spatially inhomogeneous modes of $\hat{\phi}$ is analogous to the number operator representation of the harmonic oscillator. The representation we build now is, using the same analogy, the position wave function representation. The ingredients in the construction are (i) a space $\mathcal{S}'$ of distributional field configurations, (ii) a Gaussian integration measure $d\mu$ on $\mathcal{S}'$, (iii) a representation of the (spatially inhomogeneous modes of the) field and its conjugate momentum at a fixed time in terms of differential operators on $L^2(\mathcal{S}', d\mu)$. We are adopting the approach of Glimm and Jaffe [13].

We want a representation in which the field operator at a fixed time $t = T$, denoted by $\hat{\phi}(T)$, acts by multiplication. The spatially homogeneous mode of $\hat{\phi}(T)$ can be treated in the usual way for a point particle. We focus on the spatially inhomogeneous modes in what follows. Let $\mathcal{S}$ denote the pre-Hilbert space of smooth real functions f on the circle with vanishing integral:

$$\int_{S^1} f = 0. \tag{4.1}$$

We introduce the scalar product for $f, g \in \mathcal{S}$:

$$(f, g) = \int_{S^1} fg. \tag{4.2}$$

We can write, for $f \in \mathcal{S}$,

$$f = \sum_{n \neq 0} f_n e_n, \tag{4.3}$$

where

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{-inx} \in \mathcal{S}, \quad f_{-n} = f_n^*, \quad n \neq 0 \tag{4.4}$$

and $\{f_n\}$ is a rapidly decreasing sequence of complex numbers, that is,

$$\lim_{n \to \infty} n^p f_n = 0 \quad \forall p > 0. \tag{4.5}$$

Thus $\mathcal{S}$ can be identified with the set of rapidly decreasing complex numbers and is a nuclear space, as described, *e.g.,* in [15, 13].

Let $\mathcal{S}'$ denote the real vector space of linear functions on the nuclear space $\mathcal{S}$ which are continuous in the Frechet topology. This will be the "quantum configuration space" used to define the Schrödinger representation. (We shall see that this configuration space can be reduced in size, however.) Each $Q \in \mathcal{S}'$ is determined by its Fourier components:

$$Q(f) = \sum_{n \neq 0} Q_n f_n, \tag{4.6}$$

where

$$Q_n = Q(e_n), \quad Q_{-n} = Q_n^*. \tag{4.7}$$

The sequence $\{Q_n\}$ is slowly increasing, that is, the sequence of real numbers $\{|Q_n|\}$, $n = 1, 2, \ldots$, increases no faster than a polynomial in n.

Given $\mathcal{S}$ and $\mathcal{S}'$ we are in a position to construct an integration measure on $\mathcal{S}'$. As shown in [13], a Gaussian integration measure on $\mathcal{S}'$ is uniquely determined by a choice of a continuous, positive definite bilinear form $\mathbf{C} : \mathcal{S} \times \mathcal{S} \to \mathbf{R}$,

$$f, g \in \mathcal{S} \to \mathbf{C}(f, g) = (f, Cg). \tag{4.8}$$

Our strategy is to find the form of C dictated by the quantization of §3 and define the representation space for the spatially inhomogeneous modes of the scalar field operator as $L^2(\mathcal{S}', d\mu)$, where $d\mu$ is the Gaussian measure defined by C. A general formalism for finding the form of C appropriate to a given Fock representation can be found in [17]. A straightforward application of such techniques to the case at hand yields, for $f \in \mathcal{S}$,

$$Cf = \frac{1}{8}\sqrt{\frac{\pi}{2}} \sum_{n \neq 0} |H_0(|n|T)|^2 f_n e^{-inx}. \tag{4.9}$$

Because $|H_0(z)|$ decays like $\frac{1}{\sqrt{z}}$ as $z \to \infty$ the Fourier components of Cf are rapidly decreasing so that $C\colon \mathcal{S} \to \mathcal{S}$. Because $|H_0(|n|T)|$ is bounded and positive definite for fixed T and $n > 0$ it follows that $\mathbf{C}$ is continuous and positive definite on $\mathcal{S} \times \mathcal{S}$ as it should be. We remark that both the covariance and its associated measure depend upon the choice of time T. We denote the Gaussian integration measure defined by the covariance (4.9) by $d\mu_T$.

What is the nature of the Gaussian integration defined by the covariance (4.9)? On the cylinder set defined by $\{Q_{-N}, Q_{-N+1}, \ldots, Q_{-1}, Q_1, \ldots, Q_{N-1}, Q_N\} \in \mathcal{R}$ the covariance defines a measure given in terms of the Lebesgue measure dQ by

$$d\mu_T\Big|_{\mathcal{R}} = \prod_{j=1}^{N} \frac{4}{\pi^2 |H_0(|j|T)|^2} \exp\left\{ -\frac{4}{\pi |H_0(|j|T)|^2} |Q_j|^2 \right\} dQ_j dQ_j^*. \tag{4.10}$$

The Schrödinger representation is defined upon the Hilbert space $L^2(\mathcal{S}', d\mu_T)$. We still have to represent the field operators. It is convenient to work with the Fourier components of the field operators at time T. Using the pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{S}'$ and $\mathcal{S}$, we have

$$\hat{\phi}_n = \langle \hat{\phi}(T), e_{-n} \rangle, \quad \hat{P}_n = \langle \hat{P}_\phi(T), e_n \rangle. \tag{4.11}$$

Let us consider the following representation:

$$\hat{\phi}_n \Psi(Q) = Q_n \Psi(Q_n), \tag{4.12}$$

$$\hat{P}_n \Psi(Q) = \left(\frac{1}{i} \frac{\partial \Psi(Q)}{\partial Q_n} - |n|T \frac{H_1^*(|n|T)}{H_0^*(|n|T)} Q_{-n} \Psi(Q) \right) \tag{4.13}$$

We need to check the commutation relations

$$[\hat{\phi}_n, \hat{P}_m] = i\delta_{nm}\hat{I}, \quad [\hat{\phi}_n, \hat{\phi}_m] = 0 = [\hat{P}_n, \hat{P}_m], \tag{4.14}$$

and the reality conditions

$$\hat{\phi}_n^\dagger = \hat{\phi}_{-n}, \quad \hat{P}_n^\dagger = \hat{P}_{-n}. \tag{4.15}$$

The commutation relations easily follow from the fact that $\hat{P}$ differs from the operation of differentiation with respect to Q by the gradient of a function of Q. Now consider

the reality conditions. The conditions for $\hat{\phi}_n$ are obviously satisfied. For $\hat{P}_n$ we have

$$
\begin{aligned}
\int d\mu_T \Phi^* \hat{P}_n \Psi &= \int d\mu_T \Big[(\frac{1}{i}\partial_{-n}\Phi)^* \Psi + \frac{4}{\pi i |H_0(|n|T)|^2} Q_{-n} \Phi^* \Psi \\
&\quad - |n|T \frac{H_1^*(|n|T)}{H_0^*(|n|T)} Q_{-n} \Phi^* \Psi \Big] \\
&= \int d\mu_T \Big[(\frac{1}{i}\partial_{-n}\Phi)^* \Psi - |n|T \frac{H_1(|n|T)}{H_0(|n|T)} Q_{-n} \Phi^* \Psi \Big] \ (4.16) \\
&= \int d\mu_T \{ (\frac{1}{i}\partial_{-n} - |n|T \frac{H_1^*(|n|T)}{H_0^*(|n|T)} Q_n) \Phi \}^* \Psi \\
&= \int d\mu_T (\hat{P}_{-n}\Phi)^* \Psi.
\end{aligned}
$$

Here we used the Wronskian identity

$$
H_0(z) H_1^*(z) - H_0^*(z) H_1(z) = \frac{4}{i\pi z}. \tag{4.17}
$$

This representation for the field operators is such that

$$
\hat{a}_n = \frac{\sqrt{\pi}}{2} H_0^*(|n|T) \partial_{-n}, \tag{4.18}
$$

and

$$
\hat{a}_n^* = \hat{a}_n^\dagger = -\frac{\sqrt{\pi}}{2} H_0(|n|T) \partial_n + \frac{2}{\sqrt{\pi}} \frac{1}{H_0^*(|n|T)} Q_{-n}. \tag{4.19}
$$

Note, in particular, the "vacuum state" is $\Psi[Q] = 1$ in this representation so the Gaussian measure defines the probability distribution of the field values (at time T) for this state.

We remark that a parallel construction can be used to construct the Schrödinger "momentum representation" in which the momentum operators at a given time are diagonal. We shall not pursue this here.

Equivalence of the Schrödinger representation with the Fock representation (see below) in conjunction with results of [9] imply the field operators ϕ and P_ϕ can be defined as self-adjoint operators. One thus has the usual probability interpretation, *e.g.*, given a set $\mathcal{V} \subset \mathcal{S}'$ and a wave function $\Psi \in L^2(\mathcal{S}', d\mu_T)$, the probability $\mathbf{P}(\mathcal{V})$ for finding $\phi(T)$ with "values" in $\mathcal{V}$ is given by

$$
\mathbf{P}(\mathcal{V}) = \int_{\mathcal{V}} d\mu_T[Q] \, |\Psi[Q]|^2. \tag{4.20}
$$

Physical states of the Gowdy model must satisfy a quantum form of the constraint (2.32). As discussed, *e.g.*, in [9], this amounts to defining physical states as those belonging to the kernel of

$$
\hat{K} = \sum_{n \neq 0} n \hat{a}_n^\dagger a_n. \tag{4.21}
$$

This space of states is a Hilbert subspace spanned by the subset of the number operator basis with vanishing total field momentum. One can simply identify these states with their Schrödinger representatives to obtain the space of physical states. It is straightforward to check that this is equivalent to defining physical states as functions $\Psi[Q]$ which are invariant under the group of phase transformations

$$
Q_n \to e^{in\alpha} Q_n, \quad \alpha \in \mathbf{R}. \tag{4.22}
$$

Because the group is compact, the physical state wave functions are still square-integrable so this restriction will not cause any complications in what follows.

We have seen that for each time $t = T$ there is a Gaussian integration measure $d\mu_T$ and a corresponding Schrödinger representation on $L^2(\mathcal{S}', d\mu_T)$. If we define a linear transformation which identifies the occupation number states in $\mathcal{F}$ with those in $L^2(\mathcal{S}', d\mu_T)$ it is straightforward to see this transformation defines a unitary equivalence of the Fock and Schrödinger representations (see Theorem 6.3.4 in [13]). Schrödinger representations constructed at any two times are therefore also equivalent. This equivalence of the different Schrödinger representations may seem surprising at first sight, given the failure of unitary implementability of time evolution. However, the latter result arises because time evolution for this model can be viewed as a non-trivial change in the complex structure which defines the Fock representation. The Schrödinger representations at different times all correspond to a fixed choice of complex structure, *i.e.,* a single Fock representation and are therefore equivalent. Still, the failure of implementability does have implications for the Schrödinger representation: it prevents the mutual continuity of the Gaussian measures constructed at different times [18, 19]. Indeed, in the next section we shall show that the one parameter family of Gaussian measures we have constructed are all mutually singular.

5. Support of the measure and regularity properties of the field

The Schrödinger representation is defined in terms of functions on the space of distributions $\mathcal{S}'$, which suggests typical field configurations are rather singular. On the other hand, Gaussian measures are typically supported on a smaller space than $\mathcal{S}'$. For example, the Wiener measure for paths in one dimension is supported within a space of Hölder continuous paths. Here we will give some results on the support of the probability measure μ_T featuring in the Schrödinger representation. This will, in particular, give us some information about the regularity properties of typical field configurations.

We will apply the Minlos theorem in the form given in [20] (see also [21]) to our Gaussian measure. This theorem gives the following information. Let $\mathbf{H}$ be a Hilbert space with norm $||\cdot||$ and let A be a Hilbert-Schmidt operator on $\mathbf{H}$. Define the norm $||\cdot||_A = ||A \cdot ||$. If the covariance is continuous in $||\cdot||_A$, so that

$$(f, Cf) \leq \text{const.} ||f||_A^2, \tag{5.1}$$

then the corresponding Gaussian measure lies in the topological dual $\mathbf{H}'$.

For our Hilbert spaces we will use a class of Sobolev spaces. Recall that for any $s \in \mathbf{R}$ the Sobolev spaces $H^s(\mathbf{S}^1)$ can be defined in terms of the Fourier transform of $Q \in \mathcal{S}'$ as [22]:

$$H^s(\mathbf{S}^1) = \left\{ Q \in \mathcal{S}' \mid \sum_n (1+n^2)^s |Q_n|^2 < \infty \right\}. \tag{5.2}$$

The topological dual of H^s is H^{-s}.

In the Minlos theorem choose $\mathbf{H} = H^\epsilon(\mathbf{S}^1)$, where $\epsilon > 0$. Define the linear operator A on $\mathbf{H}$ by

$$(AQ)_n = \frac{Q_n}{(1+n^2)^{\frac{1}{4}+\frac{\epsilon}{2}}}. \tag{5.3}$$

It is easy to check that A is Hilbert-Schmidt. The norm defined by A is the norm for $H^{-\frac{1}{2}}(\mathbf{S}^1)$:

$$||Q||_A^2 = \sum_{n \neq 0} (1 + n^2)^{-\frac{1}{2}} |Q_n|^2. \tag{5.4}$$

We next check that the covariance (4.9) is continuous in this norm. We have that

$$(1 + n^2)^{-\frac{1}{2}} \geq \frac{\text{const.}}{|n|}, \tag{5.5}$$

And we have [23]

$$|H_0(z)|^2 \leq \frac{\text{const.}}{|z|}, \tag{5.6}$$

so that for any fixed $T > 0$

$$\sum_{n \neq 0} |H_0(|n|T)|^2 |f_n|^2 \leq \text{const.} \sum_{n \neq 0} (1 + n^2)^{-\frac{1}{2}} |f_n|^2 \tag{5.7}$$

as desired. We conclude that the Gaussian measure lies in $H^{-\epsilon}(\mathbf{S}^1)$ for any $\epsilon > 0$.

We remark that the subspace $L^2(\mathbf{S}^1) = H^0(\mathbf{S}^1) \subset H^{-\epsilon}(\mathbf{S}^1)$ has measure zero. This can be seen explicitly as follows. Consider the function

$$\chi_\alpha[Q] = \exp\left\{ -\frac{4\alpha}{\pi} \sum_{n=1}^{\infty} |Q_n|^2 \right\}, \quad \alpha \geq 0. \tag{5.8}$$

If $Q \in L^2(\mathbf{S}^1)$ then $\lim_{\alpha \to 0} \chi_\alpha = 1$; otherwise $\lim_{\alpha \to 0} \chi_\alpha = 0$. Thus $\lim_{\alpha \to 0} \chi_\alpha$ is the characteristic function on the set of square-integrable functions. We have

$$\mu_T(L^2(\mathbf{S}^1)) = \lim_{\alpha \to 0} \int d\mu_T \chi_\alpha[Q] \tag{5.9}$$

$$= \lim_{\alpha \to 0} \lim_{N \to \infty} \int d\mu_T \exp\left\{ -\frac{4\alpha}{\pi} \sum_{n=1}^{N} |Q_n|^2 \right\}. \tag{5.10}$$

(Interchange of integration and limits is permitted by the monotone convergence theorem.) We denote the integral in (5.10) by $I(N, \alpha)$; it can be computed via integration on the cylinder set defined earlier (*cf.* (4.10)) with $\mathcal{R} = \mathbf{R}^{2N}$. We have

$$I(N, \alpha) = \int_{\mathbf{R}^{2N}} \prod_{j=1}^{N} dQ_j dQ_j^*$$

$$\times \frac{4}{\pi^2 |H_0(|j|T)|^2} \exp\left\{ -\frac{4}{\pi} (\alpha + \frac{1}{|H_0(|j|T)|^2}) |Q_j|^2 \right\} \tag{5.11}$$

$$= \prod_{j=1}^{N} \frac{1}{1 + \alpha |H_0(|j|T)|^2}.$$

Using the asymptotic behavior of the Hankel function it follows that

$$\lim_{N \to \infty} I(N, \alpha) = 0 \tag{5.12}$$

so that $\mu_T(L^2(\mathbf{S}^1)) = 0$. Thus we can say that "typical" configurations of the quantized Gowdy field $\hat{\phi}(T)$ are slightly more singular than square-integrable functions but are not as singular as, say, the delta function.

The preceding results are valid for any element of the family of measures μ_T, $T > 0$. But this does not imply the support of the measure is the same for all time. It is not. More precisely, recall that two measures μ and μ' on a space M are said to be mutually singular if there exists a set $A \subset M$ such that $\mu(A) = 0$ and $\mu'(M \setminus A) = 0$. We shall now show that the Gaussian measures defined at two different times are mutually singular.

We will borrow methods from [24], which are designed to deduce finer support properties of Gaussian measures than can be obtained via the Minlos theorem. To begin with, it is straightforward to modify the proof of their Proposition 1 — which deals with an infinite product of identical Gaussian measures — to obtain the following result for the measure defined by (4.9). Let $\{\Delta_k\}$, $\Delta_k > 1$, $k = 1, 2, \ldots$ be a given sequence. Consider the set

$$Z_T(\Delta) = \{Q \in \mathcal{S}' \mid \exists\, N \in \mathbf{Z}^+ \text{ such that when } n > N \tag{5.13}$$

$$|Q_n| < |H_0(|n|T)|\sqrt{\frac{\pi}{4}\ln \Delta_n}\}. \tag{5.14}$$

The μ_T measure of $Z_T(\Delta)$ is unity if

$$\sum_{k=1}^{\infty} \frac{1}{\Delta_k \sqrt{\ln \Delta_k}} < \infty, \tag{5.15}$$

while the measure of $Z_T(\Delta)$ vanishes if the series diverges.

Next (still following the strategy of [24]), we note the sequence $\Delta_n = n^\alpha$ satisfies (5.15) for any $\alpha > 1$, while it fails to satisfy (5.15) for $\alpha = 1$. Thus $\mu_T(Z_T(\{n^\alpha\})) = 1$ for $\alpha > 1$ and the measure vanishes on the set with $\alpha = 1$.

Consider two times T_1 and T_2 with corresponding measures μ_{T_1} and μ_{T_2}. It is straightforward to verify that

$$Z_{T_1}(\{n\}) = Z_{T_2}(\{n^\xi\}), \tag{5.16}$$

where

$$\xi = \left|\frac{H_0(|n|T_1)}{H_0(|n|T_2)}\right|^2. \tag{5.17}$$

As $n \to \infty$ we have

$$\xi \sim \frac{T_2}{T_1},$$

so that $\Delta_n = n^\xi$ satisfies (5.15) for any $T_2 > T_1$. We see then that

$$\mu_{T_1}(Z_{T_1}(\{n\})) = 0, \quad \mu_{T_2}(Z_{T_1}(\{n\})) = 1. \tag{5.18}$$

It is also easy to see that

$$\mu_{T_2}(Z_{T_1}(\{n^\xi\})) = 1 \tag{5.19}$$

It is now easy construct a set disjoint from $Z_{T_1}(\{n\})$ within $Z_{T_1}(\{n^\xi\}))$ which has vanishing μ_{T_2} measure. Indeed, we have

$$\mu_{T_2}\left(Z_{T_1}(\{n^\xi\}) \setminus Z_{T_1}(\{n\})\right) = 0. \tag{5.20}$$

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