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SOME REMARKS ON GRAVITATIONAL ANALOGS OF MAGNETIC CHARGE

C. G. TORRE

*Department of Physics
Utah State University
Logan, UT 84322-4415
USA*

Abstract:

Existing mathematical results are applied to the problem of classifying closed p -forms which are locally constructed from Lorentzian metrics on an n -dimensional orientable manifold M ($0 < p < n$). We show that the only closed, non-exact forms are generated by representatives of cohomology classes of M and $(n - 1)$ -forms representing n -dimensional (with n even) generalizations of the conservation of “kink number”, which was exhibited by Finkelstein and Misner for $n = 4$. The cohomology class that defines the kink number depends only on the diffeomorphism equivalence class of the metric, but a result of Gilkey implies that there is no representative of this cohomology class which is built from the metric, curvature and covariant derivatives of curvature to any finite order.

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1. Introduction.

Let M be a smooth n -dimensional manifold and let g be a Lorentzian metric on M . Let α be a p -form, $0 < p < n$, locally constructed from the metric and its derivatives. We say that α is a *locally conserved p -form* if $d\alpha = 0$ for all choices of g . Here d is the exterior derivative on M . If Σ is a p -dimensional closed, oriented submanifold of M , then, for any given metric, the integral of α over Σ depends only on the homology class of Σ . Wald has called such integrals “gravitational analogs of magnetic charge”. Of course, if there is a $(p - 1)$ -form β that is locally constructed from the metric and its derivatives such that $\alpha = d\beta$, then α is trivially closed and the corresponding charge will vanish for all metrics. Let us then define a *topological conservation law* [1] as an equivalence class of locally conserved p -forms; two locally conserved p -forms, α and α' , are equivalent if there is some $(p - 1)$ -form β that is locally constructed from the metric and its derivatives such that $\alpha - \alpha' = d\beta$.

Wald has given a clear and rather complete discussion of topological conservation laws in a general field theory [2]. In particular, he has shown that when the fields of interest are Lorentzian metrics, the charge of a topological conservation law can only depend on the homotopy class of the metric g on M . Then, using results of Unruh [3], Wald is able to conclude that for $n = 4$ there are no topological conservation laws which are covariantly constructed using the metric, polynomials in the curvature and covariant derivatives of curvature to any finite order. This result is in accord with general results of Gilkey [4,5]. Gilkey classifies “natural” conservation laws, which are closed forms modulo exact forms, all of which are constructed covariantly from the metric, curvature and covariant derivatives of curvature. Gilkey’s work implies that the only natural conservation laws are generated by the Pontrjagin forms. These natural conservation laws only exist when $n > 4$.

Quite some time ago, Finkelstein and Misner [6] studied homotopy classes of asymptotically flat Lorentzian metrics on $M = R^4$ and showed that, given a metric, one can associate an integer to any asymptotically spacelike hypersurface in M . This integer, later dubbed the “kink number” [7], represents the number of times light cones tumble as one traverses the hypersurface. The kink number is a homotopy invariant of g and is thus unchanged by any continuous deformation of the hypersurface. Given a metric, the work of [7] shows how to evaluate this kink number. By analogy with similar results from other field theories, it is reasonable to suppose that the kink number corresponds to a topological conservation law, *i.e.*, is a gravitational analog of magnetic charge. That this is so is, to some extent, implicit in the formula for the kink number given in [7]. We would like to make this fact explicit as well as provide some additional results on topological conservation laws for field theories based on Lorentzian metrics on a general class of n -dimensional manifolds.

Our goals in this paper are as follows.

- (i) Give an interpretation of the kink number as a topological conservation law, that is, as a cohomology class in the Euler-Lagrange complex [8] associated with Lorentzian metrics on any orientable, even-dimensional manifold.
- (ii) Show that all topological conservation laws are generated by cohomology classes of M and the kink conservation law.
- (iii) Show that while the cohomology class that defines the kink number conservation law depends only on the diffeomorphism equivalence class of the metric, there is no naturally constructed representative of this equivalence class. In particular, this result shows how the kink number in dimension four manages to evade the rather stringent results of [2] and, more generally, [4,5].

The main technical tool that we shall use is the variational bicomplex [8], which is specifically tailored to analyze structures such as topological conservation laws. We shall describe the results we need from the bicomplex in §2. In §3 we apply these results to the topological conservation laws built from Lorentzian metrics and exhibit a representative of the topological conservation law corresponding to the kink number. We also explain the sense in which this conservation law is unique. In §4 we point out that our construction of the differential form representing the kink conservation law is not natural, *i.e.*, the locally conserved form is not built from a universal expression in the metric, curvature, and covariant derivatives of curvature. This leaves open the possibility that one can find another representative of this conservation law that is naturally constructed. However, the results of Gilkey imply that there is no natural representative of this topological conservation law.

2. Topological conservation laws and the variational bicomplex.

To begin, we need a more precise definition of a topological conservation law. For our purposes it is best to give this definition in the context of the variational bicomplex, although this is certainly not necessary, see for example [2]. Our treatment is taken from that of Anderson [8,9].

Let $\pi: E \rightarrow M$ be the bundle of Lorentzian metrics over the n -dimensional manifold M . A section $g: M \rightarrow E$ defines a metric on M . Of course we assume that M admits global Lorentzian metrics. If M is non-compact it always admits a Lorentzian metric; if M is compact it admits a Lorentzian metric if and only if the Euler number of M is zero [10]. We denote by $\pi_M: J^\infty(E) \rightarrow M$ the infinite jet bundle of metrics. Here the bundle is interpreted as having base space M . There is also a projection $\pi_E: J^\infty(E) \rightarrow E$. A section g of E has a canonical lift to a section $j^\infty(g): M \rightarrow J^\infty(E)$ called the jet of g . For a general description of jet bundles see [11].

Given local coordinates x^i on $U \subset M$ we have local coordinates on $J^\infty(\pi^{-1}(U))$ defined

by

$$(x^i, g_{ij}, g_{ij,k}, g_{ij,kl}, \dots), \quad (2.1)$$

where g_{ij} are the components of g in the coordinates x^i and for any section g of E

$$g_{ij,k_1 \dots k_l}(j^\infty(g)) = \frac{\partial^l g_{ij}(x)}{\partial x^{k_1} \dots \partial x^{k_l}}.$$

A differential form ω on $J^\infty(E)$ is called a *contact form* if for every section $g: M \rightarrow E$,

$$[j^\infty(g)]^* \omega = 0.$$

The set of contact forms is a differential ideal in the ring $\Omega^*(J^\infty(E))$ of all differential forms on $J^\infty(E)$. In the local coordinates (2.1) the contact ideal is spanned locally by the contact 1-forms

$$\theta_{ij,k_1 \dots k_l} = dg_{ij,k_1 \dots k_l} - g_{ij,k_1 \dots k_l m} dx^m, \quad l = 0, 1, 2, \dots \quad (2.2)$$

Here, and in all that follows, d is the exterior derivative on $J^\infty(E)$.

The contact ideal defines a connection on $J^\infty(E)$. In particular, a vector X at a point $\sigma \in J^\infty(E)$ is said to be π_M -vertical if $(\pi_M)_* X = 0$ at $\pi_M(\sigma)$; X is said to be horizontal at σ if $X \lrcorner \omega = 0$ for all contact forms ω at σ . A p -form γ on $J^\infty(E)$ is said to be of type (r, s) , where $r + s = p$, if at each point of $J^\infty(E)$

$$\gamma(X_1, X_2, \dots, X_p) = 0$$

whenever more than s of the vectors X_1, X_2, \dots, X_p are π_M -vertical, or more than r of the vectors are horizontal. The space of type (r, s) forms is denoted $\Omega^{r,s}(J^\infty(E))$. In the local coordinates (2.1) a type (r, s) form is a sum of terms of the form

$$f[g] dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge \Theta,$$

where Θ is a wedge product of s contact forms (2.2) and $f[g]$ is a function on $J^\infty(\pi^{-1}(U))$ depending on the metric and its derivatives to some finite order.

There is a direct sum decomposition

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)),$$

and we let $\pi^{r,s}: \Omega^p(J^\infty(E)) \rightarrow \Omega^{r,s}(J^\infty(E))$ denote the projection to $\Omega^{r,s}(J^\infty(E))$, where $p = r + s$. The exterior derivative on $J^\infty(E)$,

$$d: \Omega^p(J^\infty(E)) \rightarrow \Omega^{p+1}(J^\infty(E)),$$

splits into a horizontal and vertical piece via

$$\begin{aligned} d &= d_H + d_V, \\ d_H: \Omega^{r,s}(J^\infty(E)) &\rightarrow \Omega^{r+1,s}(J^\infty(E)) \\ d_V: \Omega^{r,s}(J^\infty(E)) &\rightarrow \Omega^{r,s+1}(J^\infty(E)) \end{aligned}$$

where, for a p -form $\gamma \in \Omega^{r,s}(J^\infty(E))$,

$$d_H\gamma = \pi^{r+1,s}(d\gamma)$$

and

$$d_V\gamma = \pi^{r,s+1}(d\gamma).$$

As an example, in local coordinates the horizontal exterior derivative of a function $f: J^\infty(E) \rightarrow R$ takes the form

$$d_H f = \left(\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial g_{ij}} g_{ij,m} + \frac{\partial f}{\partial g_{ij,k}} g_{ij,km} + \dots \right) dx^m = (D_m f) dx^m, \quad (2.3)$$

and the vertical exterior derivative of f takes the form

$$d_V f = \frac{\partial f}{\partial g_{ij}} \theta_{ij} + \frac{\partial f}{\partial g_{ij,k}} \theta_{ij,k} + \dots$$

In particular, because

$$dg_{ij} = \theta_{ij} + g_{ij,k} dx^k, \quad (2.4)$$

we have that

$$d_V g_{ij} = \theta_{ij}, \quad (2.5)$$

and

$$d_H g_{ij} = g_{ij,k} dx^k = (D_k g_{ij}) dx^k. \quad (2.6)$$

The differential operator D_i in (2.3) and (2.6) is the *total derivative* operator. It is not too hard to see that the horizontal exterior derivative corresponds to the usual exterior derivative on M , with partial differentiation replaced by total differentiation. More precisely, if $g: M \rightarrow E$ is a metric, $\alpha \in \Omega^{r,0}(J^\infty(E))$, and d_M is the exterior derivative on M , then we have,

$$j^\infty(g)^*(d_H \alpha) = d_M[j^\infty(g)^*(\alpha)].$$

In this way the notion of “exterior derivative of a form locally constructed from the metric” is made precise in terms of d_H acting on forms in $\Omega^{r,0}(J^\infty(E))$.

The identity $d^2 = 0$ now decomposes into

$$d_H^2 = 0, \quad d_V^2 = 0, \quad d_H d_V = -d_V d_H,$$

so that the de Rham complex $\Omega^*(J^\infty(E))$ on the jet bundle decomposes into a double complex called the *variational bicomplex*. Topological conservation laws arise as cohomology classes on the “bottom edge” of the bicomplex. More precisely, a *locally conserved p-form* is a form $\alpha \in \Omega^{p,0}(J^\infty(E))$ such that

$$d_H \alpha = 0. \quad (2.7)$$

If α is exact, then there is a form $\beta \in \Omega^{p-1,0}(J^\infty(E))$ such that

$$\alpha = d_H \beta. \quad (2.8)$$

The restriction of the de Rham complex on $J^\infty(E)$ to the forms in $\Omega^{p,0}(J^\infty(E))$, $0 < p < n$, will be called the *Euler-Lagrange complex* [12], and will be denoted $\mathcal{E}^*(J^\infty(E))$. Cohomology classes in $\mathcal{E}^*(J^\infty(E))$ are forms which are d_H -closed (2.7) modulo forms that are d_H -exact (2.8). Thus a *topological conservation law* is a cohomology class in $\mathcal{E}^*(J^\infty(E))$.

We now present two results from the theory of the variational bicomplex that vastly simplify the computation of all topological conservation laws. Let $H^p(\Omega^*(J^\infty(E)))$ denote the p^{th} cohomology of the de Rham complex on $J^\infty(E)$. Similarly, denote by $H^p(\mathcal{E}^*(J^\infty(E)))$ the p^{th} cohomology of the Euler-Lagrange complex. It can be shown [8,9] that there is an isomorphism between these vector spaces. More precisely, define a map $\Psi: \Omega^p(J^\infty(E)) \rightarrow \mathcal{E}^p(J^\infty(E))$ for $0 < p < n$ by

$$\Psi(\alpha) = \pi^{p,0}(\alpha). \quad (2.9)$$

The induced map

$$\Psi^*: H^p(\Omega^*(J^\infty(E))) \rightarrow H^p(\mathcal{E}^*(J^\infty(E))) \quad (2.10)$$

is an isomorphism. Next, it can be shown that the projection $\pi_E: J^\infty(E) \rightarrow E$ is a homotopy equivalence, and hence the de Rham cohomology of $J^\infty(E)$ is isomorphic to the de Rham cohomology of E . Thus *the topological conservation laws are in one to one correspondence with the cohomology classes of E*. In detail, the correspondence just stated is as follows. Let α be a closed p -form representing a nontrivial cohomology class in $\Omega^p(E)$. The form α can be pulled back via π_E to give a representative of a nontrivial cohomology class in $\Omega^p(J^\infty(E))$. The map Ψ in (2.9) then defines a representative of a nontrivial cohomology class in $\mathcal{E}^p(J^\infty(E))$. The theory of the variational bicomplex tells us that all topological conservation laws arise in this manner.

3. Classification of topological conservation laws.

The theory of the variational bicomplex reduces the task of computing the cohomology classes of the Euler-Lagrange complex to that of the bundle of metrics. According to Steenrod [13], there is a deformation retraction $\varphi: E \rightarrow E'$ of the bundle of metrics E to

a bundle $\pi': E' \rightarrow M$ which has the same bundle data as E except the fiber F' is the real projective space RP^{n-1} . This deformation retraction corresponds to the construction of a line element field from a given Lorentzian metric and an auxiliary Riemannian metric [14]. Thus we have the isomorphism

$$H^*(E) = H^*(E'). \quad (3.1)$$

To begin, let us assume that M is parallelizable, in which case the bundle of metrics is trivial, that is, $E = M \times F$, and hence $E' = M \times F'$. The Kunneth formula [15] then shows that the cohomology of E' is given by

$$H^*(E') = H^*(M) \otimes H^*(F'). \quad (3.2)$$

The cohomology classes of M are smooth invariants of M . The closed forms on E representing this cohomology can be obtained by pulling back representatives on M using π . These locally conserved forms are manifestly independent of the metric. The cohomology of F' , *i.e.*, $H^p(RP^{n-1})$, $p > 0$, is trivial if n is odd, and when n is even the only nontrivial class is at form degree $n - 1$:

$$H^{n-1}(RP^{n-1}) = R, \quad n \text{ even}. \quad (3.3)$$

A representative of $H^{n-1}(F')$ can be obtained by taking, *e.g.*, the standard volume form on the $(n - 1)$ -sphere, which projects to give a volume form on RP^{n-1} (when n is even) via the usual antipodal projection from S^{n-1} to RP^{n-1} . According to (3.2), this form on RP^{n-1} corresponds to a representative of a topological conservation law. A formula for this locally conserved form can be constructed as follows. Keeping in mind that we are assuming for the moment that M is parallelizable, fix a global trivialization $e: E' \rightarrow M \times F'$. Let $\pi_{F'}: M \times F' \rightarrow F'$ denote the projection to the fiber defined by the trivialization. Assuming n is even, (i) Take a volume form Ω on $F' = RP^{n-1}$ and pull it back to E' via $\pi_{F'}$. Using φ , pull the resulting form back to $M \times F$. The resulting form is then pulled back to $J^\infty(M \times F)$ using the projection π_E . Finally, apply the map Ψ to construct the representative α of the topological conservation law:

$$\alpha = \pi^{n-1,0} \{ \pi_E^* [\varphi^* (\pi_{F'}^* \Omega)] \} = \pi^{n-1,0} \{ (\pi_{F'} \circ \varphi \circ \pi_E)^* \Omega \}. \quad (3.4)$$

Having constructed the closed $(n - 1)$ -form α , we can compute the charge $Q[g]$ associated with a metric as follows. Given a metric $g: M \rightarrow E$ with jet $j^\infty(g): M \rightarrow J^\infty(E)$, we pull α back to give an $(n - 1)$ -form on M which we integrate over a closed, oriented $(n - 1)$ -dimensional submanifold Σ

$$Q[g] = \int_\Sigma [j^\infty(g)]^* \alpha. \quad (3.5)$$

Up to a numerical factor, this integral represents the degree of the map from Σ to RP^{n-1} defined by the metric. When $n = 4$ and $\Sigma = S^3$, the integral (3.5) corresponds to the formula proposed in [7] for the kink number. Accordingly, we shall call the conservation law represented by α the *kink conservation law*.

When M is not parallelizable, *i.e.*, when E' is not trivial we can repeat the construction above using a *local* trivialization, in which $\pi'^{-1}(U) = U \times F'$. However, there may be obstructions to patching together the local representative (3.4) of the cohomology class on $U \times F'$ to give a global representative of a class on E' . In order to generalize the kink conservation law to non-parallelizable manifolds we should find a closed $(n - 1)$ -form on E' whose restriction to any fiber generates the cohomology (3.3) of the fiber. Moreover, if we can do this, then (3.1) and the Leray-Hirsch theorem [15] imply that we still have the isomorphism

$$H^*(E) = H^*(M) \otimes H^*(F'). \quad (3.6)$$

Because F' is cohomologically trivial when n is odd, it immediately follows that the Leray-Hirsch theorem is applicable in this case, and we have that

$$H^p(E) = H^p(M), \quad n \text{ odd.}$$

When n is even, the obstructions to constructing a global $(n - 1)$ -form on E' which restricts to generate the cohomology (3.3) are (i) orientability of the bundle E' and (ii) the Euler class of E' . This result follows directly from the discussion of sphere bundles in [15], which is easily adapted to the case where the sphere is replaced by RP^{n-1} . In essence, the results of [15] are unchanged because $H^*(S^{n-1}) = H^*(RP^{n-1})$ when n is even. The Euler class of E' vanishes because we assume that E admits a global section, that is, a global Lorentzian metric, which implies that E' admits a global section. Furthermore, because the transition functions of E' are induced by those of the tangent bundle, E' is orientable if and only if M is orientable. Hence, if M is orientable, a representative of the kink conservation law can be defined globally, and the isomorphism (3.6) holds.

Thus, aside from cohomology classes of M , the kink number arises as an additional topological conservation law when n is even and M is orientable. All topological conservation laws for Lorentzian metrics on orientable manifolds are generated by these conservation laws.

4. General covariance of the conservation law.

Our construction (3.4) of a (local) representative α of the kink conservation law depends on the deformation retraction φ and the choice of (local) trivialization. This dependence means that α cannot be “naturally” constructed from the spacetime metric and its derivatives. In order to make this discussion more precise, we phrase it in terms of the behavior of α with respect to the action of spacetime diffeomorphisms on the metric [16].

Let $\Psi: M \rightarrow M$ be a diffeomorphism. The map Ψ lifts to give a bundle map $\Psi_E: E \rightarrow E$ via $(x, g) \rightarrow (\Psi(x), \Psi_*g)$. The bundle map Ψ_E , in turn, lifts by prolongation [11] to give a bundle map $\text{pr } \Psi: J^\infty(E) \rightarrow J^\infty(E)$. Let us consider a p -form ρ locally constructed from the metric and its derivatives. Such a form is a map from $J^\infty(E)$ into the bundle of p -forms on M , $\rho: J^\infty(E) \rightarrow \Omega^p(M)$. We say that ρ is a *natural p -form* if for every point $\sigma \in J^\infty(E)$ and for every diffeomorphism Ψ

$$\rho(\text{pr } \Psi(\sigma)) = (\Psi_*\rho)(\sigma).$$

The notion of a natural p -form gives a precise characterization of a “ p -form covariantly constructed from the metric and its derivatives”. Of course, the property of being natural can be generalized to any type of tensor field. Natural tensor fields are defined on any manifold by universal formulas involving the metric and its derivatives. It is an old result of Thomas that natural tensor fields are locally constructed from the metric, the curvature, and covariant derivatives of the curvature to some order [17]. If M is oriented, then we should restrict attention to orientation preserving diffeomorphisms. In this case, natural tensor fields are constructed in a tensorial fashion from the metric, curvature, covariant derivatives of curvature, and the volume form defined by the metric.

It is straightforward to verify that α in (3.4) is not a natural $(n-1)$ -form. For example, in local coordinates (2.1) on $J^\infty(\pi^{-1}(U))$ the components of α are locally constructed from the metric and its first derivatives only (this follows from (2.4)–(2.6)). Thomas’s result then implies that α is not a natural form. In the bundle language we have been using, we say that α does not behave naturally under the map $\text{pr } \Psi$. Thus, while α is globally defined on any orientable even-dimensional manifold, its definition depends on the choice of manifold. On the other hand, given the isomorphism (2.10), the bundle diffeomorphism $\text{pr } \Psi$ cannot change the cohomology class of α . Thus, we conclude that while α will change unnaturally under the action of a diffeomorphism, it can only do so by the addition of a d_H -exact $(n-1)$ -form locally constructed from the metric and its derivatives:

$$\alpha(\text{pr } \Psi(\sigma)) - (\Psi_*\alpha)(\sigma) = d_H\tau(\sigma).$$

In other words, the charge Q in (3.5) is diffeomorphism invariant even if its integrand is not.

The diffeomorphism invariance of the equivalence class of α suggests that it may be possible to find a natural representative of this conservation law. In other words, is there a form $\beta \in \Omega^{n-2,0}(J^\infty(E))$ such that

$$\alpha' = \alpha + d_H\beta$$

is a natural $(n-1)$ -form with respect to orientation preserving diffeomorphisms? To answer this question we observe that such a natural d_H -closed p -form $\alpha' \in \Omega^{p,0}(J^\infty(E))$

falls under the scope of the theorem of Gilkey [4,5]. Gilkey's theorem serves to classify natural d_H -closed p -forms ρ modulo forms $d_H\gamma$ where γ is a natural $(p-1)$ -form. In this setting of *equivariant cohomology*, Gilkey asserts that if ρ is a natural d_H -closed p -form, then there is a natural $(p-1)$ -form γ such that

$$\rho = \kappa + d_H\gamma,$$

where κ is a *characteristic form*, *i.e.*, an element of the algebra generated by the Pontrjagin forms [18]. This means that all natural forms which are d_H -closed but are not d_H of a natural form are of even degree $4k$, where $k = 1, 2, \dots$. We have seen that the kink conservation law only exists in odd degree. Hence there is no natural representative of the kink conservation law.

The results of this paper are independent of any field equations that might be imposed on the metric, *e.g.*, the vacuum Einstein equations. In the presence of field equations, new conservation laws may be obtained since the differential forms need only be closed, say, when the Einstein tensor vanishes. The variational bicomplex for Lorentzian metrics can be pulled back to the jet bundle of Einstein metrics and the conservation laws can be classified using spinor methods [19,20]. The results of this investigation will be presented elsewhere [21].

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