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Problem Set 3

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PROBLEM SET 3

Problem 3.1

Verify that

$$q(x,t) = A\cos(kx)\cos(\omega t) + B\sin(kx)\sin(\omega t)$$

solves the 1-d wave equation. Here A and B are constants, and $\omega = \frac{2\pi v}{\lambda} = kv$. If $A = B$, show that this solution can be written in the form

$$q(x,t) = A\cos[k(x - vt)].$$

Problem 3.2

Let $q(x,t) = te^x$. Define new independent variables: $u = x + vt$, $s = x - vt$ and find the form of $\tilde{q}(u,s) = q(x(u,s), t(u,s))$ in the new variables. Compute $\frac{\partial q}{\partial t}$ and $\frac{\partial q}{\partial x}$ using (a) direct calculation via $q(x,t)$ and (b) the chain rule via $\tilde{q}(u,s)$ and its u,s derivatives. Show that you get the same result.

Problem 3.3

Verify by direct computation using the chain rule that $q(x,t) = f(x + vt) + g(x - vt)$ solves the one-dimensional wave equation for any choice of the functions f and g.

Problem 3.4

Find the solution to the wave equation corresponding to the following initial data:

(a) $q(x,0) = 0$, $\frac{\partial q(x,0)}{\partial t} = 0$,

(b) $q(x,0) = \alpha x$, $\frac{\partial q(x,0)}{\partial t} = 0$, $\alpha =$ constant,

(c) $q(x,0) = 0$, $\frac{\partial q(x,0)}{\partial t} = \beta x$, $\beta =$ constant.

(d) $q(x,0) = \alpha x$, $\frac{\partial q(x,0)}{\partial t} = \beta x$, $\alpha, \beta =$ constant.

Problem 3.5

In (7.26) we present the general form of the solution to the wave equation in terms of arbitrary initial data. Verify directly that (a) this is a solution to the wave equation, and (b) it matches the initial data as advertised. (For part (a) you will need to use the Leibniz rule for differentiation of an integral.)

Problem 3.6

Determine the functions f and g in (7.13) associated with the standing wave solution

$$q(x,t) = A\sin(kx)\,\cos(kvt).$$

Determine all values of A and k such that this solution satisfies (i) the boundary conditions (7.17), and (ii) the boundary conditions (7.19).

Problem 3.7

Show that the general solution to the wave equation, $q(x,t) = f(x+vt) + g(x-vt)$, satisfies periodic boundary conditions $q(0,t) = q(L,t)$ if (7.19) holds. As an example, show that the solution $q = at$ satisfies these conditions on f and g (for any constant a).

Problem 3.8

Consider three different approximations to the derivative $f'(x)$ of a function $f(x)$ in terms of a small parameter $h << 1$:

(1) Forward difference: $\Delta_1 f = \frac{f(x+h)-f(x)}{h}$

(2) Backward difference: $\Delta_2 f = \frac{f(x)-f(x-h)}{h}$

(3) Central difference: $\Delta_3 f = \frac{f(x+\frac{1}{2}h)-f(x-\frac{1}{2}h)}{h}$

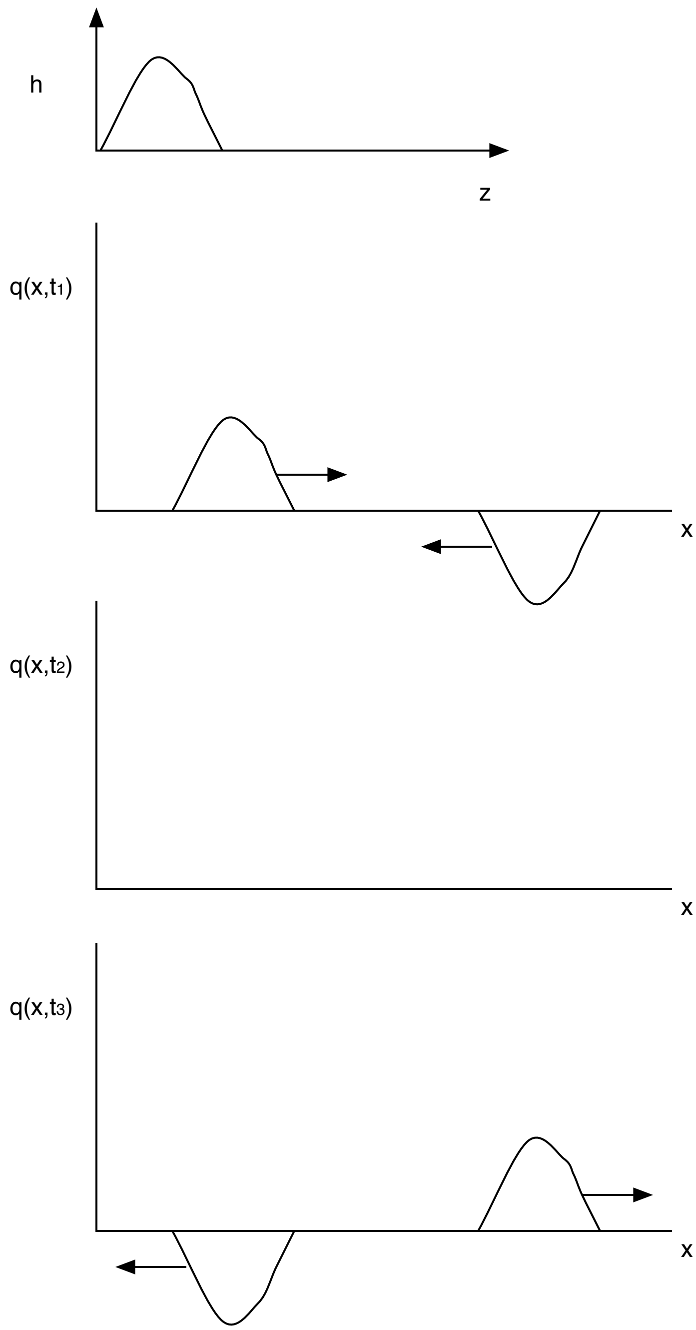
Use Taylor's theorem to show that the error in the derivative introduced by not taking the limit as $h \to 0$ is of order h for $\Delta_1 f$ and $\Delta_2 f$, but is of order h^2 for $\Delta_3 f$, and hence $\Delta_3 f$ is the more accurate approximation of the derivative.

Problem 3.9

Suppose $h(z)$ is a given smooth function such that

$$h(z) = 0 \quad z \leq 0, \text{ or } z \geq 1.$$

You can think of $h(z)$ as defining a "pulse" of finite length. Build a solution to the wave equation which corresponds to two such pulses, one of them inverted, which start out moving toward each other, which then collide and cancel each other out, and then propagate off to arbitrarily large values of $|x|$. (See the following figure.)


h
z
q(x,t1)
x
q(x,t2)
x
q(x,t3)
x