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Gravitational observables and local symmetries

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Using a recent classification of local symmetries of the vacuum Einstein equations, it is shown that there can be no observables for the vacuum gravitational field (in a closed universe) built as spatial integrals of local functions of Cauchy data and their derivatives.

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A long-standing open problem in Einstein's general theory of relativity is to give an invariant characterization of the state of the (vacuum) gravitational field in terms of quantities measurable at a single instant of time. Finding such a characterization constitutes the well-known "problem of observables" in Hamiltonian relativity [1]. A precise formulation of this problem is as follows. Let Γ denote the phase space for general relativity. To fix ideas, let us choose Γ to be the cotangent bundle over the space of Riemannian metrics on a compact three-dimensional manifold Σ. A point in phase space can be fixed by specifying a pair (q_{ab},p^{ab}), where q_{ab} is a metric on Σ and p^{ab} is a symmetric tensor density on Σ. A point $x\in\Gamma$ defines a state of the gravitational field if and only if it lies in the subspace $\overline{\Gamma}\subset\Gamma$ defined as the locus of points satisfying the Hamiltonian and momentum constraints [2,3]

$$\mathcal{H}=0=\mathcal{H}_a \ . \tag{1}$$

$\mathcal{H}$ and $\mathcal{H}_a$ are often called the "super-Hamiltonian" and "super-momentum." Viewing the constraints as vanishing of functions on Γ we can express them as

$$H(N)=0=H(\mathbf{N}) \quad \forall N,\mathbf{N} \ . \tag{2}$$

Here $H(N)$ is the super-Hamiltonian smeared with a "lapse function," which is any function on Σ; $H(\mathbf{N})$ is the super-momentum smeared with a vector field on Σ, often called the "shift vector":

$$H(N)=\int_\Sigma N\mathcal{H} \ ,$$

$$H(\mathbf{N})=\int_\Sigma N^a\mathcal{H}_a \ .$$

While each point of $\overline{\Gamma}$ defines a gravitational field, the description is rather redundant: infinitely many points in $\overline{\Gamma}$ define the same gravitational field [1]. As is well known, for each point $x\in\overline{\Gamma}$ and for each choice of lapse and shift there is a one-parameter family of points on $\overline{\Gamma}$ that are physically equivalent to x. This curve of redundancy is the flow through x of the Hamiltonian vector field defined by the constraint function $H(N)+H(\mathbf{N})$. For each N and $\mathbf{N}$, $H(N)+H(\mathbf{N})$ represents a Hamiltonian for the Einstein equations, so the flow connecting physically equivalent canonical data represents time evolution. Infinitesimally, $H(N)$ generates the canonical transformation of the phase space data induced by a normal deformation of Σ (now thought of as embedded in the Einstein space) specified at each point by N. Similarly, $H(\mathbf{N})$ provides the infinitesimal canonical transformation of the data induced by a tangential deformation of Σ specified by $\mathbf{N}$. Normal and tangential deformations of the hypersurface can be viewed as the action on the hypersurface of infinitesimal diffeomorphisms of the spacetime manifold $\mathcal{M}$. The corresponding canonical transformations represent the change in the canonical data as they are carried by the (infinitesimal) diffeomorphism from point to point in the Einstein space for which they are the Cauchy data.

From the above discussion it is clear that a nonredundant characterization of the state of the gravitational field involves finding functions on $\overline{\Gamma}$ invariant under the flow generated by $H(N)+H(\mathbf{N})$. Such "observables" are functions of Cauchy data that are invariant under infinitesimal spacetime diffeomorphisms modulo the Einstein equations. More succinctly, the observables are constants of motion for the Einstein equations. A mathematical characterization is easily found: the observables are equivalence classes of functions $F:\Gamma\rightarrow R$ that have weakly vanishing Poisson brackets with the constraint functions $H(N), H(\mathbf{N})$ for all N and $\mathbf{N}$:

$$[F,H(N)]|_{\overline{\Gamma}}=0=[F,H(\mathbf{N})]|_{\overline{\Gamma}} \ . \tag{3}$$

Two functions F_1 and F_2 are equivalent if their difference vanishes on $\overline{\Gamma}$:

$$F_1 \sim F_2 \Longleftrightarrow (F_1-F_2)|_{\overline{\Gamma}}=0 \ .$$

If Σ is open, and asymptotically flat boundary conditions are included in the definition of Γ, then the Arnowitt-Deser-Misner (ADM) energy, momentum, and angular momentum provide examples of observables. Clearly this handful of constants of motion is inadequate to characterize completely the state of the gravitational field. If Σ is compact without boundary there are *no known observables.* In the classical theory the scarcity of known observables is perhaps only a technical annoyance. This annoyance becomes a stumbling block when the rules of Dirac constraint quantization are applied to construct a quantum theory of gravity [4]. Here observables play a key role, and their scarcity hampers progress in quantum gravity. Here we will show that the complexity of the Einstein equations prohibits the simplest class of putative observables from existing. Henceforth, unless otherwise stated, we will assume the universe is closed, i.e., Σ is compact without boundary.

If one could integrate the Einstein equations and find an internal time, then in principle a complete set of observables could be found [5]. Unfortunately, it is unlikely that the general solution of the Einstein equations will be available any time soon, and it is quite problematic to isolate internal spacetime variables from Γ [6]. A direct systematic search for observables would seem to be intractable if only because of the bewildering array of ways to attempt their construction. Nevertheless, let us begin such a search. The simplest class of functions on Γ that one can consider are the *local functionals,* built as integrals over Σ of local functions of the canonical variables (q_{ab},p^{ab}) and their derivatives. By "local functions" is meant that at a given point $x\in\Sigma$ the function being integrated depends on the canonical variables and their derivatives up to some finite order *at x*. For example, the constraint functions $H(N)$ and $H(\mathbf{N})$ are local functionals; they are observables too, but they are equivalent to zero. In the asymptotically flat context the energy, momentum, and angular momentum observables can be viewed as local functionals. So we would like to answer the question: Are there any (nontrivial) observables for closed universes built as local functionals of the canonical data? As we shall see, the answer is no. The key to showing this is to use the fact that if a local functional is an observable, then there must be a corresponding local "hidden symmetry" for the Einstein equations.

Let $F:\Gamma\rightarrow R$ be such an observable. Because it is a local functional, we can rigorously assert that (3) is equivalent to the Poisson brackets relations

$$[F,\mathcal{H}_\alpha(x)]=\int_\Sigma dy\ \Lambda_\alpha^\beta(x,y)\mathcal{H}_\beta(y) \ ,$$

where $\Lambda_\alpha^\beta(x,y)$ is built from local functions of the canonical variables, δ functions, and derivatives of δ functions to some finite order; we have defined $\mathcal{H}_\alpha=(\mathcal{H},\mathcal{H}_a)$. Corresponding to F is the Hamiltonian vector field V_F defined by

$$V_F=\int_\Sigma\left[\frac{\delta F}{\delta p^{ab}}\frac{\delta}{\delta q_{ab}}-\frac{\delta F}{\delta q_{ab}}\frac{\delta}{\delta p^{ab}}\right] .$$

V_F is the infinitesimal generator of a one-parameter family of canonical transformations mapping admissible Cauchy data to other admissible data, i.e., mapping solutions at any given time to other solutions at that time. Infinitesimally, the canonical transformation is given by

$$\begin{aligned}\delta q_{ab}&=V_F(q_{ab})=\frac{\delta F}{\delta p^{ab}} \ ,\\ \delta p^{ab}&=V_F(p^{ab})=-\frac{\delta F}{\delta q_{ab}} \ .\end{aligned}\tag{4}$$

Because F is a local functional, the components of V_F associated with the chart (q_{ab},p^{ab}), given by $\delta F/\delta p^{ab}$ and $-\delta F/\delta q_{ab}$, are local functions of the canonical variables.

Now, let $(q_{ab}(t),p^{ab}(t))$ denote a solution to the Hamilton equations for a given choice of lapse and shift $N^\alpha=(N(t),\mathbf{N}(t))$. This means that, at each t, $(q_{ab}(t),p^{ab}(t))$ satisfy the constraints (1) and the evolution equations defined by the Hamiltonian $H(N)+H(\mathbf{N})$. Because of the requirement (3), the infinitesimal transformation $(\delta q_{ab},\delta p^{ab},\delta N^\alpha)$, given by (4) and

$$\delta N^\alpha(y)=\int_\Sigma dx\ N^\beta(x)\Lambda_\beta^\alpha(x,y) \ , \tag{5}$$

satisfies the Hamilton equations linearized about the solution $(q_{ab}(t),p^{ab}(t),N^\alpha(t))$.

The spacetime metric g_{ab} which solves the Einstein equations is constructed algebraically from $q_{ab}(t)$ and $N^\alpha(t)$. Conversely, given a spacetime Einstein metric, one can reconstruct the one-parameter family $(q_{ab}(t),p^{ab}(t),N^\alpha(t))$ algebraically (and hence locally) from the spacetime metric and its first derivatives [7]. Note in particular that, in a solution to the Hamilton equations the canonical momentum $p^{ab}(t)$ is constructed algebraically from the three-metric, the lapse and shift, and their first derivatives. Therefore, the infinitesimal transformation generated by F will correspond to a change δg_{ab} in the spacetime metric that is a local function of g_{ab} and a finite number of its derivatives at a point. It is straightforward to see that δg_{ab} satisfies the spacetime form of the Einstein equations linearized about g_{ab}. In this fashion the observable generates an infinitesimal map of solutions to solutions. Local transformations of this type mapping solutions to solutions are called "generalized symmetries" by mathematicians.

Recently all generalized symmetries of the vacuum Einstein equations have been classified [8]. They consist of a trivial scaling symmetry and the familiar diffeomorphism symmetry. The former cannot be implemented as a symplectic map of Γ, while the latter is generated by the constraint functions themselves. Because there are no other symmetries, there can be no observables (save the trivial constraints) built as local functionals of the canonical variables.

A more explicit proof of this relies on the connection between symmetries and conservation laws. An observable $\hat{F}$ that is built as a local functional corresponds to a local differential conservation law, i.e., a spacetime three-form σ that is closed by virtue of the Einstein equations. To see this we first note that $\hat{F}$ is, by definition, an integral over Σ of a spatial three-form $\hat{\sigma}$ built locally from $x \in \Sigma$, the canonical variables (q_{ab}, p^{ab}) and their derivatives:

$$\hat{F}[q,p] = \int_\Sigma \hat{\sigma}(x,q,p,\partial_x q,\partial_x p,\ldots)\ . \qquad (6)$$

Because of (3), if we evaluate $\hat{\sigma}$ on any solution $(q_{ab}(t), p^{ab}(t))$ then $\hat{F}$ is independent of t. This will be true for solutions constructed using any lapse and shift. As before, we can translate this result into spacetime form in terms of the Einstein metric g_{ab} defined by $(q_{ab}(t), N(t), \mathbf{N}(t))$. From this point of view we obtain from $\hat{\sigma}$ a spacetime three-form $\sigma(x,g,\partial_x g,\ldots)$ built locally from $x \in \mathcal{M}$, the spacetime metric and its derivatives. We thus obtain a functional of g_{ab} via

$$F[g] = \int_\Sigma \sigma(x,g,\partial_x g,\ldots)\ , \qquad (7)$$

where now Σ is viewed as a spacelike hypersurface rather than an abstract three-manifold. Equation (3) implies that the value of $F[g]$ is independent of the choice of Σ when g_{ab} satisfies the vacuum Einstein equations. Therefore the exterior derivative of σ vanishes when the Einstein equations are satisfied.

As a by-product of the symmetry classification of [8] it was shown that all weakly closed three-forms are weakly equivalent to identically (i.e., strongly) closed three-forms [9]. Thus, because of the trivial nature of the symmetries of the vacuum Einstein equations, local conservation laws are essentially topological in nature. The proper setting for understanding this is the variational bicomplex [10] associated with the jet bundle of metrics over spacetime. In that context it can be shown that an identically closed three-form σ built locally from the spacetime metric and its derivatives (as well as the spacetime position) can be written as the sum of an exact form and a representative σ_0 of the cohomology class of σ:

$$\sigma = d\alpha + \sigma_0\ , \qquad (8)$$

where α and σ_0 are also local functions of the metric and its derivatives. The relevant cohomology is the de Rham cohomology of the bundle of metrics over spacetime. We need not explore this cohomology here; although the integral of σ_0 over a hypersurface *is* a constant of motion, it is a trivial one becuase this functional of the metric is conserved irrespective of whether or not the Einstein equations are satisfied. Therefore only the two-form α can lead to nontrivial local observables.

In the asymptotically flat context, the structure of spatial infinity allows nontrivial conservation laws, namely, that of energy, momentum, and angular momentum, to be encoded in α. In detail, the integral of σ over Σ involves an integral of α over the "sphere at infinity," and this leads to the ADM observables (for appropriate choices of α) [11]. If spacetime is diffeomorphic to $R \times \Sigma$ with $\partial\Sigma = 0$ then no asymptotic region can be used to construct nontrivial constants of motion (built as local functionals) because now the integral over $d\alpha$ vanishes identically. In other words, for closed universes "on shell," the only possible conservation laws derive from the topology of the bundle of metrics over spacetime—this is the information contained in σ_0—and have nothing to do with the Einstein equations *per se.* Thus there can be no nontrivial observables for closed universes constructed as local functionals.

It would seem then that observables must be constructed in a more complicated fashion than a local functional. Unfortunately, there does not appear to be any way of systematically identifying "nonlocal conservation laws" for the Einstein equations. In many examples nonlocal conservation laws for partial differential equations are closely tied to the integrability of those equations. A well-known attribute of an integrable system of partial differential equations is the existence of infinitely many generalized symmetries. Modulo the diffeomorphism symmetry, which is physically trivial, the Einstein equations fail to pass this test and so one can expect little luck in finding such nonlocal conservation laws based on some sort of integrability. Indeed, there is a result of Kuchař that rules out any observables built as linear functionals of the ADM momenta [12]. One encouraging recent result [13] shows that the holonomy group of the Ashtekar connection on a given hypersurface is almost a constant of motion. For the meaning of "almost" see [14]. Clearly this type of observable is quite nonlocal. One can hope (but it is only a hope) that the results of [13] in the context of the Ashtekar canonical formalism are the hint of some structure that can be used to find nonlocal conservation laws, at least in principle. In practice, it is possible that perturbative methods for defining observables can be devised. This is really an important possibility. Given the scarcity of exactly soluble quantum field theories, it is to be expected that a quantum theory of gravity would need a perturbative definition at some point. So, while it seems possible to find the exact quantum states [4], it may be necessary to approximate the dynamical information contained in the observables. Hopefully, such a perturbation theory will be better behaved than its weak-field counterpart. Failing this, it appears that the standard rules for canonical quantization of constrained systems, in which the observables play a central role, will have to be improved or modified to avoid the problem of observables.

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