# SPINORS, JETS, AND THE EINSTEIN EQUATIONS

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Abstract. Many important features of a field theory, *e.g.*, conserved currents, symplectic structures, energy-momentum tensors, *etc.*, arise as tensors locally constructed from the fields and their derivatives. Such tensors are naturally defined as geometric objects on the jet space of solutions to the field equations. Modern results from the calculus on jet bundles can be combined with a powerful spinor parametrization of the jet space of Einstein metrics to unravel basic features of the Einstein equations. These techniques have been applied to computation of generalized symmetries and "characteristic cohomology" of the Einstein equations, and lead to results such as a proof of non-existence of "local observables" for vacuum spacetimes and a uniqueness theorem for the gravitational symplectic structure.

## 1 Introduction

This is a survey of results of work performed largely in collaboration with Ian Anderson (Dept. of Mathematics, Utah State University). The presentation will be somewhat informal; rigorous statements and proofs of our results will be presented elsewhere [Torre and Anderson (1993)], [Anderson and Torre (1994)], [Anderson and Torre (1995)]. In the very broadest terms, our efforts are intended to help answer the question: "In what ways are the Einstein equations special?" There are of course a plethora of special features of the Einstein equations which have been uncovered since the advent of general relativity. Here are some examples. The vacuum equations in 4 dimensions are, up to specification of the cosmological constant, the only second order partial differential equations one can write down for a metric which are "generally covariant" and can be derived from a variational principle. Despite the complexity of the field equations, a large number of exact solutions are known. Indeed, certain reductions of the equations (self-dual equations, stationary-axisymmetric vacuum and electrovac equations) admit transitive symmetry groups and are in some sense "integrable". Special features that are of particular relevance to the physical viability of the Einstein equations include theorems guaranteeing that the Cauchy problem is well posed, existence and uniqueness (up to diffeomorphisms) of solutions, existence and positivity of conserved energy-momentum in the asymptotically flat context, *etc*. Finally, a feature of the Einstein equations which is especially relevant for attempts at quantization is that the equations constitute a constrained Hamiltonian system.

The modern geometric theory of differential equations provides systematic means through which one can characterize certain important structural features of any set of field equations. These structural features arise as geometric objects on the *jet space* of solutions to the field equations [Saunders (1989)], [Olver (1993)] and are neatly analyzed in terms of the *variational bicomplex* [Anderson (1992)], [Olver (1993)]. Such jet space techniques have been used in the analysis of a number of differential equations of applied mathematics and mathematical physics. It is now possible to apply these techniques to study special features of the vacuum Einstein equations; the results of such investigations are the subject of this paper. In particular, I will report on a systematic classification of *generalized symmetries* and *generalized conservation laws* for the vacuum equations. Roughly speaking, generalized symmetries are infinitesimal transformations constructed locally from the metric and its derivatives mapping solutions to solutions. Generalized conservation laws are closed but not exact differential forms constructed locally from solutions to the field equations and/or solutions to the linearized equations. The generalized conservation laws correspond to conserved volume, surface, or line integrals associated to solutions to the Einstein equations. In the mathematical literature, the generalized conservation laws go by the name "characteristic cohomology" of the field equations [Bryant and Griffiths (1993)] (for a BRST approach to characteristic cohomology see [Barnich *et al* (1994)]).

To begin our survey of generalized symmetries and conservation laws for the Einstein equations, we should first improve our understanding of the jet space of solutions to the vacuum equations.

## 2 Jet Space

The jet space of solutions to the Einstein equations can be viewed as the beginning of an answer to the question "What data are freely specifiable *at a point* of a vacuum spacetime?" This question is similar in spirit to the kind of question one asks when investigating the Cauchy problem, where one seeks the data which can be freely specified on a Cauchy surface. Of course the two questions are quite different mathematically but, to carry the analogy a little further, one can think of the jet space of solutions as something of an analog of Cauchy data for analytic solutions. Indeed, to construct an analytic solution to the Einstein equations, one can pick coordinates $x^i$ about some point $x^i = 0$ and write down a power series expansion:

$$g_{ij}(x) = g_{ij}(0) + g_{ij,k}(0)x^k + \frac{1}{2!}g_{ij,kl}(0)x^k x^l + \cdots . \tag{1}$$

Evidently, formal power series expansions of a metric about a point $x^i$ are determined by the data

$$(x^i, g_{ij}, g_{ij,k}, g_{ij,kl}, \dots). \tag{2}$$

Such data define a point in the *jet space of metrics* $\mathcal{J}$. Of course, to define a metric via power series about some point, one should enforce some convergence criterion on the metric and its derivatives at the given point; the jet space is, however, defined without such convergence criteria. A more precise way to introduce the jet space of metrics is to view it as a bundle whose base space is spacetime and whose typical fiber is the space of values of the metric and its derivatives at a point. A metric defines a cross section of this bundle. Moreover, Borel's theorem (see, *e.g.*, [Kahn (1980)]) implies that given a point $(x^i, g_{ij}, g_{ij,k}, g_{ij,kl}, \dots) \in \mathcal{J}$, there is always a smooth metric which has that data.

Let us remark that while we have defined $\mathcal{J}$ using coordinates and coordinate derivatives of the metric, the jet space is in fact a coordinate-independent object. It is possible to give a coordinate-free definition of $\mathcal{J}$ as a fiber bundle over spacetime along the lines mentioned above. In particular, it is possible to replace coordinates and coordinate derivatives with globally defined derivative operators (see, *e.g.*, [Wald (1990)]), but for simplicity we shall stick with an informal local treatment.

Having defined the jet space of metrics we would now like to see what points in jet space are allowed by the Einstein equations. To do this, we need a slightly better parametrization of $\mathcal{J}$. To this end, given coordinates $x^i$, define variables $\Gamma^{(k)}$ and $Q^{(k)}$, where

$$\Gamma^{(k)} \longrightarrow \Gamma^i_{j_0 j_1 j_2 \cdots j_k} := \Gamma^i_{(j_0 j_1, j_2 \cdots j_k)}, \qquad k = 1, 2, \dots \tag{3}$$

and

$$Q^{(k)} \longrightarrow Q_{ab, c_1 c_2 \cdots c_k} := g_{am} g_{bn} \nabla_{(c_3} \cdots \nabla_{c_k} R^m{}_{c_1}{}^n{}_{c_2)} \qquad k = 2, 3, \dots, \tag{4}$$

where $\nabla$ is the torsion-free derivative operator compatible with $g_{ab}$ and $R_{abcd}$ is the Riemann tensor. It can be shown that $\Gamma^{(k)}$ and $Q^{(k)}$ are algebraically independent at any given point of spacetime. The variables $\Gamma^{(k)}$ carry the coordinate-dependent information in the $k^{th}$ partial derivatives of the metric. In particular, all of the variables $\Gamma^{(k)}$ vanish at the origin of a geodesic coordinate chart. The variables $Q^{(k)}$ contain all spacetime-geometric information in the $k^{th}$ partial derivatives of the metric. In particular, the curvature tensor and all of its covariant derivatives can be uniquely expressed in terms of the variables $Q^{(k)}$. The tensors $Q^{(k)}$ were apparently introduced by Penrose [Penrose (1960)] and are closely related to Thomas's "normal metric tensors" [Thomas (1934)].

Our first result is that the variables

$$(x^i, g_{ij}, \Gamma^{(1)}, \Gamma^{(2)}, \dots, Q^{(2)}, Q^{(3)}, \dots) \tag{5}$$

uniquely parameterize points in the jet bundle. In other words, given the data (5) one can reconstruct the metric and all of its derivatives at the point labeled $x^i$. The data (5) are freely specifiable at a point of a (pseudo-) Riemannian manifold.

The Einstein tensor can be viewed as a collection of functions on $\mathcal{J}$, and the Einstein equations can be viewed as defining a subspace (actually a submanifold) $\mathcal{E} \hookrightarrow \mathcal{J}$. Because the Einstein equations are geometrically defined, they introduce relations only among the variables $Q^{(k)}$. In fact, the vacuum equations uniquely fix the traces of these tensor in terms of their trace-free parts [Anderson and Torre (1994)]. Thus a point in the jet space of Einstein metrics is defined by the variables

$$(x^i, g_{ij}, \Gamma^{(1)}, \Gamma^{(2)}, \dots, \widetilde{Q}^{(2)}, \widetilde{Q}^{(3)}, \dots), \tag{6}$$

where $\widetilde{Q}^{(k)}$ denotes the completely trace-free part of tensors (4) with respect to the metric $g_{ij}$. The data (6) are freely specifiable at a point of a Ricci-flat spacetime.

These coordinates on $\mathcal{E}$ can be interpreted in terms of a power series expansion of an Einstein metric as follows. If we are trying to build an Einstein metric by Taylor series we (i) specify the spacetime point $x^i$ around which the series is being developed, (ii) specify the metric components $g_{ij}$ at $x^i$, (iii) specify the variables $\Gamma^{(k)}$; this fixes the coordinate system in which the metric is being built, (iv) specify the variables $\widetilde{Q}^{(k)}$, which supplies the geometric content of the Einstein metric. Of course, this procedure leaves open the question of convergence of the series.

The parametrization (6) of $\mathcal{E}$ turns out to be somewhat unwieldy in applications, primarily because of the need to remove so many traces. A much more useful parametrization, which only exists in four dimensions, uses a spinor representation of the variables $\widetilde{Q}^{(k)}$. Let $\Psi_{ABCD}$ and $\overline{\Psi}_{A'B'C'D'}$ denote the Weyl spinors [Penrose (1960)]. Fix a soldering form $\sigma_a^{AA'}$ such that, for a given $g_{ij}$,

$$g_{ij} = \sigma_i^{AA'}\sigma_{jAA'}. \tag{7}$$

It can be shown that the variables $\widetilde{Q}^{(k)}$ are uniquely parametrized by the soldering form, the spinor variables

$$\Psi^{(k)} \longleftrightarrow \Psi^{J_1'\cdots J_{k-2}'}_{J_1\cdots J_{k+2}} = \nabla^{(J_1'}_{(J_1}\cdots\nabla^{J_{k-2}')}_{J_{k-2}}\Psi_{J_{k-1}J_kJ_{k+1}J_{k+2})}, \tag{8}$$

and their complex conjugates $\overline{\Psi}^{(k)}$. Thus we obtain a spinor parametrization of $\mathcal{E}$ in terms of

$$(x^i, g_{ij}, \Gamma^{(1)}, \Gamma^{(2)}, \dots, \Psi^{(2)}, \overline{\Psi}^{(2)}, \Psi^{(3)}, \overline{\Psi}^{(3)}, \dots). \tag{9}$$

The spinor aficionado will recognize that our spinor parametrization of $\mathcal{E}$ is closely related to Penrose's notion of an "exact set of fields" [Penrose (1960)].

The parametrization (9) of the jet space of solutions to the Einstein equations is an important technical tool needed to classify symmetries and conservation laws. More generally, these variables allow us to address problems of the following type. Suppose we are interested in finding a tensor field $T = T(x, g, \partial g, \dots)$, locally constructed from the metric and its derivatives to some order, which satisfies some local differential relations,

$$DT = 0, \tag{10}$$

when the Einstein equations hold. As an example, suppose we wanted to find a conserved current for the Einstein equations. This would be a vector field $j^a = j^a(x, g, \partial g, \dots)$ such that

$$\nabla_a j^a = 0 \qquad \text{when } G_{ab} = 0. \tag{11}$$

The tensors $T$ and $DT$ can be viewed as a collection of functions on $\mathcal{J}$ and the differential relation (10) says that the function $DT$ vanishes on $\mathcal{E}$. If we express the relation (10) in the coordinates (9), then (10) must hold *identically.* The power of spinor analysis can now be brought to bear on classifying all such solutions $T$ to this identity up to terms which vanish on $\mathcal{E}$. As we shall see, classifications of generalized symmetries and generalized conservation laws are precisely problems of this type.

To be honest, it is a bit of an over-simplification to say that these problems can be solved in a straightforward manner given the coordinates (9). The results to be given below are in fact intricately related through a mathematical structure on $\mathcal{J}$ (or $\mathcal{E}$) called "the variational bicomplex" [Anderson (1992)]. Because my goal here is to emphasize results rather than techniques, I will not be able to say more about the variational bicomplex in this paper. Suffice it to say that the bicomplex is an indispensable tool in the analysis of any set of field equations, and this mathematical structure is playing a vital role "behind the scenes" in all that follows.

## 3 Generalized Symmetries

A generalized symmetry is an infinitesimal transformation constructed locally from the relevant fields and their derivatives to some order which maps solutions of the field equations to other solutions. Generalized symmetries are of interest because they provide methods of generating new solutions from known solutions, their existence is necessary for the existence of local conservation laws, and because of their role in complete integrability of a variety of partial differential equations [Olver (1993)]. Before giving results from our classification of generalized symmetries of the Einstein equations, let us first have a look at an elementary example of a dynamical system that admits a generalized symmetry.

Consider the Kepler problem, which can be described by the non-linear system of ordinary differential equations:

$$\mathbf{r}'' = -k\frac{\mathbf{r}}{r^3}, \tag{12}$$

where $\mathbf{r}$ is the relative position of two masses in space, $k$ is a constant, and a prime denotes a time derivative. A point in the jet space $\mathcal{J}$ for this problem is defined by the variables $(t, \mathbf{r}, \mathbf{r}', \mathbf{r}'', \dots)$. The equations (12) and their time derivatives define the jet space $\mathcal{E} \hookrightarrow \mathcal{J}$ of solutions. Coordinates on $\mathcal{E}$ are $(t, \mathbf{r}, \mathbf{r}')$; these variables parametrize the extended velocity phase space for the Kepler problem. There are a couple of obvious symmetries of the equations (12), namely, rotational symmetry and time translation symmetry. These symmetries are usually called "point symmetries" or "Lie symmetries". The point symmetries are distinguished by the fact that they can be defined as groups of transformations of the underlying space of independent and dependent variables $(t, \mathbf{r})$ only, without involving derivatives of $\mathbf{r}$. By contrast, the following infinitesimal transformation necessarily involves derivatives of $\mathbf{r}$:

$$\delta\mathbf{r} = 2(\lambda \cdot \mathbf{r})\mathbf{r}' - (\lambda \cdot \mathbf{r}')\mathbf{r} - (\mathbf{r} \cdot \mathbf{r}')\lambda. \tag{13}$$

Here $\lambda$ is a fixed, time-independent vector. It is straightforward to check that if $\mathbf{r}(t)$ satisfies (12) then, to first order in $\lambda$, $\mathbf{r}(t) + \delta\mathbf{r}(t)$ also satisfies (12). The transformation (13) represents a first-order generalized symmetry of the equations (12)[1]. I think you will agree that it is somewhat remarkable that the relatively simple system of equations (12) admits such a complicated "hidden symmetry". The three-parameter family of symmetries given in (13) are responsible for the

[1] Note that all generalized symmetries of (12) can be expressed as first-order symmetries if we use the equations of motion. This property of generalized symmetries does not generalize to partial differential equations

existence of a conserved vector, known as the Laplace-Runge-Lenz vector $\mathbf{A}$ (see, for example, [Goldstein(1990)]):

$$\mathbf{A} = \mathbf{r}' \times (\mathbf{r} \times \mathbf{r}') - k\frac{\mathbf{r}}{r}. \tag{14}$$

Conservation of the Laplace-Runge-Lenz vector reflects a special feature of the Kepler problem: all its bound orbits are closed. The only other central force that has this special property is that of an isotropic oscillator, and in this case there is again a generalized symmetry and (tensor) conservation law which is responsible.

It is natural to wonder if the Einstein equations will admit any hidden generalized symmetries as do many simpler systems of non-linear differential equations. To find such symmetries we must look for an infinitesimal transformation

$$\delta g_{ab} = h_{ab}(x, g, \partial g, \dots) \tag{15}$$

mapping solutions to solutions. This means that $h_{ab}$ must satisfy the linearized equations,

$$-\nabla^c\nabla_c h_{ab} - \nabla_a\nabla_b(g^{cd}h_{cd}) + 2\nabla^c\nabla_{(a}h_{b)c} = 0, \tag{16}$$

when the metric $g_{ab}$, out of which $h_{ab}$ is built, satisfies the Einstein equations $G_{ab} = 0$. Because we are only interested in transformations between solutions, any two symmetries that are equal when the field equations hold can be considered equivalent. So, for example, the Einstein tensor defines a generalized symmetry, $h_{ab} = G_{ab}$ but the transformation it generates is trivial, *i.e.*, we identify this symmetry with $h_{ab} = 0$. As described earlier, the classification of on-shell generalized symmetries is exactly the kind of problem that can be fruitfully attacked using the spinor parametrization of $\mathcal{E}$. In detail, the symmetry transformation $h_{ab}$ is viewed as a function on $\mathcal{J}$ (we are actually only interested in the restriction of $h_{ab}$ to $\mathcal{E}$). We view (16) as requiring that certain functions on $\mathcal{J}$ built from $h_{ab}$ vanish when restricted to $\mathcal{E}$. That is, as a function on $\mathcal{E}$, $h_{ab}$ should satisfy the identity (16). As an identity on $\mathcal{E}$, (16) must hold for all values of the coordinates (9); analysis of this requirement leads to the following result.

Let $h_{ab} = h_{ab}(x, g, \partial g, \dots)$ be a generalized symmetry of the vacuum Einstein equations in four spacetime dimensions. Then there is a constant $c$ and a covector $V_a = V_a(x, g, \partial g, \dots)$ such that, modulo terms that vanish when $G_{ab} = 0$, the symmetry must take the following form:

$$h_{ab} = c\, g_{ab} + \nabla_a V_b + \nabla_b V_a. \tag{17}$$

This form of $h_{ab}$ represents a combination of two types of symmetry transformations. The term $c\, g_{ab}$ corresponds to a scale symmetry admitted by the vacuum equations. If we allowed for a cosmological constant, this symmetry would be absent. The terms $\nabla_a V_b + \nabla_b V_a$ correspond to the infinitesimal change in the metric arising from the pull-back by a 1-parameter family of (local) spacetime diffeomorphisms generated by $V^a$. Thus the most general symmetry is a combination of a constant scaling and a "gauge transformation". Of course, both of these symmetries are well-known and we conclude that the vacuum Einstein equations admit no "hidden local symmetries".

## 4 Generalized Conservation Laws

The symmetries we have found do not have any non-trivial conservation laws associated with them. Noether's theorem then suggests that, aside from possible topological conservation laws, there are no conserved currents for the Einstein equations built locally from the metric and its derivatives. This is in fact true, but it does not follow directly from our symmetry classification because it is *a priori* possible to have symmetries that are on-shell trivial and nevertheless generate non-trivial conservation laws. For completely non-degenerate systems of PDE's it is known that every non-trivial conserved current follows from a non-trivial generalized symmetry [Olver (1993)], but the Einstein equations do not qualify as a completely non-degenerate system owing to their general covariance. The problem of rigorously classifying conserved currents for the vacuum Einstein equations can be solved using the variational bicomplex and our spinor techniques. In fact, it is possible to generalize the analysis and classify all closed forms locally built from vacuum metrics as well as a large class of closed forms locally built from vacuum metrics and solutions of the linearized equations. We begin with closed forms built locally from vacuum metrics (see [Wald (1990)] for a general discussion of identically closed forms locally built from fields).

Let $\omega = \omega(x, g, \partial g, \dots)$ be a $p$-form locally built from the metric and its derivatives to some order. We assume $p < 4$. If $d\omega = 0$ when the vacuum Einstein equations hold, we say that $\omega$ represents a *generalized conservation law* for the vacuum equations. The importance of a generalized conservation law stems from the fact that its integral over a closed[2] $p$-dimensional submanifold $\Sigma$ is independent of the choice of $\Sigma$ (up to homology) when the metric satisfies the field equations. Thus the integral

$$Q[g] = \int_\Sigma \omega(x, g, \partial g, \dots) \tag{18}$$

is a conserved charge characterizing vacuum spacetimes. As we are only interested in the values of the conserved charges for solutions of the field equations, we will identify any generalized conservation laws which are equal when the field equations hold. In other words, viewing the generalized conservation laws as functions on $\mathcal{J}$, we are only interested in their restriction to $\mathcal{E}$.

Of course, if (on $\mathcal{E}$) there exists a $(p-1)$-form $\eta = \eta(x, g, \partial g, \dots)$ such that

$$\omega = d\eta, \tag{19}$$

then $\omega$ is identically closed and $Q[g] = 0$ for any metric $g$. We will call such exact forms *trivial conservation laws.*

If $\omega$ is a closed 3-form, then its Hodge dual is a conserved current; the conserved charge is then a volume integral of a local density. In field theory this is the way in which conserved charges typically arise. However, interesting conserved quantities do sometimes arise not as volume integrals but instead as surface—or even line—integrals. For example, if we restrict our attention to vacuum (regions of) spacetimes admitting a Killing vector field $k^a$, then the Komar 2-form,

$$\kappa_{ab} = \epsilon_{ab}{}^{cd}\nabla_c k_d, \tag{20}$$

[2] With suitable boundary conditions, the submanifold can be open or have boundaries.

defines a generalized conservation law, as does the twist 1-form

$$\tau_a = k^b \kappa_{ab}. \tag{21}$$

Is there a field-theoretic explanation for the existence of these generalized conservation laws? Are there any generalized conservation laws admitted by the vacuum Einstein equations which can be defined without assuming the existence of a Killing vector? We shall answer the latter question now and return to the first question toward the end of this article.

We wish to classify closed forms locally built from Ricci-flat metrics. Once again we are looking for some functions on $\mathcal{J}$ satisfying a certain differential relation on $\mathcal{E}$. Using spinor techniques and basic properties of the variational bicomplex we obtain the following results. If $\omega = \omega(x, g, \partial g, \dots)$ is a $p$-form locally constructed from the metric, and $\omega$ is closed when the Einstein equations hold then, modulo terms which vanish when the field equations are satisfied, there exists a $(p-1)$-form $\eta = \eta(x, g, \partial g, \dots)$ and a constant $c$ such that

$$\begin{aligned} &\omega = \text{const.}, && p = 0; \\ &\omega = d\eta, && p = 1, 2; \\ &\omega = c\,\sigma + d\eta, && p = 3. \end{aligned} \tag{22}$$

In (22) $\sigma$ is an *identically* closed 3-form locally built from the metric and its first derivatives. This 3-form is a representative of the non-trivial cohomology class at degree 3 which exists on the bundle of Lorentzian metrics over spacetime [Torre (1995)]. The corresponding conserved charge is the "kink-number" of the spacetime, which was discussed by Finkelstein and Misner quite some time ago [Finkelstein and Misner (1959)]. Roughly speaking, the kink number counts the number of times light cones tumble as one traverses a hypersurface. Because $\sigma$ is identically closed, this conservation law exists for any gravitational theory in which one employs a Lorentzian metric. Aside from this single topological conservation law, the vacuum Einstein equations admit no non-trivial generalized conservation laws.

There is an immediate corollary of this result which is relevant for the Hamiltonian formulation of general relativity on closed universes [Torre(1993)]. Recall that the Einstein equations can be viewed as defining a constrained Hamiltonian system. The constraints limit the possible range of canonical data and generate canonical transformations which are Hamiltonian expressions of the action of spacetime diffeomorphisms on the Cauchy data. It is of interest, especially in attempts to canonically quantize the theory, to find functions on the gravitational phase space which are invariant under these canonical transformations, *i.e.*, which have vanishing Poisson brackets with the constraint functions. These "gauge invariant" functions on phase space are commonly called the *observables* of the theory. Because, for closed universes, the Hamiltonian of general relativity is a linear combination of the constraint functions, the observables are constants of the motion. But we have, in effect, just classified a large set of constants of motion for the Einstein equations, and so we have a classification of a particular type of observables.

Let us define a *local observable* $\mathcal{O}$ as an observable on the gravitational phase space which is constructed as an integral over a compact $p$-dimensional manifold $\Sigma$ of a differential $p$-form $\omega$, where $\omega$ is locally built from the canonical coordinates and momenta and their derivatives to any finite order. Because the canonical variables are related by local formulas to the spacetime metric it follows that, on solutions (in which $\Sigma$ is embedded in spacetime), $\mathcal{O}$ is a conserved charge of the type (18). But there are no such conserved charges except the kink number, which must always vanish in spacetimes that admit a foliation by spacelike hypersurfaces [Finkelstein and Misner (1959)]. We conclude that there are no local observables for vacuum spacetimes with compact Cauchy surfaces. This is not to say that there are no observables, but only that the simplest class of observables that one naturally tries to construct for general relativity are simply not there.

Let us now turn to generalized conservation laws built from solutions $\gamma_{ab}$ to the linearized Einstein equations. An important example is provided by the pre-symplectic current for the vacuum equations. Here we present it as a 3-form:

$$\Omega_{abc} = \left(\gamma^m_{[a}\nabla^n\hat{\gamma}^p_b - \hat{\gamma}^m_{[a}\nabla^n\gamma^p_b\right)\epsilon_{c]mnp}. \tag{23}$$

The pre-symplectic current is locally built from the metric, a pair of metric perturbations $\gamma$ and $\hat{\gamma}$, and their first derivatives. $\Omega$ is closed when the metric and perturbations satisfy the Einstein equations and linearized equations respectively. Thus its integral over a compact hypersurface,

$$\Xi(\gamma,\hat{\gamma}) = \int_\Sigma \Omega, \tag{24}$$

is independent of the choice of $\Sigma$ up to homology[2]. Up to a normalization, $\Xi(\gamma,\hat{\gamma})$ is the value of the pre-symplectic form on the space of solutions to the vacuum Einstein equations when acting on a pair of tangent vectors $(\gamma,\hat{\gamma})$ (see, for example, [Ashtekar (1990)], and references therein).

Integrable systems of PDE's often admit inequivalent symplectic structures [Olver (1993)]. In addition, it is known that the existence of generalized conservation laws depending on three or more solutions to the linearized equations is closely related to applicability of Darboux's method of integration [Anderson (1992)]. Thus this sort of conservation law—depending on one or more solutions to the linearized equations— can expose important features of a set of field equations, and it is natural to ask if the Einstein equations admit any other generalized conservation laws of this type.

To present our classification of generalized conservation laws depending on solutions to the linearized equations we must introduce some notation. Let $\omega^{(p,q)}$ denote a spacetime $p$-form locally constructed from the metric and its derivatives as well as from $q$ solutions of the linearized equations and their derivatives. The linearized solutions must appear in a skew $q$-multilinear fashion. For example, the pre-symplectic current $\Omega_{abc}$ would be denoted $\omega^{(3,2)}$ in this notation, and the forms classified in (22) would be denoted $\omega^{(p,0)}$. We now look for such forms that are closed when the Einstein equations and their linearization are satisfied by the metric and perturbations. To find such forms, we must generalize our spinor parametrization of $\mathcal{E}$ to include the jet space of solutions to the linearized equations, but this is relatively straightforward. Using techniques from the variational bicomplex we obtain the following results.

Let $\omega^{(p,q)}$, $q > 0$ and $p < 4$, be a generalized conservation law for the vacuum Einstein equations in four spacetime dimensions. Then there exist constants $b$ and $c$ and forms $\eta^{(p-1,q)}$ locally constructed from the metric and perturbations such that, modulo terms which vanish when the Einstein and linearized Einstein equations hold, we have:

$$
\begin{aligned}
\omega^{(0,q)} &= 0; \\
\omega^{(p,q)} &= d\eta^{(p-1,q)}, \qquad p = 1, 2; \\
\omega^{(3,q)} &= d\eta^{(2,q)}, \qquad q > 2; \\
\omega^{(3,2)} &= b\,\Omega + d\eta^{(2,2)}; \\
\omega^{(3,1)} &= c\,\Theta + d\eta^{(2,1)}.
\end{aligned} \tag{25}
$$

In this last equation we have denoted by $\Theta$ a 3-form $\Theta_{abc}$, which is defined as

$$\Theta_{abc} = \epsilon_{abc}{}^{d}\nabla^{e}\left(\gamma_{de} - g_{de}g^{mn}\gamma_{mn}\right). \tag{26}$$

$\Theta$ is closed when $g_{ab}$ and $\gamma_{ab}$ satisfy the Einstein equations and their linearization respectively. The integral of $\Theta$ over a hypersurface is again independent of the choice of hypersurface and defines the canonical 1-form, or pre-symplectic potential, on the space of solutions to the Einstein equations. This means that, viewing the conserved charge defined by $\Theta$ as a 1-form on the infinite-dimensional space of solutions acting on a tangent vector $\gamma$, the exterior derivative of the charge is the symplectic 2-form on the space of solutions. Aside from the symplectic current and its associated potential, there are no other non-trivial generalized conservation laws built from solutions of the linearized equations as described above. Note that this result establishes a uniqueness theorem for the gravitational pre-symplectic structure in the sense that any such structure which can be constructed as the spatial integral of a closed, locally constructed 3-form is a multiple of (24).

## 5 Discussion

We have classified the generalized symmetries and generalized conservation laws of the vacuum Einstein equations in four dimensions. Our results indicate that, from the vantage point of geometric structures on the jet space of solutions, one can see only a handful of "special features" of the vacuum equations. Still, let us summarize our results and what they tell us about the vacuum equations.

The generalized symmetries include a constant scale symmetry and a diffeomorphism symmetry. The scale symmetry simply indicates that there are no length scales set by the vacuum equations; this symmetry is absent if one modifies the equations using dimensionful constants, *e.g.*, if one includes a cosmological term in the equations. The diffeomorphism symmetry reflects the general covariance of the Einstein equations. We expect this symmetry to be present in any generally covariant system of field equations. Aside from these well-known transformations, the Einstein equations are devoid of symmetry.

The only closed–not–exact form locally constructed from Ricci-flat metrics corresponds to a topological conservation law—the conservation of "kink number". This conservation law reflects the non-trivial topology of the bundle of Lorentzian metrics over spacetime and will arise in any system of field equations for a Lorentzian metric. If the metric is Riemannian, this conservation law is absent. The absence of any other conserved 3-forms can be traced back to the absence of suitable generalized symmetries. But this does not explain the dearth of lower-degree conservation laws.

Remarkably, it is possible to give a rather simple theory of lower-degree conservation laws in a general field theory [Anderson and Torre (1995)], which can be thought of as somewhat analogous to Noether's theory of conserved currents [Olver (1993)]. With some mild technical assumptions it is possible to show that in order for a set of Lagrangian field equations to admit lower-degree conservation laws two conditions must be met. First, the theory must be a "gauge theory", that is, it must admit some form of *gauge transformation*, where we define a gauge transformation as a generalized symmetry built from arbitrary functions of spacetime. Second, the solutions to the field equations must be such that they always allow for *gauge symmetries*, that is, there always exists a gauge transformation that leaves each solution invariant[3]. Thus, the gauge transformation of the Einstein equations is the diffeomorphism symmetry ((17) with $c = 0$), and a gauge symmetry of a solution $g_{ab}$ to the field equations would be a diffeomorphism which does not change that solution. The infinitesimal gauge symmetry is then generated by a *generalized Killing vector field*, that is, a vector field locally constructed from the metric and its derivatives to some order which satisfies the Killing equations when the metric is Ricci-flat. The generic solution to the vacuum equations admits no Killing vector fields. More precisely, it is possible to show that there are no generalized Killing vector fields, and so we can say that the absence of lower-degree conservation laws for the Einstein equations reflects the absence of isometries of generic solutions. If we consider reductions of the Einstein equations obtained by demanding the solutions always admit a Killing vector, then the general theory leads to lower-degree conservation laws such as shown in (20) and (21).

Finally, we have classified generalized conservation laws built locally from solutions to the linearized equations and found only the symplectic current and its "potential". These conservation laws reflect the variational properties of the Einstein equations. As is well-known [Ashtekar (1990)], a conserved symplectic current arises for any field equations derivable from a Lagrangian. Thus the conserved 3-form (23) reflects the fact that the Einstein equations can be derived from the Einstein-Hilbert Lagrangian. The essential uniqueness of the pre-symplectic current leads to a uniqueness result for variational principles for the vacuum Einstein equations, which will be presented elsewhere. Normally, the current defining the symplectic potential for a system of Lagrangian field equations is not conserved. However, it is not hard to show that the current defining the symplectic potential *is* conserved provided the Lagrangian can be chosen to vanish when the field equations hold. Thus the closed 3-form (26) reflects the fact that the Einstein-Hilbert Lagrangian vanishes on Ricci-flat metrics.

In this article I have discussed structural features of the vacuum Einstein equations which can be uncovered using spinor–jet space techniques. The techniques that were used can be generalized to analyze related systems of equations, specifically (i) reductions of the Einstein equations such as obtained by restricting attention to solutions with one or more Killing vector fields; and (ii) the Einstein equations with matter couplings, *e.g.*, the Einstein-Maxwell equations. As mentioned above, we expect non-trivial lower-degree conservation laws to arise when one analyzes the Einstein-Killing equations, and a classification of such conservation laws is currently in progress. Matter couplings can also induce lower degree conservation laws; for example, the Einstein-Maxwell equations admit a 2-form conservation law.

[3]The gauge symmetry of a solution will in general vary with the choice of solution.

More intriguing, if perhaps on somewhat less firm physical ground, are matter couplings dictated by Kaluza-Klein reductions of higher-dimensional vacuum relativity. For example, the classical reduction from five to four dimensions corresponds to an Einstein-Maxwell-scalar field theory. This reduction is dictated by the assumption that the five-dimensional vacuum theory admits a Killing vector field, which, on general grounds, indicates the existence of "characteristic cohomology". The details of such investigations will be presented elsewhere.

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