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The Schwarzschild Solution and Timelike Geodesics

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Abstract

General Relativity is the standard theory of the gravitational interaction. It allows us to calculate the motions and interactions of particles in a non-Euclidean space-time. This presentation will present the derivation of the Schwarzschild metric tensor field by finding a solution of the Einstein Equation for a non-rotating, static vacuum. A general form of the metric for a static, spherically symmetric spacetime will be used to calculate the Riemann curvature tensor and subsequently the Ricci tensor and Ricci scalar which will then be used to find a vaccum solution to the Einstein Equation. Once the solutions of the Einstein equation are found, we can study the geodesic equation. This lets us find the orbits for massive particles moving around and into a black hole. Overall, this presentation provide an examination of the basic calculations that are done in General Relativity and shows how matter moves in a curved space-time.

Einstein's General Theory of Relativity has produced a massive change in how we view the universe and how we calculate the motion of objects in the universe. Einstein was revolutionary in that he was able to develop an accurate theory of time and space that was later confirmed by observation. This theory involved using Riemannian Geometry to describe gravitation as the motion of particles in a curved spacetime. This is quite different from the classical theory of gravity outlined by Issac Newton and others. Perhaps the biggest paradigm shift caused by Einstein's General Theory of Relativity is that particles move in the straightest possible lines (geodesics) within this curved spacetime rather than particles being affected by gravity while moving in a Euclidian spacetime.

We begin by finding the metric tensor field for a static, non-rotating Schwartzchild black hole. A metric tensor field is symmetric bilinear form on the tangent space that satisfies the conditions of an inner product at each point [1]. Once this metric has been found, equations of motion – the geodesic equation – around this black hole can be calculated. There are three types of geodesic for which equations of motion can be calculated: timelike, spacelike, and null. Two of these, the timelike and null, correspond to known types of particle. In this paper we solve the geodesic equation for a radially infalling particle, showing that the particle reaches the horizon in a finite them. Then we compute the solution for circular orbits around a Schwartzchild black hole, showing that there is a minimum radius related to the mass of the black hole.

The Schwarzschild solution Using a static, spherically symmetric metric with all components being functions of the radius allows us to put the line element in the form $ds^2 = -f^2(r)dt^2 + h^2(r)dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2$. The coeffecient matrix, denoted $g_{\mu\nu}$ is as follows:

$$g_{\mu\nu} = \begin{pmatrix} -f^2(r) & 0 & 0 & 0 \\ 0 & h^2(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \tag{1}$$

In order to solve for the functions $f^2(r)$ and $h^2(r)$, the Einstein tensor for this metric will need to be found. The Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R$, where $R_{\mu\nu}$is the Ricci curvature tensor, $g_{\mu\nu}$ is the metric tensor field, and R is the Ricci scalar. The Ricci curvature tensor and the Ricci scalar are contractions of the Riemannian curvature tensor. The Riemannian curvature tensor gives a measure the curvature of a Riemannian manifold by encoding the transformation of a vector $\vec{V}$ under parallel transport along a closed loop in the manifold [2].

The method used to find the Riemannian curvature tensor is to first find the Christoffel connection of the metric which is found from a given metric tensor field and its derivatives [3]. Although the Christoffel connection is not present in the Einstein equation, it is an essential step on the way to calculating the Riemannian curvature of a given metric tensor field. Specifically, the Christoffel connection is given by [3]:

$$\Gamma^{\alpha}_{\ \mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \tag{2}$$

Substituting in the metric $g_{\mu\nu}$ above, we find that the Christoffel symbols are:

$$
\begin{aligned}
\Gamma^{t}{}_{tr} &= \Gamma^{t}{}_{rt} = \frac{f'}{f} \\
\Gamma^{r}{}_{tt} &= \frac{ff'}{h^2} \\
\Gamma^{r}{}_{rr} &= \frac{h'}{h} \\
\Gamma^{r}{}_{\theta\theta} &= -\frac{r}{h^2} \\
\Gamma^{r}{}_{\phi\phi} &= -\frac{r\sin^2(\theta)}{h^2} \\
\Gamma^{\theta}{}_{\theta r} &= \Gamma^{\theta}{}_{r\theta} = \frac{1}{r} \\
\Gamma^{\theta}{}_{\phi\phi} &= -\sin(\theta)\cos(\theta) \\
\Gamma^{\phi}{}_{\theta\phi} &= \Gamma^{\phi}{}_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)} \\
\Gamma^{\phi}{}_{\phi r} &= \Gamma^{\phi}{}_{r\phi} = \frac{1}{r}
\end{aligned}
\tag{3}
$$

where these equations are in terms of the functions f, h in the metric $g_{\mu\nu}$. A prime denotes a derivative with respect to r, $f' \equiv \frac{df}{dr}$. Using these expressions for the Christoffel symbols, the Riemann curvature tensor is given by:

$$
R^{\alpha}_{\ \beta\mu\nu} = \Gamma^{\alpha}_{\ \beta\nu,\mu} - \Gamma^{\alpha}_{\ \beta\mu,\nu} + \Gamma^{\alpha}_{\ \mu\rho}\Gamma^{\rho}_{\ \beta\nu} - \Gamma^{\alpha}_{\ \nu\rho}\Gamma^{\rho}_{\ \beta\mu} \tag{4}
$$

Substituting to find the components of the Riemann curvature tensor,

$$
\begin{aligned}
R^{t}{}_{rtr} &= -R^{t}{}_{rrt} = -\frac{f''h - f'h'}{fh} \\
R^{t}{}_{\theta t\theta} &= -R^{t}{}_{\theta\theta t} = -\frac{f'r}{fh^2} \\
R^{t}{}_{\phi t\phi} &= -R^{t}{}_{\phi\phi t} = -\frac{f'r\sin^2(\theta)}{fh^2} \\
R^{r}{}_{trt} &= -R^{r}{}_{ttr} = \frac{(f''h - f'h')f}{h^3} \\
R^{r}{}_{\theta r\theta} &= -R^{r}{}_{\theta\theta r} = \frac{rh'}{h^3} \\
R^{r}{}_{\phi r\phi} &= -R^{r}{}_{\phi\phi r} = \frac{r\sin^2(\theta)h'}{h^3} \\
R^{\theta}{}_{t\theta t} &= -R^{\theta}{}_{tt\theta} = \frac{ff'}{rh^2} \\
R^{\theta}{}_{r\theta r} &= -R^{\theta}{}_{rr\theta} = \frac{h'}{rh} \\
R^{\theta}{}_{\phi\theta\phi} &= -R^{\theta}{}_{\phi\phi\theta} = \frac{\sin^2(\theta)(h^2-1)}{h^2} \\
R^{\phi}{}_{t\phi t} &= -R^{\phi}{}_{tt\phi} = -\frac{f'r\sin^2(\theta)}{fh^2} \\
R^{\phi}{}_{r\phi r} &= -R^{\phi}{}_{rr\phi} = \frac{h'}{rh} \\
R^{\phi}{}_{\theta\phi\theta} &= -R^{\phi}{}_{\theta\theta\phi} = \frac{h^2-1}{h^2}
\end{aligned}
\tag{5}
$$

By contracting the Riemann curvature tensor on its first and third indices, the Ricci curvature tensor can be found. The resulting symmetric tensor is the first object to appear in Einstein's field equation. Using the components found for the Riemann curvature tensor, the Ricci curvature tensor can be calculated as the contraction:

$$R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu} \tag{6}$$

It is worth noting that the contraction on the first and third indices is the only contraction that we care about for these purposes. All other contractions either vanish, or simplify down to $\pm R_{\mu\nu}$[5, 6]. Carrying out the sums, we find

$$\begin{aligned}
R_{tt} &= \frac{f}{rh^3}\left(hf'' - rh'f' + 2hf'\right) && (7)\\
R_{rr} &= \frac{1}{rfh}\left(-hf'' + rh'f' + 2fh'\right) && (8)\\
R_{\theta\theta} &= \frac{1}{fh^3}\left(fh^3 + rh'f - rf'h - fh\right) && (9)\\
R_{\phi\phi} &= \frac{\sin^2\theta}{fh^3}\left(fh^3 + rh'f - rf'h - fh\right) && (10)
\end{aligned}$$

Now we are ready to write and solve the Einstein equation [7],

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{11}$$

where R is the Ricci scalar defined by $R = g^{\mu\nu} R_{\mu\nu}$. By taking the trace of the Einstein tensor, we can show that $R = 0$. Thus, solving the Einstein Equation can be done by setting $R_{\mu\nu} = 0$.

The equations to be solved are therefore,

$$\begin{aligned}
0 &= \frac{f}{rh^3}\left(hf'' - rh'f' + 2hf'\right)\\
0 &= \frac{1}{rfh}\left(-hf'' + rh'f' + 2fh'\right)\\
0 &= \frac{1}{fh^3}\left(fh^3 + rh'f - rf'h - fh\right)\\
0 &= \frac{\sin^2\theta}{fh^3}\left(fh^3 + rh'f - rf'h - fh\right)
\end{aligned}$$

Since $R_{\mu\nu} = 0$, $R_{tt} + R_{rr}$ can be added together to get:

$$\begin{aligned}
R_{tt} + R_{rr} &= 2\left(fh' + hf'\right)\\
&= 2\frac{d}{dr}\left(fh\right)
\end{aligned}$$

which implies $fh = K$ where K is a constant. Cancelling the prefactor on the final two equations, we see that $R_{\theta\theta} = 0$ and $R_{\phi\phi} = 0$ give the same equation. Because the Bianchi identity relates the four equations, there is only one more equation to solve.

Eliminating $h = \frac{K}{f}$ from $R_{\theta\theta} = 0$,

$$\begin{aligned}
0 &= \frac{K^3}{f^2} - \frac{Kf'r}{f} - \frac{Kf'r}{f} - K \\
&= K\left(\frac{K^2}{f} - \frac{2f'r}{f} - 1\right) \\
2f\dot{f}r &= K^2 - f^2 \\
\frac{2f}{(K^2 - f^2)}\frac{df}{dr} &= \frac{1}{r} \\
\frac{2f}{(K^2 - f^2)}df &= \frac{1}{r}dr
\end{aligned} \tag{12}$$

Integrating yields:

$$\int \frac{2f}{(K^2 - f^2)}df = \int \frac{1}{r}dr \tag{13}$$

$$-\ln\left(f^2 - K^2\right) = \ln\frac{r}{C} \tag{14}$$

Solving:

$$f^2 = K^2 + \frac{C}{r} \tag{}$$

where C is the integration constant.

The function $h^2(r)$ can be found by squaring the equation $f(r)h(r) = K$ and substituting our solution for $f^2(r)$.

$$f^2(r)h^2(r) = K^2 \Rightarrow h^2(r) = \frac{K^2}{K^2 + \frac{C}{r}} \tag{15}$$

Now the components of the original line element can be written as

$$ds^2 = -\left(K^2 + \frac{C}{r}\right)dt^2 + \frac{K^2}{K^2 + \frac{C}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{16}$$

It is beneficial to simplify the time part of the equation to $-K^2\left(1 + \frac{W}{r}\right)dt^2$ where $W = \frac{C}{K^2}$. Then rescaling the time, $\bar{t} = K^2 t \Rightarrow d\bar{t} = K^2 dt$ puts the line element in the simpler form,

$$-d\tau^2 = ds^2 = -\left(1 + \frac{W}{r}\right)dt^2 + \frac{dr^2}{1 + \frac{W}{r}} + r^2 d\Omega^2 \tag{17}$$

To find the constant W, we compare to Newton's law of universal gravitation. At a large enough radius from the spherical mass, we can model the motion of a particle as if the particle was under the influence of Newtonian gravity, neglecting smaller effects evident in stronger gravity. We follow a particle with mass much less than the mass of the spherical body from rest at a large distance. The geodesic equation is

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha{}_{\beta\mu}u^\beta u^\mu \tag{18}$$

and the initial 4-velocity of the particle is:

$$u_0^\alpha = \left(c\frac{dt}{d\tau}, 0, 0, 0\right) \tag{19}$$

By substituting this initial value into the geodesic equation for the metric, only the radial component remains. Looking at this radial component of the geodesic equation with the initial condition we get:

$$\frac{du^r}{d\tau} = -\,\Gamma^1{}_{00}\left(u_0^t\right)^2 \tag{20}$$

Having already calculated the value of

$$\begin{aligned}\Gamma^r{}_{tt} &= \frac{ff'}{h^2}\\ &= -\frac{W}{2r^2}\left(1+\frac{W}{r}\right)\end{aligned}$$

we need to only find the value of u_0^t. The norm of the 4-velocity is

$$\begin{aligned}-1 &= g_{\mu\nu}u^\mu u^\nu && (21)\\ &= -\left(1+\frac{W}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1+\frac{W}{r}}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2 sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 && (22)\end{aligned}$$

For an initially stationary particle, $\frac{dr}{d\tau} = \frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = 0$, it is trivial to see that $\frac{dt}{d\tau} = \frac{1}{\sqrt{1+\frac{W}{r}}}$. Therefore, we can insert this value into the geodesic equation and find:

$$\frac{du^r}{d\tau} = \left(1+\frac{W}{r}\right)\frac{W}{2r^2}\frac{c^2}{\left(1+\frac{W}{r}\right)} \approx \frac{c^2W}{2r^2} \tag{23}$$

At a large enough radius, $\frac{dt}{d\tau} \approx 1$ which means that $dt \approx d\tau$ and

$$\frac{du^r}{dt} \approx \frac{c^2W}{2r^2}$$

From Newton's universal law of gravitation, the force due to gravity between two masses is radial with magnitude

$$F^r_{grav} = -G\frac{Mm}{r^2} \tag{24}$$

Substituting this into the second law, m cancels to give

$$\frac{du^r}{dt} = -\frac{GM}{r^2}$$

Given that this is a system of a point particle starting at a large radius from a spherically symmetric mass, a comparison of the radial acceleration of the point particle found from the geodesic equation with the radial acceleration found from Newton's universal law of gravitation can be made. Equating the accelerations,

$$\frac{GM}{r^2} = \frac{c^2W}{2r^2} \tag{25}$$

where M is the mass of the central spherical body. Solving for the constant W:

$$W = \frac{2GM}{c^2} \tag{26}$$

We see that W is the escape velocity for a black hole.

Now, writing $\frac{2GM}{rc^2}$ as $\frac{2M}{r}$, we have the complete line element of a Schwartzchild black hole,

$$-d\tau^2 = ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

geodesic calculations can be made.

Particles do not orbit each other due to some gravitational force, but rather masses curve the spacetime around them particles move in the straightest possible lines within this curved spacetime. These straight lines are called geodesics. The geodesic equation produces four differentiable equations that describe the motion of a particle in a space with a given metric tensor field and some initial conditions [8]. We calculate two motions will be calculated by having a particle of negligible mass compared to the black hole: radial infall from rest, and a circular orbit. These will be calculated with units in which $G = c = 1$ so that $M = \frac{GM}{c^2}$.

The first motion will be calculated with the particle starting from rest. Essentially, the particle will be dropped from a radius $r_0 \gg r_s$ (where r_s is the Schwartzchild radius of the black hole) and fall radially inward. The initial conditions of this particle will be:

$$\begin{aligned} x_0^\alpha &= \left(0, r_0, \frac{\pi}{2}, 0\right) \\ u_0^\alpha &= \left(u_0^t, 0, 0, 0\right) \end{aligned} \tag{27}$$

The value of u_0^t can be found by looking at the norm $g_{\mu\nu} u_0^\mu u_0^\nu = -1$. Solving for u_0^t:

$$u_0^t = \frac{1}{\sqrt{1 - \frac{2M}{r_0}}} \tag{28}$$

Using the Christoffel symbols from above, the geodesic equations are:

$$\begin{aligned} \frac{d^2x^t}{d\tau^2} &= -\frac{2M}{r^2\left(1 - \frac{2M}{r}\right)} u^t u^r \\ \frac{d^2x^r}{d\tau^2} &= -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)\left(u^t\right)^2 + \frac{M}{r\left(1 - \frac{2M}{r}\right)}\left(u^r\right)^2 - \left(1 - \frac{2M}{r}\right)\left(u^\theta\right)^2 - \left(1 - \frac{2M}{r}\right) r \sin^2(\theta)\left(u^\phi\right)^2 \\ \frac{d^2x^\theta}{d\tau^2} &= -\frac{2}{r} u^r u^\theta + \sin(\theta)\cos(\theta)\left(u^\phi\right)^2 \\ \frac{d^2x^\phi}{d\tau^2} &= -\frac{2}{r} u^r u^\phi - \frac{2\cos(\theta)}{\sin(\theta)} u^\theta u^\phi \end{aligned} \tag{29}$$

By writing $\frac{d^2x^t}{d\tau^2} = \frac{du^t}{d\tau}$ and $u^r = \frac{dr}{d\tau}$, integrating the first of the geodesic equations using the initial conditions becomes:

$$\begin{aligned} \frac{du^t}{d\tau} &= -\frac{2M}{r^2\left(1 - \frac{2M}{r}\right)} u^t \frac{dr}{d\tau} \\ \int_{u_0^t}^{u^t} \frac{du^t}{u^t} &= -\int_{r_0}^{r} \frac{2M}{r^2\left(1 - \frac{2M}{r}\right)} dr \\ \ln\left(\frac{u^t}{u_0^t}\right) &= \ln\left(\frac{1 - \frac{2M}{r_0}}{1 - \frac{2M}{r}}\right) \\ u^t &= u_0^t \left(\frac{1 - \frac{2M}{r_0}}{1 - \frac{2M}{r}}\right) \end{aligned} \tag{30}$$

Integrating $\frac{d^2x^\theta}{d\tau^2}$ and $\frac{d^2x^\phi}{d\tau^2}$ with the initial conditions shows that $u^\theta = u^\phi = 0$ and therefore $\frac{d^2x^\theta}{d\tau^2} = \frac{d^2x^\phi}{d\tau^2} = \textit{constant}$ which is understandable because intuitively the particle is expected to fall radially only and is not expected to have any angular changes in this reference frame.

The radial component of the geodesic equation can be found by looking at the normal of the 4-velocity and inserting the values for u^t, u^θ, and u^ϕ.

$$
\begin{aligned}
-1 &= -\big(1-\frac{2M}{r}\big)\big(u^t\big)^2 + \frac{(u^r)^2}{\big(1-\frac{2M}{r}\big)} + r^2\big(u^\theta\big) + r^2\sin^2(\theta)\big(u^\phi\big)^2 \\
&= -\frac{1-\frac{2M}{r_0}}{1-\frac{2M}{r}} + \frac{(u^r)^2}{1-\frac{2M}{r}} \\
\frac{(u^r)^2}{1-\frac{2M}{r}} &= \frac{1-\frac{2M}{r_0}}{1-\frac{2M}{r}} - 1 \\
u^r &= \sqrt{\Big(1-\frac{2M}{r_0}\Big) - \Big(1-\frac{2M}{r}\Big)} \\
&= -\sqrt{\frac{2M}{r} - \frac{2M}{r_0}}
\end{aligned}
\tag{31}
$$

This yields all components of the 4-velocity,

$$
\begin{aligned}
u^t &= u^t_0\left(\frac{1-\frac{2M}{r_0}}{1-\frac{2M}{r}}\right) \\
u^r &= -\sqrt{\frac{2M}{r} - \frac{2M}{r_0}} \\
u^\theta &= 0 \\
u^\phi &= 0
\end{aligned}
\tag{32}
$$

The radial equation can be integrated to find the proper time for the particle to start at r_0 and fall to the horizon at $r = 2M$.

$$
\begin{aligned}
\frac{dr}{d\tau} &= -\sqrt{\frac{2M}{r} - \frac{2M}{r_0}} \\
\tau &= -\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2M}{r} - \frac{2M}{r_0}}} \\
&= r_0\left(\sqrt{\frac{r_0}{2M}}\left(\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{2M}{r_0}}\right) + \sqrt{1-\frac{2M}{r_0}}\right)
\end{aligned}
\tag{33}
$$

Assuming that the initial radius is much larger than $2M$, $r_0 \gg 2M$, the proper time that it would take for the particle to fall from r_0 to $2M$ would be approximately

$$
\begin{aligned}
c\tau &= r_0\left(\sqrt{\frac{r_0}{2M}}\left(\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{2M}{r_0}}\right) + \sqrt{1-\frac{2M}{r_0}}\right) \\
\tau &= \frac{r_0}{c}\left(\sqrt{\frac{r_0}{2M}}\frac{\pi}{2} + 1\right) \\
&\approx \frac{\pi}{2c}\sqrt{\frac{r_0^3 c^2}{2GM}} \\
&= \sqrt{\frac{\pi^2 r_0^3}{8GM}}
\end{aligned}
\tag{34}
$$

Using this, a value for r_0 can be substituted and the proper time elapsed for the particle to fall into the event horizon of the black hole can be found. For example, take $r_0 = 8M$. Then we have

$$
\begin{aligned}
\tau &= 8M\left(\sqrt{\frac{8M}{2M}}\left(\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{2M}{8M}}\right) + \sqrt{1 - \frac{2M}{8M}}\right) \\
\tau &= 8M\left(2\Big(\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{2}\right)\Big) + \sqrt{\frac{3}{4}}\right) \\
&= 8M\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) \\
&\approx 9.83M
\end{aligned}
\tag{35}
$$

Notice that in this calculation, M has not been expanded and therefore $M = \frac{GM}{c^2}$.

For the second basic geodesic calculation, we will be looking into a circular orbit. Unlike the previous calculation, the radius will be held constant and the particle will only have a time component and angular components. Additionally, being that the orbit is circular, the particle will only be moving in the ϕ-direction (The geodesic equation shows that a particle starting at $\theta = \frac{\pi}{2}$ will stay there). For this, the initial conditions will be:

$$\begin{aligned} x^\alpha &= \left(0, r_0, \frac{\pi}{2}, 0\right) \\ u_0^\alpha &= \left(u_0^t, 0, 0, u_0^\phi\right) \end{aligned} \tag{36}$$

Knowing that the particle position is constant in the r and θ direction, the equations of motion can be found from the t and ϕ components of the geodesic equation as well as the norm of the 4-velocity.

$$\begin{aligned} \frac{du^t}{d\tau} &= -\frac{2M}{r^2\left(1 - \frac{2M}{r}\right)} u^t u^r \\ \frac{du^\phi}{d\tau} &= -\frac{2}{r} u^r u^\phi \\ -1 &= -\left(1 - \frac{2M}{r}\right)\left(u^t\right)^2 + r^2 \sin^2(\theta)\left(u^\phi\right)^2 \end{aligned} \tag{37}$$

Solving for the time component yeilds:

$$\begin{aligned} \frac{du^t}{d\tau} &= -\frac{2M}{r^2\left(1 - \frac{2M}{r}\right)} u^t u^r = 0 \\ \frac{du^t}{d\tau} &= 0 \\ u^t &= u_0^t \end{aligned} \tag{38}$$

Solving for the phi component yeilds:

$$\begin{aligned} \frac{du^\phi}{d\tau} &= -\frac{2}{r}\frac{dr}{d\tau} u^\phi \\ \int_{u_0^\phi}^{u^\phi} \frac{du^\phi}{u^\phi} &= -\int_{r_0}^{r} \frac{2}{r} dr \\ \ln\left(\frac{u^\phi}{u_0^\phi}\right) &= -2\ln\left(\frac{r}{r_0}\right) \\ \frac{u^\phi}{u_0^\phi} &= \left(\frac{r_0}{r}\right)^2 \\ u^\phi r^2 &= u_0^\phi r_0^2 = L \\ u^\phi &= \frac{L}{r^2} \end{aligned} \tag{39}$$

Since r is constant, u^ϕ is constant and the constant L is the angular momentum per unit mass. These values can be plugged into the norm of the 4-velocity to find the value for u^t.

$$\begin{aligned} -1 &= -\left(1 - \frac{2M}{r_0}\right)\left(u_0^t\right)^2 + \left(\frac{L}{r_0}\right)^2 \\ 1 + \left(\frac{L^2}{r_0^2}\right) &= \left(1 - \frac{2M}{r_0}\right)\left(u_0^t\right)^2 \\ u_0^t &= \sqrt{\frac{1 + \left(\frac{L^2}{r_0^2}\right)}{1 - \frac{2M}{r_0^2}}} \end{aligned} \tag{40}$$

The value for u^ϕ can be found by substituting these values into the r component of the geodesic equation.

$$\begin{aligned}
\frac{du^r}{d\tau} &= -\frac{M}{r_0^2}\left(1-\frac{2M}{r_0}\right)\left(u_0^t\right)^2 + \frac{M}{r_0\left(1-\frac{2M}{r_0}\right)}\left(u_0^r\right)^2 - \left(1-\frac{2M}{r_0}\right)\left(u_0^\theta\right)^2 - \left(1-\frac{2M}{r_0}\right)\sin^2(\theta) r_0 \left(u_0^\phi\right)^2 \\
0 &= \frac{M}{r_0^2}\left(1-\frac{2M}{r_0}\right)\left(\frac{1+\frac{L^2}{r_0^2}}{1-\frac{2M}{r_0^2}}\right) - \left(1-\frac{2M}{r_0}\right) r_0 \left(\frac{L}{r_0}\right)^2 \\
\frac{M}{r_0^2} + \frac{M}{r_0^2}\frac{L^2}{r_0^2} &= \frac{L^2}{r_0^3} - 2M\frac{L^2}{r_0^4} \\
\frac{M}{r_0^2} &= \frac{L^2}{r_0^3} - \frac{M}{r_0^2}\frac{L^2}{r_0^2} - 2M\frac{L^2}{r_0^4} \\
\frac{L^2}{r_0^4} &= \frac{M}{r_0^2\left(r_0-3M\right)} \\
\frac{L}{r_0^2} &= \sqrt{\frac{M}{r_0^2\left(r_0-3M\right)}} \\
u_0^\phi &= \sqrt{\frac{Mr_0^2}{r_0-3M}}
\end{aligned} \tag{41}$$

From equation 39, it can be seen that $L = \sqrt{\frac{Mr_0^4}{r_0^2(r_0-3M)}}$ and this value for L can be substituted into the equation for u_0^t to find a value without the L-term.

$$\begin{aligned}
u_0^0 &= \sqrt{\frac{1+\frac{L^2}{r_0^2}}{1-\frac{2M}{r_0}}} \\
&= \sqrt{\frac{1+\frac{1}{r_0^2}\frac{Mr_0^4}{r_0^2(r_0-3M)}}{1-\frac{2M}{r_0}}} \\
&= \sqrt{\frac{r_0 + r_0\left(\frac{M}{r_0-3M}\right)}{r_0-2M}} \\
&= \sqrt{\frac{\frac{r_0^2-3Mr_0}{2M-r_0} + \frac{Mr_0}{r_0-3M}}{r_0-2M}} \\
&= \sqrt{\frac{\frac{r_0^2-2Mr_0}{r_0-3M}}{r_0-2M}} \\
&= \sqrt{\frac{r_0}{r_0-2M}\frac{r_0-2M}{r_0-3M}} \\
u_0^0 &= \sqrt{\frac{r_0}{r_0-3M}}
\end{aligned} \tag{42}$$

With all of the components for the 4-velocity found, the final 4-velocity is:

$$u^\alpha = \sqrt{\frac{1}{r-3M}}\left(\sqrt{r}, 0, 0, \sqrt{M}r\right) \tag{43}$$

This diverges at $r = 3M$ showing that there are no timelike orbits that exist at or within $r = 3M$.

These circular orbits at $3M$ are not stable. If the particle were to be bumped or jostled either way in the radial direction, the particle would either fall into the hole or fly off. Other orbits are stable, however, depending on the initial conditions. By doing a calculation such as the one done above, but not leaving the radius fixed, one can show that small perturbations of circular or elliptical orbits lead to nearby orbits. In the elliptical orbits, the perihelion and aphelion precesses. Such a calculation shows, famously, that the excess perihelion precession of Mercury is a consequence of General Relativity [9].

This has been an exercise in the basic calculations that can be made in Einstein's General Theory of Relativity. This is just the beginning of what can be done. There are many more complex calculations that can be made. General Relativity has given rise to an entirely new idea of how particles in our universe interact with one another. In the future, I would like to calculate motions around more complex objects such as non-static, rotating, and charged black holes as well as other, more theoretical, masses. It would be interesting to calculate the geodesics of a particle moving around a torus or a non-orientable object such as a Klein bottle.

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