Taylor’s Theorem and Taylor Series (Appendix A)

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Appendix A. Taylor’s Theorem and Taylor Series

Taylor’s theorem and the Taylor series constitute one of the more important tools used by mathematicians, physicists and engineers. They provide a means of approximating a function in the vicinity of any chosen point in terms of polynomials. To begin, we present Taylor’s theorem, which is an identity satisfied by any function \( f(x) \) that has continuous derivatives of, say, order \( (n+1) \) on some interval \( a \leq x \leq b \). Taylor’s theorem asserts that

\[
f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \ldots + \frac{1}{n!} f^{(n)}(a)(x - a)^n + R_{n+1},
\]

where \( R_{n+1} \) – the remainder – can be expressed as

\[
R_{n+1} = \frac{1}{(n+1)!} (x - a)^{n+1} f^{(n+1)}(\xi),
\]

for some \( \xi \) with \( a \leq \xi \leq b \). Here we are using the notation

\[
f^{(k)}(c) = \frac{d^k f}{dx^k} \bigg|_{x=c}.
\]

The number \( \xi \) is not arbitrary; it is determined (though not uniquely) via the mean value theorem of calculus. For our purposes we just need to know that it lies between \( a \) and \( b \). The equation (A.1) is an identity; it involves no approximations.

The idea is that for many functions the value of \( n \) can be chosen to make the remainder sufficiently small compared to the polynomial terms so that we can omit the remainder to a good approximation. In this case we get Taylor’s approximation:

\[
f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \ldots + \frac{1}{n!} f^{(n)}(a)(x - a)^n.
\]

Typically, the approximation is reasonable provided \( x \) is close enough to \( a \) and none of the derivatives of \( f \) get too large in the region of interest. As you can see, if \( (x - a) \) is small, \( i.e., \, x - a << 1 \), successive powers of \( (x - a) \) become smaller and smaller so that one need only keep a few terms in the polynomial expansion to get a good approximation.

If you can prove that

\[
\lim_{n \to \infty} R_n = 0,
\]

then it makes sense to consider expressing \( f(x) \) as a power series:

\[
f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x - a)^n,
\]

which is known in this context as the Taylor series for \( f \). (Note that here we use the definitions \( 0! = 1 \) and \( f^{(0)}(x) = f(x) \).) Normally the Taylor series of a function will
converge in some neighborhood of \( x = a \) and diverge outside of this neighborhood.\(^*\) In any case, for a sufficiently “well-behaved” function, one can usually get a good approximation to the function using (A.4) even with \( n \) being relatively small. How small \( n \) needs to be depends, in large part, on how big \( (x - a) \) is. Often times one can get away with just choosing \( n = 1 \) or perhaps \( n = 2 \) for \( x \) sufficiently close to \( a \).

As a simple example, consider the sine function \( f(x) = \sin(x) \). Let us approximate the sine function in the vicinity of \( x = 0 \), so that we are taking \( a = 0 \) in the above formulas. The \textit{zeroth-order} approximation amounts to using \( n = 0 \) in (A.4). We get

\[
\sin(x) \approx \sin(0) = 0.
\]

This is obviously not a terribly good approximation. But you can check (using your calculator \textit{in radian mode}) that if \( x \) is nearly zero, so is \( \sin(x) \). A better approximation, the \textit{first-order} approximation, arises when \( n = 1 \) in (A.4). We get (exercise)

\[
\sin(x) \approx \sin(0) + \cos(0)x = x.
\]

Again, you can check this approximation on your calculator. If \( x \) is kept sufficiently small (in radians), this approximation does a pretty good job. As \( x \) gets larger the approximation gets less accurate. For example, at \( x = 0.1 \) the error in the approximation is about 0.2\%. At \( x = 0.75 \), the error is about 10\%. The \textit{second-order} approximation is identical to the first-order approximation, as you can check explicitly (exercise). The third-order approximation (exercise),

\[
\sin(x) \approx x - \frac{1}{6}x^3
\]

is considerably better than the first-order approximation. It gives good results out to, say, \( x = 1.7 \), where the error is about 11\%. Incidentally, the remainder term for the sine function satisfies (A.5) (exercise), and we can represent the analytic function \( \sin(x) \) by its (everywhere convergent) Taylor series:

\[
\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \cdots + \frac{1}{n!}x^n + \cdots, \quad n \text{ odd}.
\]

\(^*\) As usual, “convergence” in this context means that the sequence of partial sums approaches the stated value in the limit. Functions that can be represented by a convergent Taylor series are called \textit{analytic}.  

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