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# On Signifiable Computability: Part I: Signification of Real Numbers, Sequences, and Types

**Vladimir A. Kulyukin**

Department of Computer Science, Utah State University, Logan, UT 84322, USA; vladimir.kulyukin@usu.edu

**Abstract:** Signifiable computability aims to separate what is theoretically computable from what is computable through performable processes on computers with finite amounts of memory. Real numbers and sequences thereof, data types, and instances are treated as finite texts, and memory limitations are made explicit through a requirement that the texts be stored in the available memory on the devices that manipulate them. In Part I of our investigation, we define the concepts of signification and reference of real numbers. We extend signification to number tuples, data types, and data instances and show that data structures representable as tuples of discretely finite numbers are signifiable. From the signification of real tuples, we proceed to the constructive signification of multidimensional matrices and show that any data structure representable as a multidimensional matrix of discretely finite numbers is signifiable.

**Keywords:** computability theory; theory of recursive functions; number theory; real numbers; real number sequences

**MSC:** 03D75; 03D80

## 1. Introduction

In classical computability, computation is described with formalisms such as $\lambda$-calculus (cf., e.g., Rogers, 1988, Ch. 1, [1]) and the Turing Machine (cf., e.g., Davis et al., 1994 [2], Ch. 6) that do not place memory limitations on physical or abstract computers on which the formalized computation is performed and confine computation to natural numbers. Yet, memory limitations are fundamental in distinguishing what is theoretically possible from what is computationally feasible, because scientific computation on real numbers is routinely performed on devices with finite amounts of memory. A theory of signifiable computability proposed in our investigation aims to distinguish what is computable without any memory limitations from what is computable on a finite amount of memory available for computation and to characterize the computability of functions on real numbers with and without memory limitations. To that end, data types and instances, programs, and program states are treated as finite texts constructed with formalisms (e.g., the standard decimal notation on `Unicode`). The texts are formed to *signify* or, equivalently, *designate* intuitive objects such as numbers, data types and structures, programs, and program states in a given universe of discourse. When the formed texts are stored within the available memory units, they signify or designate the corresponding objects on finite memory, or, equivalently, the texts finite memory-signify (*FM-signify*) or finite memory-designate (*FM-designate*) those objects. No finite memory device (FMD) can mechanically manipulate or, equivalently, *transform* texts unless the latter are stored in its memory cells.

We introduce the concepts of *discretely finite* and *discretely infinite* numbers. A discretely finite number is signified completely by a text constructed with a formalism on an alphabet in that the *signs* (i.e., elements of the alphabet) of the designating text completely coincide with the elements of the number they signify. A discretely infinite number is a number for which no such signification is possible. E.g., texts such

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as "2.7182818284" and "3.1415926535" in the standard decimal notation on Unicode signify the numbers 2.7182818284 and 3.1415926535. However, when constructed to approximate the transcendental numbers $e$ and $\pi$, respectively, these texts only *reference* these discretely infinite numbers.

In Part I, we show that signification and FM-signification extend to number tuples (i.e., finite sequences of numbers), data types, and instances and show that data structures representable as tuples of discretely finite numbers are signifiable. Consequently, all standard data structures of computer science (e.g., lists, arrays, tuples, queues, stacks, hash tables, priority queues, and heaps) whose elements are discretely finite numbers are signifiable. We also show that any data structure representable as a multidimensional matrix of discretely finite numbers (e.g., trees and graphs) is signifiable. We conclude Part I by defining a Gödel numbering of texts to map texts to unique positive integers. A principal objective of Part I of our investigation is to lay a foundation for a subsequent axiomatization of signifiable computability.

## 2. Prolegomena

This section briefly introduces the basic conceptual and formal apparatus used in this article. The reader may skip this section on first reading and return to it as necessary. The statements $S_1 \subset S_2$ and $S_1 \subseteq S_2$ mean that $S_1$ is a proper subset and a subset of the set $S_2$, respectively. The symbol $\emptyset$ denotes the empty set. $S_1$ and $S_2$ are *equivalent* if there is a one-to-one (1–1) correspondence between them, or, in symbols, $S_1 \sim S_2$, whereby each element of $S_1$ is uniquely paired with an element of $S_2$, and vice versa. $|S|$ denotes the cardinality of $S$. If $S$ is finite, $|S| \in \mathbb{N} = \{0, 1, 2, \ldots\}$. $\mathbb{Z}$, $\mathbb{Z}^-$, $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of whole numbers ($0 \in \mathbb{Z}$), negative whole numbers ($0 \notin \mathbb{Z}^-$), positive whole numbers ($0 \notin \mathbb{Z}^+$), rational numbers, and real numbers, respectively. $S$ is *enumerably finite* if $S \sim S' \wedge |S'| \in \mathbb{N}$, where $\wedge$ designates the logical *and*. $S$ is *enumerably infinite* if $S \sim \mathbb{N}$. $S$ is *enumerable* if it is enumerably finite or enumerably infinite. Per Cantor's diagonalization (cf., e.g., Kleene, 1952, [3], Ch. I, § 2), $\mathbb{R}$ is not enumerable. If $T_{i_1}, T_{i_2}, \ldots, T_{i_k}$, $k \in \mathbb{Z}^+$ are sets, then

$$T_{i_1} \times T_{i_2} \times \cdots \times T_{i_k} = \{(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) | t_{i_1} \in T_{i_1}, t_{i_2} \in T_{i_2}, \ldots, t_{i_k} \in T_{i_k}\},$$

where $(t_{i_1}, t_{i_2}, \ldots, t_{i_k})$ is called a *k-tuple*, i.e., a finite sequence of $k$ elements.

A *finite memory device* (FMD) $\mathscr{D}$ is a computer with an enumerably finite set of memory units or, equivalently, *memory cells*. Let $\mathfrak{D}$ denote the set of all FMDs, then the function $\texttt{CCAP} : \mathfrak{D} \mapsto \mathbb{Z}^+$ maps each FMD to the number of its memory cells. $\mathscr{D}$ is a FMD if and only if $\texttt{CCAP}(\mathscr{D}) = k \in \mathbb{Z}^+$. An *elementary sign* is that which can be written in exactly one memory cell. We enclose elementary signs in double quotation marks, e.g., "0", "1", "+", etc. The elementary sign "" is the *empty sign*.

An *alphabet* is an enumerably finite, non-empty set of non-empty elementary signs. A *text* is a finite sequences of elementary signs of $\mathscr{A}$. An alphabet $\mathscr{A}$ is *decimal sufficient* if and only if $\mathscr{A}$ contains the signs of the standard decimal notation or the unique equivalents thereof, e.g., Unicode. A *formalism* $\mathscr{L}$ on an alphabet $\mathscr{A}$ is a finite set of formal rules to *form* or, equivalently, to construct texts on $\mathscr{A}$. $\mathscr{L}$ is *decimal notation sufficient* if and only if it allows for the mechanical formation of texts on a decimal sufficient alphabet $\mathscr{A}$ in the characteristic-mantissa form of the standard decimal notation or an equivalent thereof. E.g., $\mathscr{L}$ on $\mathscr{A}$ can be the floating-point notation of Lisp on Unicode (cf., e.g., Steele, 1990 [4], Ch. 2, Section 2.1.3) or the numerical notation of Perl on Unicode (cf., e.g., Lee, 2010 [5], Ch. 2, Section 1). We will hereafter assume that any $\mathscr{L}$ on $\mathscr{A}$ is decimal notation sufficient. We call an alphabet $\mathscr{A}$ decimal notation sufficient if and only if there exists a decimal notation sufficient formalism $\mathscr{L}$ on $\mathscr{A}$.

A basic text formation operation in $\mathscr{L}$ on $\mathscr{A}$ is *concatenation*, denoted as $\bigoplus_{\mathscr{A},\mathscr{L}}$. In the base case, $\bigoplus_{\mathscr{A},\mathscr{L}}$ maps two elementary signs $s_i$ and $s_j$ from $\mathscr{A}$ to the sign $s_i s_j$. The concatenation of the empty sign to any other sign $s$ results in $s$. In the recursive case, the concatenation of the elementary signs $s_{i_1} \in \mathscr{A}, s_{i_2} \in \mathscr{A}, \ldots, s_{i_n} \in \mathscr{A}, 2 < n \in \mathbb{Z}^+$, is formed by concatenating $s_{i_1}$ and $s_{i_2}$ to obtain $s_{i_1}s_{i_2}$, and then concatenating $s_{i_3}$ to the right of $s_{i_1}s_{i_2}$ to obtain $s_{i_1}s_{i_2}s_{i_3}$, and so on until all $n$ signs are concatenated to form $s_{i_1}s_{i_2}\ldots s_{i_n}$. A *complex sign* in $\mathscr{L}$ on $\mathscr{A}$ is a concatenation of at least two elementary signs from $\mathscr{A}$ in $\mathscr{L}$. The order of the sign concatenation in the formation of a text is arbitrarily assumed to be left to right, i.e.,

$$s_{i_1}s_{i_2}\ldots s_{i_n} \quad = \quad \Big(\ldots\Big(\ldots\Big(\Big(s_{i_1}\bigoplus_{\mathscr{A},\mathscr{L}} s_{i_2}\Big)\bigoplus_{\mathscr{A},\mathscr{L}} s_{i_3}\Big)\ldots\bigoplus_{\mathscr{A},\mathscr{L}} s_{i_{n-1}}\Big)\bigoplus_{\mathscr{A},\mathscr{L}} s_{i_n}\Big).$$

Two enumerably infinite sets are defined with respect to $\mathscr{L}$ on $\mathscr{A}$: $\mathscr{A}^+_{\mathscr{L}}$ and $\mathscr{A}^*_{\mathscr{L}}$. $\mathscr{A}^+_{\mathscr{L}}$ includes the elements of $\mathscr{A}$ and all texts on $\mathscr{A}$ constructed according to the rules of $\mathscr{L}$, while $\mathscr{A}^*_{\mathscr{L}} = \{\texttt{""}\} \cup \mathscr{A}^+_{\mathscr{L}}$. We say that $t$ is a text in $\mathscr{L}$ on $\mathscr{A}$ when $t \in \mathscr{A}^+_{\mathscr{L}}$. A *numeral* is a text $t \in \mathscr{A}^+_{\mathscr{L}}$ that designates a real number. E.g., if $\mathscr{L}$ is the standard decimal notation on the subset of `Unicode`

$$\mathscr{A} = \{\texttt{"0"},\texttt{"1"},\texttt{"2"},\texttt{"3"},\texttt{"4"},\texttt{"5"},\texttt{"6"},\texttt{"7"},\texttt{"8"},\texttt{"9"},\texttt{"+"},\texttt{"-"},\texttt{"."}\}, \tag{1}$$

then, omitting the subscripts $\mathscr{A}$ and $\mathscr{L}$ for brevity, we can form the numerals `"+12.7"` and `"-31.05"` as

$$\begin{array}{lcl} \Big(\Big(\Big(\Big(\texttt{"+"} \bigoplus \texttt{"1"}\Big) \bigoplus \texttt{"2"}\Big) \bigoplus \texttt{"."}\Big) \bigoplus \texttt{"7"}\Big) & = & \texttt{"+12.7"}; \\ \Big(\Big(\Big(\Big(\Big(\texttt{"-"} \bigoplus \texttt{"3"}\Big) \bigoplus \texttt{"1"}\Big) \bigoplus \texttt{"."}\Big) \bigoplus \texttt{"0"}\Big) \bigoplus \texttt{"5"}\Big) & = & \texttt{"-31.05"}. \end{array} \tag{2}$$

Since the concatenation is left to right, the parenthesization can be omitted and (2) rewritten as

$$\begin{array}{lcl} \texttt{"+"} \bigoplus \texttt{"1"} \bigoplus \texttt{"2"} \bigoplus \texttt{"."} \bigoplus \texttt{"7"} & = & \texttt{"+12.7"}; \\ \texttt{"-"} \bigoplus \texttt{"3"} \bigoplus \texttt{"1"} \bigoplus \texttt{"."} \bigoplus \texttt{"0"} \bigoplus \texttt{"5"} & = & \texttt{"-31.05"}. \end{array} \tag{3}$$

For $t_i \in \mathscr{A}^+_{\mathscr{L}}$, $1 \le i \le l$, $l > 2$, we let

$$\begin{array}{c} \bigoplus_{\mathscr{A},\mathscr{L}}\Big(t_1, t_2, \ldots, t_l\Big) = \bigoplus_{\mathscr{A},\mathscr{L}}\Big|_{i=1}^{l} t_i = \\ \bigoplus_{\mathscr{A},\mathscr{L}} t_1 \bigoplus_{\mathscr{A},\mathscr{L}} t_2 \bigoplus_{\mathscr{A},\mathscr{L}} \cdots \bigoplus_{\mathscr{A},\mathscr{L}} t_l. \end{array} \tag{4}$$

If $t \in \mathscr{A}^+_{\mathscr{L}}$, $|t|$ is the number of the elementary signs from $\mathscr{A}$ in $t$. If $s \in \mathscr{A}$, $|s| = 1$ and $|\texttt{""}| = 0$. If $|t| = n \in \mathbb{Z}^+$, then $s_1, s_2, \ldots, s_n$ designate the elementary signs of $t$ from left to right. E.g., if $t = \texttt{"+12.7"}$, $|t| = 5$, and $s_1 = \texttt{"+"}$, $s_2 = \texttt{"1"}$, $s_3 = \texttt{"2"}$, $s_4 = \texttt{"."}$, $s_5 = \texttt{"7"}$.

If $f$ is a function, $dom(f)$ and $ran(f)$ denote the domain and the range of $f$, respectively. A statement $f : S \mapsto R$ abbreviates $dom(f) = S \wedge ran(f) = R$. If $f : T_{i_1} \times T_{i_2} \times \cdots \times T_{i_k} \mapsto T_j$ and $(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) \in dom(f)$, then $f$ is *defined on* $(t_{i_1}, t_{i_2}, \ldots, t_{i_k})$ or, in symbols, $f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) \downarrow$ if and only if there exists $t_j \in T_j$ such that $f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) = t_j$ or, in symbols, $(\exists\ t_j \in T_j)\ \{f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) = t_j\}$, where $\exists$ designates the logical *existential quantifier*. If $\neg(\exists\ t_j \in T_j)\ \{f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) = t_j\}$, where $\neg$ designates the logical *not*, then $f$ is *undefined on* $(t_{i_1}, t_{i_2}, \ldots, t_{i_k})$ or, in symbols, $f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) \uparrow$. If $f : T_{i_1} \times T_{i_2} \times \cdots \times T_{i_k} \mapsto T_j$ and

$$(\forall\ (t_{i_1}, t_{i_2}, \ldots, t_{i_k}) \in T_{i_1} \times T_{i_2} \times \cdots \times T_{i_k})\ (\exists\ t_j \in T_j)\ \{f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) = t_j\},$$

where $\forall$ designates the logical *universal quantifier*, $f$ is *total* on $T_{i_1} \times T_{i_2} \times \ldots \times T_{i_k}$. If

$$(\exists\ (t_{i_1}, t_{i_2}, \ldots, t_{i_k}) \in T_{i_1} \times T_{i_2} \times \cdots \times T_{i_k})\ \{f(t_{i_1}, t_{i_2}, \ldots, t_{i_k}) \uparrow\},$$

then $f$ is *partial* on $T_{i_1} \times T_{i_2} \times \ldots, T_{i_k}$.

## 3. Data Types and Instances

A data *type* is a set of objects constructable with a finite set of rules. A data *instance* or *structure* is a member of a data type. Informally, data types are sets of intuitively conceivable objects that are generated with performable processes (cf. Kleene, 1952 [3], Ch. III, § 15). E.g., $\mathbb{N}$ is the natural number data type generatable (or, equivalently, *formable* or *constructable*) with three rules: (1) 0 is a natural number; (2) any number generated by a finite number of compositions of the successor function $s(x) = x + 1$ with itself beginning at 0, i.e., $s(0), s(s(0)), \ldots$, is a natural number; (3) no other object is a natural number. $\mathbb{Z}^-$ is the negative whole number data type whose objects are formable with three rules: (1) $-1$ is a negative whole number; (2) any number obtained by a finite number of compositions of the predecessor function $ps(x) = x - 1$ with itself beginning at $-1$ is a negative whole number; (3) no other object is a negative whole number. $\mathbb{Z}^+$ is the positive whole number data type whose objects are constructable with three rules: (1) 1 is a positive whole number; (2) any number number generated by a finite number of compositions of $s(x) = x + 1$ with itself beginning at 1 is a positive whole number; (3) no other object is a positive whole number. $\mathbb{Z}$ is the whole number data type formable by combining the rules for $\mathbb{N}$, $\mathbb{Z}^+$, and $\mathbb{Z}^-$ and the rule that no other object is a whole number. Other examples include the rational number, the binary tree, the hash table, the undirected unweighted graph, and the directed unweighted graph.

A data type is *discrete* if each instance of the type consists of an enumerable set of intuitively conceivable objects. E.g., a natural number consists of finitely many applications of $s(x)$ and a whole number consists of finitely many applications of $s(x)$ or $ps(x)$. We can analogously generate the numerator of a rational number and its non-zero denominator. The binary tree type is discrete insomuch as a binary tree structure consists of an enumerable set of nodes constructed by first generating the root node and subsequently, if the desired number of levels is greater than 0, producing finitely many nodes at each level and connecting each node to the unique parent node at the previous level as its left or right descendant so that the number of nodes at each level $l$ is at most $2^l$.

A discrete data type is *finite* if each instance of the type consists of an enumerably finite set of objects. E.g., the natural number, the whole number, the rational number, the binary tree, the hash table, and the undirected or directed unweighted graph are discrete and finite. We refer to such data types and their instances as *discretely finite*. E.g., 3 is a discretely finite natural number (i.e., $3 \in \mathbb{N}$) because $3 = s(s(s(0)))$; $-3$ is a discretely finite negative whole number (i.e., $-3 \in \mathbb{Z}^+$) because $-3 = ps(ps(-1)))$; $-3/2$ is discretely finite rational number (i.e., $-3/2 \in \mathbb{Q}$) because $-3/2 = ps(ps(-1)))/s(s(0))$. A data type is *discretely infinite* if each instance of the data type consists of an enumerably infinite set of objects.

## 4. Signification and Reference of Real Numbers

One can define $\mathbb{R}$ as an infinite set of sets, each of which represents a real number so that the sets representing individual whole or rational numbers are singletons, whereas the sets representing individual irrational numbers are infinite sets of rational numbers (cf., e.g., Kleene, 1952 [3], Ch. II, § 9). A real number can, therefore, be an infinite set of numbers conceived as the completed infinite. Under this interpretation, such statements as $x = y$, $x + 1$, $x - 1$ require one to allow for the left- or right-hand side of the equality to be the completed infinite or to add or subtract 1 to and from the completed infinite, which, albeit theoretically possible and insightful, may not be constructable with performable processes, especially when the latter are restricted to finite amounts of memory available for computation.

To the extent that each individual member of the set representing one specific real number is discretely representable, e.g., in the characteristic–mantissa form in the standard decimal notation, to that extent only do we construe that set to be a discrete subtype of $\mathbb{R}$. For, once so represented, the individual elements of each representation's characteristic, the period that separates the characteristic from the mantissa, and the individual elements of the mantissa can be enumerated. Under this interpretation, some instances of $\mathbb{R}$ are discretely finite, while others are not. E.g., the singletons representing individual whole or rational numbers whose sole elements have finite characteristics and mantissas are discretely finite, but the transcendental numbers $\pi$ and $e$ are not. It is with the aim to distinguish discretely finite and discretely infinite numbers that we introduce the following definitions.

**Definition 1.** *Let $r \in \mathbb{R}$. Then, $r$ is discretely finite in $\mathscr{L}$ on $\mathscr{A}$ if and only if there exists the set $S_r = \left\{ t | t \in \mathscr{A}^+_{\mathscr{L}} \right\} \neq \emptyset$, where each $t$ signifies or, equivalently, designates $r$ and no other number, or, in symbols,*

$$r \leftarrow \{=\}_{\mathscr{A},\mathscr{L}} \rightarrow S_r. \tag{5}$$

*If $t \in S_r$, then $t$ signifies or, equivalently, designates $r$ in $\mathscr{L}$ on $\mathscr{A}$, or, in symbols,*

$$t \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow r. \tag{6}$$

*If there is no text that designates $r$ in $\mathscr{L}$ on $\mathscr{A}$, then $r$ is not signifiable or, equivalently, not designatable in $\mathscr{L}$ on $\mathscr{A}$ and, hence, not discretely finite in $\mathscr{L}$ on $\mathscr{A}$, or, in symbols,*

$$\begin{array}{lll} r & \leftarrow \{=\}_{\mathscr{A},\mathscr{L}} \rightarrow & \emptyset; \\ r & \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{""}. \end{array} \tag{7}$$

*$\leftarrow \{=\}_{\mathscr{A},\mathscr{L}} \rightarrow$ and $\leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow$ are symmetric.*

The signification of real numbers, as implied by Definition (1), is always defined with respect to a given $\mathscr{L}$ on $\mathscr{A}$. E.g., if $\mathscr{A}$ is given in (1) and $\mathscr{L}$ on $\mathscr{A}$ is the standard decimal notation where, as we assume henceforth, the leading or trailing zero signs in the characteristic–mantissa form do not change numbers designated by numerals, then

$$\begin{array}{lll} 13 & \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{"0000013"}; \\ 13 & \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{"13.0000"}; \\ 13 & \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{"013.000"}; \\ 13 & \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{"0013.00"}; \\ 13 & \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{"00013.0"}; \\ 13.13 & \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{"13.1300"}; \\ 13.13 & \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{"013.130"}; \end{array} \tag{8}$$

and

$$\begin{array}{lllll} 13 & \leftarrow \{=\}_{\mathscr{A},\mathscr{L}} & S_{13} & = & \{\texttt{"0000013","13.0000","013.000",} \\ & & & & \texttt{"0013.00","00013.0",...}\}; \\ 13.13 & \leftarrow \{=\}_{\mathscr{A},\mathscr{L}} & S_{13.13} & = & \{\texttt{"13.1300","013.130",...}\}. \end{array} \tag{9}$$

**Definition 2.** *Let $r \in \mathbb{R}$ such that $r \leftarrow \{=\}_{\mathscr{A},\mathscr{L}} \rightarrow S_r \neq \emptyset$. Then, $r$ is FM-signifiable or, equivalently, FM-designatable in $\mathscr{L}$ on $\mathscr{A}$ on a FMD $\mathscr{D}$ if and only if there exist $S'_r \subseteq S_r$, $S'_r = \left\{ t \;\middle|\; |t| \leq \texttt{CCAP}(\mathscr{D}) \right\} \neq \emptyset$, in which case,*

$$r \leftarrow \{=\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow S'_r \tag{10}$$

*holds. If $t \in S'_r$, then $t$ FM-signifies or, equivalently, FM-designates $r$ in $\mathscr{L}$ on $\mathscr{A}$ on $\mathscr{D}$, or, in symbols,*

$$t \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow r. \tag{11}$$

*If there is no text that FM-signifies or FM-designates $r$ in $\mathscr{L}$ on $\mathscr{A}$ on $\mathscr{D}$, then $r$ is not FM-signifiable or, equivalently, not FM-designatable in $\mathscr{L}$ on $\mathscr{A}$ on $\mathscr{D}$, or, in symbols,*

$$\begin{array}{lll} r & \leftarrow \{=\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow & \emptyset; \\ r & \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow & "". \end{array} \quad (12)$$

$\leftarrow \{=\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow$ *and* $\leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow$ *are symmetric.*

A direct consequence of Definitions (1) and (2) is

**Lemma 1.** *If* $r \leftarrow \{=\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow S_r \neq \emptyset$, *then* $r \leftarrow \{=\}_{\mathscr{A},\mathscr{L}} \rightarrow S_r$.

**Definition 3.** *Let* $r \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow t \in \mathscr{A}^+_{\mathscr{L}}$. *Then, $t$ is a signifying or designating numeral of $r$ in $\mathscr{L}$ on $\mathscr{A}$. If* $r \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow t$, *then $t$ is an FM-signifying or FM-designating numeral of $r$ in $\mathscr{L}$ on $\mathscr{A}$ on $\mathscr{D}$.*

We prove

**Lemma 2.** *Let* $r \leftarrow \{=\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow S_r$, $r \in \mathbb{R}$, $|\mathscr{A}| = m \in \mathbb{Z}^+$. *Then* $|S_r| \in \mathbb{N}$.

**Proof.** If $r \leftarrow \{=\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow S_r = \emptyset$, $|S_r| = 0$. Let $t \in S_r$. Then, $1 \leq |t| \leq \texttt{CCAP}(\mathscr{D}) = k \in \mathbb{Z}^+$ so that $1 \leq |S_r| \leq \sum_{l=1}^{k} m^l$, $1 \leq l \leq k$. □

Signification, in the sense explicated in the above definitions and lemmas, is possible only with discretely finite numbers. E.g., 13 is signifiable as "13" in a decimal notation sufficient $\mathscr{L}$ on $\mathscr{A}$ in (1). Concatenating "0" any number of times to the left of "13" does not change the signification of 13, because the resultant texts still designate 13 and no other number in $\mathscr{L}$ on $\mathscr{A}$, which one can verify by eliminating the leading zero signs to obtain "13" and then obtaining the number 13 through the 13 applications of $s(n)$ starting at 0.

No text in $\mathscr{L}$ on $\mathscr{A}$ signifies a discretely infinite number so that its signs completely coincide with the individual elements of the number. E.g., let us consider the list of the first 13 approximations of the transcendental numbers $e$ and $\pi$ in (13) from the published sequences OEIS:A001113 and OEIS:A000796 in the standard decimal notation.

| | $e$ | $\pi$ |
|---|---|---|
| (1) | 2.7 | 3.1 |
| (2) | 2.71 | 3.14 |
| (3) | 2.718 | 3.141 |
| (4) | 2.7182 | 3.1415 |
| (5) | 2.71828 | 3.14159 |
| (6) | 2.718281 | 3.141592 |
| (7) | 2.7182818 | 3.1415926 |
| (8) | 2.71828182 | 3.14159265 |
| (9) | 2.718281828 | 3.141592653 |
| (10) | 2.7182818284 | 3.1415926535 |
| (11) | 2.71828182845 | 3.14159265358 |
| (12) | 2.718281828459 | 3.141592653589 |
| (13) | 2.7182818284590 | 3.1415926535897 |

(13)

Since the mantissas of $e$ and $\pi$ are infinite and non-repeating, there is no finite concatenation of the elementary signs from $\mathscr{A}$ in (1) that allows one to construct texts to signify $e$ or $\pi$ in a decimal notation sufficient $\mathscr{L}$ on $\mathscr{A}$. One can only state that texts in the infinite sets

$$\begin{array}{lll} S_e & = & \{"2.7", "2.71", "2.718", \ldots\} \\ S_\pi & = & \{"3.1", "3.14", "3.141", \ldots, \} \end{array} \quad (14)$$

*reference* $e$ and $\pi$, respectfully, without ever signifying them. While one can construct individual real numbers signified by the numerals in $S_e$ and $S_\pi$ with performable processes such as Euler's method of continued fractions for $e$ (cf., e.g., Abelson and Sussman, 1996 [6], Ch. 1, Section 1.3.3) and the Chudnovsky algorithm for $\pi$ (cf., e.g., Chudnovsky and Chudnovsky, 1988 [7] or Lorenz, 2018 [8]), the signifying numerals never completely coincide with $e$ or $\pi$.

**Definition 4.** *Let $r \in \mathbb{R}$ be discretely infinite and let $S_r = \{t | t \in \mathscr{A}^+_{\mathscr{L}}\}$ be an infinite set of numerals such that each numeral references $r$ and no other number. Then,*

$$r \leftarrow \{\approx\}_{\mathscr{A},\mathscr{L}} \rightarrow S_r \tag{15}$$

*holds and states that $r$ is referenceable in $\mathscr{L}$ on $\mathscr{A}$. If $t \in S_r$, then $t$ references $r$ in $\mathscr{L}$ on $\mathscr{A}$, or, in symbols,*

$$r \leftarrow (\approx)_{\mathscr{A},\mathscr{L}} \rightarrow t. \tag{16}$$

*If there is no numeral that references $r$ in $\mathscr{L}$ on $\mathscr{A}$, then $r$ is not referenceable in $\mathscr{L}$ on $\mathscr{A}$, or, in symbols,*

$$\begin{array}{lll} z & \leftarrow \{\approx\}_{\mathscr{A},\mathscr{L}} \rightarrow & \emptyset; \\ z & \leftarrow (\approx)_{\mathscr{A},\mathscr{L}} \rightarrow & \texttt{""}. \end{array} \tag{17}$$

*$\leftarrow \{\approx\}_{\mathscr{A},\mathscr{L}} \rightarrow$ and $\leftarrow (\approx)_{\mathscr{A},\mathscr{L}} \rightarrow$ are symmetric.*

E.g., let $s =$ "2.718" be a numeral in a decimal notation sufficient $\mathscr{L}$ on $\mathscr{A}$ in (1). If $s \in S_e$ in (14), then $e \leftarrow (\approx)_{\mathscr{A},\mathscr{L}} \rightarrow s$. However, if $s \in S_{2.718}$, where

$$S_{2.718} \quad = \quad \{\texttt{"2.718"},\texttt{"02.718"},\texttt{"2.7180"},\texttt{"002.718"},\texttt{"2.71800"},\ldots\},$$

then $2.718 \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow \texttt{"2.718"}$. We generalize these observations by proving

**Lemma 3.** *There exists $t \in \mathscr{A}^+_{\mathscr{L}}$ such that, for $r_1, r_2 \in \mathbb{R}$,*

$$r_1 \leftarrow \{=\}_{\mathscr{A},\mathscr{L}} \rightarrow S_{r_1} \ \wedge \ r_2 \leftarrow \{\approx\}_{\mathscr{A},\mathscr{L}} \rightarrow S_{r_2} \ \wedge \ t \in S_{r_1} \cap S_{r_2}.$$

**Proof.** Let $t = \texttt{"3.141"} \in \mathscr{A}^+_{\mathscr{L}}$. Let $S_\pi$ be defined in (14) and let

$$S_{3.141} = \{\texttt{"3.141"},\texttt{"03.141"},\texttt{"3.1410"},\texttt{"003.141"},\texttt{"3.14100"},\ldots\}.$$

Then,

$$3.141 \leftarrow \{=\}_{\mathscr{A},\mathscr{L}} \rightarrow S_{3.141} \ \wedge \ \pi \leftarrow \{\approx\}_{\mathscr{A},\mathscr{L}} \rightarrow S_\pi \ \wedge \ \texttt{"3.141"} \in S_{3.141} \cap S_\pi.$$

□

**Definition 5.** *Let $r \in \mathbb{R}$ such that $r \leftarrow \{\approx\}_{\mathscr{A},\mathscr{L}} \rightarrow S_r \neq \emptyset$. Then, $r$ can be FM-referenced in $\mathscr{L}$ on $\mathscr{A}$ on a FMD $\mathscr{D}$ if and only if there exists $S'_r \subseteq S_r$, $S'_r = \left\{t \;\middle|\; |t| \leq \texttt{CCAP}(\mathscr{D})\right\} \neq \emptyset$, in which case,*

$$r \leftarrow \{\approx\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow S'_r \tag{18}$$

*holds. If $t \in S'_r$, then $t$ FM-references $r$ in $\mathscr{L}$ on $\mathscr{A}$ on $\mathscr{D}$, or, in symbols,*

$$t \leftarrow (\approx)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow r. \tag{19}$$

*If there is no text that FM-references $r$ in $\mathscr{L}$ on $\mathscr{A}$ on $\mathscr{D}$, then $r$ cannot be FM-referenced in $\mathscr{L}$ on $\mathscr{A}$ on $\mathscr{D}$, or, in symbols,*

$$\begin{array}{ll} r & \leftarrow \{\approx\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow \quad \emptyset; \\ r & \leftarrow (\approx)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow \quad \texttt{""}. \end{array} \tag{20}$$

$\leftarrow \{\approx\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow$ *and* $\leftarrow (\approx)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow$ *are symmetric.*

We elaborate on Lemma (3) by showing

**Lemma 4.** *There exists* $t \in \mathscr{A}^+_{\mathscr{L}}$ *such that, for some* $r_1, r_2 \in \mathbb{R}$ *and a FMD* $\mathscr{D}$*,*

$$t \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow r_1 \ \wedge \ t \leftarrow (\approx)_{\mathscr{A},\mathscr{L}} \rightarrow r_2.$$

**Proof.** Let $\texttt{CCAP}(\mathscr{D}) \geq 13$ and $t = \texttt{"3.14159265358"} \in \mathscr{A}^+_{\mathscr{L}}$. Then, since $|t| = 13$, we have $t \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow 3.14159265358$ and $t \leftarrow (\approx)_{\mathscr{A},\mathscr{L}} \rightarrow \pi$. □

E.g., let $\mathscr{D}$ be an Ubuntu 18.04 LTS computer with a CPU `@` 2.40GHz x 4 and 8 GB of RAM and 127 GB of hard disk space. Let $\mathscr{A}$ be `Unicode` and $\mathscr{L}$ be `Lisp` as implemented in GNU CLISP 2.49.60+. Then, on this computer, $t = \texttt{"3.1415926535897932385"}$ signifies 3.1415926535897932385 and references $\pi$ in `Lisp` on `Unicode` because $|t| \leq \texttt{CCAP}(\mathscr{D})$.

The next theorem shows a consequence of $\texttt{CCAP}(\mathscr{D}) \in \mathbb{Z}^+$, for a FMD $\mathscr{D}$.

**Theorem 1.** *The set of real numbers FM-signifiable in* $\mathscr{L}$ *on* $\mathscr{A}$ *on a FMD* $\mathscr{D}$ *is enumerably finite.*

**Proof.** Let $\texttt{CCAP}(\mathscr{D}) = k \in \mathbb{Z}^+$ and let $T_1, \ldots, T_k$ be an enumeration such that

$$T_j = \left\{ t \in \mathscr{A}^+_{\mathscr{L}} \middle| \ t \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow r \in \mathbb{R} \ \wedge \ |t| = j \right\}, 1 \leq j \leq k.$$

Let $\mathbf{T}_{\mathscr{A},\mathscr{L},\mathscr{D}} = \cup_{j=1}^{k} T_j$. By Lemma (2), $\left|\mathbf{T}_{\mathscr{A},\mathscr{L},\mathscr{D}}\right| \in \mathbb{N}$. Let

$$\mathbf{R}_{\mathscr{A},\mathscr{L},\mathscr{D}} = \left\{ r \in \mathbb{R} \middle| \ r \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow t \in \mathbf{T}_{\mathscr{A},\mathscr{L},\mathscr{D}} \right\}.$$

Then, $\left|\mathbf{R}_{\mathscr{A},\mathscr{L},\mathscr{D}}\right| \in \mathbb{N}$, since $\left|\mathbf{R}_{\mathscr{A},\mathscr{L},\mathscr{D}}\right| \leq \left|\mathbf{T}_{\mathscr{A},\mathscr{L},\mathscr{D}}\right|$. □

$\mathbf{T}_{\mathscr{A},\mathscr{L},\mathscr{D}}$ can be reduced to $\mathbf{T}'_{\mathscr{A},\mathscr{L},\mathscr{D}}$ such that if $t' \in \mathbf{T}'_{\mathscr{A},\mathscr{L},\mathscr{D}}$, then $t'$ FM-signifies a unique $r \in \mathbb{R}$ on $\mathscr{D}$ in $\mathscr{L}$ on $\mathscr{A}$. To effect this reduction, we prove

**Lemma 5.** *There exists* $\mathbf{T}'_{\mathscr{A},\mathscr{L},\mathscr{D}} \subseteq \mathbf{T}_{\mathscr{A},\mathscr{L},\mathscr{D}}$ *such that* $\mathbf{T}'_{\mathscr{A},\mathscr{L},\mathscr{D}} \sim \mathbf{R}_{\mathscr{A},\mathscr{L},\mathscr{D}}$*.*

**Proof.** Omitting for brevity the subscripts $\mathscr{A}, \mathscr{L}, \mathscr{D}$, we observe that if $\mathbf{T} = \emptyset$, then $\mathbf{R} = \emptyset$. Let $|\mathbf{T}| = n \in \mathbb{Z}^+$ and let $t_{i_1}, \ldots, t_{i_n}$ be a a non-descending sorting of $\mathbf{T}$ by length so that $|t_{i_j}| \leq |t_{i_{j+1}}|$, $1 \leq j < n$. Let $t_{i_1} \leftarrow (=) \rightarrow r_{i_1}$ and $S_{r_{i_1}} = \{t_{i_1}\}$. Let $t_{i_2} \leftarrow (=) \rightarrow r_{i_2}$. If $r_{i_2} = r_{i_1}$, let $S_{r_{i_1}} = \{t_{i_1}, t_{i_2}\}$. If $r_{i_2} \neq r_{i_1}$, let $S_{r_{i_2}} = \{t_{i_2}\}$. When we reach $t_{i_j}$, $1 < j \leq n$, we have the sets $S_{r_1}$, …, $S_{r_k}$, $1 \leq k < j$, constructed thus far. Let $t_{i_j} \leftarrow (=) \rightarrow r_{i_j}$. If $r_{i_j} = r_{i_z}$, $1 \leq z < j$, $S_{r_{i_z}} = S_{r_{i_z}} \cup \{t_{i_j}\}$. If $r_{i_j} \neq r_{i_z}$, then $S_{r_{i_j}} = \{t_{i_j}\}$. We reduce every non-singleton set in $S_{r_{i_1}}$, …, $S_{r_{i_n}}$, if there are any, to a singleton by retaining a numeral shortest in length. Thus, $\mathbf{T}' = \cup_{j=1}^{k} S_{r_j}$ and $|\mathbf{T}'| = |\mathbf{R}|$. □

## 5. Signification of Data Types and Instances

**Definition 6.** *A data type* $\mathbb{T}$ *is signifiable or, equivalently, designatable in* $\mathscr{L}$ *on* $\mathscr{A}$ *if and only if there is* $t \in \mathscr{A}^+_{\mathscr{L}}$ *that signifies or, equivalently, designates in* $\mathscr{L}$ *on* $\mathscr{A}$ *only* $\mathbb{T}$ *and no other data type, or, in symbols,*

$$\mathbb{T} \Leftarrow \between_{\mathscr{A},\mathscr{L}} \Rightarrow t. \tag{21}$$

*If there is no such* $t$*, then* $\mathbb{T}$ *is not signifiable or, equivalently, not designatable in* $\mathscr{L}$ *on* $\mathscr{A}$*, or, in symbols,*

$$\mathbb{T} \quad \Leftarrow\between_{\mathscr{A},\mathscr{L}}\Rightarrow \quad \texttt{""}. \tag{22}$$

$\Leftarrow\between_{\mathscr{A},\mathscr{L}}\Rightarrow$ *is symmetric.*

**Definition 7.** *A data type* $\mathbb{T}$ *is FM-signifiable or, equivalently, FM-designatable in* $\mathscr{L}$ *on* $\mathscr{A}$ *if and only if there exists a FMD* $\mathscr{D}$ *such that there is a* $t \in \mathscr{A}^{+}_{\mathscr{L}}$*,* $|t| \leq \texttt{CCAP}(\mathscr{D})$*, that signifies or, equivalently, designates in* $\mathscr{L}$ *on* $\mathscr{A}$ *on* $\mathscr{D}$ *only* $\mathbb{T}$ *and no other data type, or, in symbols,*

$$\mathbb{T} \Leftarrow\between_{\mathscr{A},\mathscr{L},\mathscr{D}}\Rightarrow t. \tag{23}$$

*If there is no such* $t$*, then* $\mathbb{T}$ *is not signifiable or, equivalently, not designatable in* $\mathscr{L}$ *on* $\mathscr{A}$ *on* $\mathscr{D}$*, or, in symbols,*

$$\mathbb{T} \quad \Leftarrow\between_{\mathscr{A},\mathscr{L},\mathscr{D}}\Rightarrow \quad \texttt{""}. \tag{24}$$

*The relation* $\Leftarrow\between_{A}\Rightarrow$ *is symmetric.*

Let us extend the alphabet $\mathscr{A}$ in (1) to the alphabet $\mathscr{B}$ so that

$$\mathscr{B} \quad = \quad \mathscr{A} \cup \{\texttt{"}\Diamond\texttt{"}, \texttt{"}|_a\texttt{"}, \texttt{"}_a|\texttt{"}, \texttt{";"}, \texttt{"}\triangledown\texttt{"}, \texttt{"}\Box\texttt{"}, \texttt{"}\blacktriangleleft\texttt{"}, \texttt{"}\circledast\texttt{"}\}. \tag{25}$$

Let the array data type $\mathbb{A}$ be a set of objects, each of which is a finite sequence of discretely finite numbers, and let

$$\begin{array}{lll} \mathbb{A} & \Leftarrow\between_{\mathscr{B},\mathscr{L}}\Rightarrow & \texttt{"}\Diamond 1117\texttt{"}; \\ \mathbb{N} & \Leftarrow\between_{\mathscr{B},\mathscr{L}}\Rightarrow & \texttt{"}\Diamond 1123\texttt{"}; \\ \mathbb{Z} & \Leftarrow\between_{\mathscr{B},\mathscr{L}}\Rightarrow & \texttt{"}\Diamond 1129\texttt{"}; \\ \mathbb{Z}^{-} & \Leftarrow\between_{\mathscr{B},\mathscr{L}}\Rightarrow & \texttt{"}\Diamond 1151\texttt{"}; \\ \mathbb{Z}^{+} & \Leftarrow\between_{\mathscr{B},\mathscr{L}}\Rightarrow & \texttt{"}\Diamond 1153\texttt{"}; \\ \mathbb{R} & \Leftarrow\between_{\mathscr{B},\mathscr{L}}\Rightarrow & \texttt{"}\Diamond 1163\texttt{"}. \end{array} \tag{26}$$

If $\texttt{CCAP}(\mathscr{D}) \geq 5$, then

$$\begin{array}{lll} \mathbb{A} & \Leftarrow\between_{\mathscr{B},\mathscr{L},\mathscr{D}}\Rightarrow & \texttt{"}\Diamond 1117\texttt{"}; \\ \mathbb{N} & \Leftarrow\between_{\mathscr{B},\mathscr{L},\mathscr{D}}\Rightarrow & \texttt{"}\Diamond 1123\texttt{"}; \\ \mathbb{Z} & \Leftarrow\between_{\mathscr{B},\mathscr{L},\mathscr{D}}\Rightarrow & \texttt{"}\Diamond 1129\texttt{"}; \\ \mathbb{Z}^{-} & \Leftarrow\between_{\mathscr{B},\mathscr{L},\mathscr{D}}\Rightarrow & \texttt{"}\Diamond 1151\texttt{"}; \\ \mathbb{Z}^{+} & \Leftarrow\between_{\mathscr{B},\mathscr{L},\mathscr{D}}\Rightarrow & \texttt{"}\Diamond 1153\texttt{"}; \\ \mathbb{R} & \Leftarrow\between_{\mathscr{B},\mathscr{L},\mathscr{D}}\Rightarrow & \texttt{"}\Diamond 1163\texttt{"}. \end{array} \tag{27}$$

Formal symbolic systems utilize finite sets of syntactic rules to separate legal and illegal texts (cf., e.g., Kleene, 1952 [3], Ch. IV; Genesereth and Nilsson, 1987 [9], Ch. 2, Section 2.2). To that end, we can introduce the following rules to distinguish legal type significations from illegal ones in $\mathscr{L}$ on $\mathscr{B}$:

1. "◇1117", "◇1123", "◇1129", "◇1151", "◇1153", "◇1163" in (26) designate $\mathbb{A}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^{-}$, $\mathbb{Z}^{+}$, and $\mathbb{R}$, respectively;
2. "$|_a$" and "$_a|$" designate the beginning and end, respectively, of a finite sequence of discretely finite numbers, i.e., a tuple;
3. ";" designates a separator of numerals in signified tuples;
4. The signs formed by concatenating a finite number of the signs from { "1", "2", "3", "4", "5", "6", "7", "8", "9" } to the right of "▽" designate variables, e.g., "▽1", "▽2", "▽13", "▽171";

5. "□" designates a white space symbol; the addition of any number of "□" to the left or right of $t \in \mathscr{B}^{+}_{\mathscr{L}}$ does not change the object designated or referenced by $t$;
6. "⊛" designates the end of a statement;
7. If $t \in \mathscr{B}^{+}_{\mathscr{L}}$ consists of a variable name followed by " ◀ " followed by another sign followed by "⊛" with arbitrarily many "□" inserted left or right of each sign in $t$, then $t$ is interpreted to mean that the variable designated by the sign to the left of "◀" is assigned the value designated or referenced by the sign to the right of "◀".

E.g., "♢1117▽17 ⊛ ▽17 ◀ $|_a 1;2;3_a|$ ⊛ " consists of two statements: "♢1117▽17 ⊛ " and "▽17 ◀ $|_a 1;2;3_a|$ ⊛ ". The first statement is interpreted to mean that the variable designated by "▽17" is of the type designated by "♢1117", i.e., $\mathbb{A}$. The second statement is interpreted to mean that the variable designated by "▽17" is assigned the value designated by"$|_a 1;2;3_a|$", i.e., the 3-tuple $(1,2,3)$.

We assume from now on that any decimal sufficient alphabet $\mathscr{A}$ includes the unique signs $\trianglelefteq$, $\trianglerighteq$, $\mp$ designating the beginning of a tuple, the end of a tuple, and a number separator inside a tuple, respectively. E.g., in $\mathscr{B}$ in (25), $\trianglelefteq$ = "$|_a$", $\trianglerighteq$ = "$_a|$", and $\mp$ = ";". We call a formalism $\mathscr{L}$ on $\mathscr{A}$ that contains these signs *tuple sufficient* and, hereafter, assume that $\mathscr{L}$ on $\mathscr{A}$ is tuple sufficient. We call an alphabet $\mathscr{A}$ tuple sufficient if and only if there exists a tuple sufficient $\mathscr{L}$ on $\mathscr{A}$.

**Definition 8.** *Let $t_i \in \mathscr{A}^{+}_{\mathscr{B}}$, $1 \le i \le l$, $l > 1$. $t_i \ne \trianglelefteq$, $t_i \ne \trianglerighteq$, $t_i \ne \mp$. Then,*

$$\bigoplus_{\mathscr{A},\mathscr{L}}\left(t_1, \mp, t_2, \mp, \ldots, \mp, t_l\right)$$

*is the unmarked concatenation of $t_1, \ldots, t_l$ in $\mathscr{L}$ on $\mathscr{A}$ and*

$$\bigoplus_{\mathscr{A},\mathscr{L}}\left(\trianglelefteq, \bigoplus_{\mathscr{A},\mathscr{L}}\left(t_1, \mp, t_2, \mp, \ldots, \mp, t_l\right), \trianglerighteq\right)$$

*is the marked concatenation of $t_1, \ldots, t_l$ in $\mathscr{L}$ on $\mathscr{A}$.*

E.g., in $\mathscr{L}$ on $\mathscr{B}$, "`1;2;3`" is the unmarked concatenation of "`1`", "`2`", and "`3`", whereas "$|_a$1;2;3$_a|$" is the marked concatenation of "`1`", "`2`", and "`3`".

**Definition 9.** *A k-tuple $\mathfrak{r} = (r_1, \ldots, r_k) \in \mathbb{R}^k$, $1 < k \in \mathbb{Z}^+$, is signifiable or, equivalently, designatable in $\mathscr{L}$ on $\mathscr{A}$ if and only if (a) $\mathfrak{r}$ is of type $\mathbb{T}$ and $\mathbb{T} \Leftarrow\between_{\mathscr{A},\mathscr{L}}\Rightarrow t_{\mathbb{T}} \in \mathscr{A}^{+}_{\mathscr{L}}$; and (b) $r_i \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow t_i$, $t_i \in \mathscr{A}^{+}_{\mathscr{L}}$, $1 \le i \le k$. If (a) and (b) hold, then $\mathfrak{r}$ is signifiable or, equivalently, designatable in $\mathscr{L}$ on $\mathscr{A}$ by*

$$t_{\mathfrak{r}} = \bigoplus_{\mathscr{A},\mathscr{L}}\left(\trianglelefteq, \bigoplus_{\mathscr{A},\mathscr{L}}\left(t_1, \mp, t_2, \mp, \ldots, \mp, t_k\right), \trianglerighteq\right)$$

*or, in symbols, $\mathfrak{r} \leftarrow \doteqdot_{\mathscr{A},\mathscr{L}} \rightarrow t_{\mathfrak{r}}$. If $\mathfrak{r}$ is not signifiable or, equivalently, not designatable in $\mathscr{L}$ on $\mathscr{A}$, then $\mathfrak{r} \leftarrow \doteqdot_{\mathscr{A},\mathscr{L}} \rightarrow$ "". The relation $\leftarrow \doteqdot_{\mathscr{A},\mathscr{L}} \rightarrow$ is symmetric.*

**Definition 10.** *A k-tuple $\mathfrak{r} = (r_1, \ldots, r_k) \in \mathbb{R}^+$, $1 < k \in \mathbb{Z}^+$, is FM-signifiable or, equivalently, FM-designatable in $\mathscr{L}$ on $\mathscr{A}$ on a FMD $\mathscr{D}$ if and only if (a) $\mathfrak{r} \leftarrow \doteqdot_{\mathscr{A},\mathscr{L}} \rightarrow t_{\mathfrak{r}} \in \mathscr{A}^{+}_{\mathscr{L}}$ and (b) $|t_{\mathfrak{r}}| \le \texttt{CCAP}(\mathscr{D})$. If (a) and (b) hold, then $\mathfrak{r} \leftarrow \doteqdot_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow t_{\mathfrak{r}}$. If $\mathfrak{r}$ is not FM-signifiable or, equivalently, not FM-designatable in $\mathscr{L}$ on $\mathscr{A}$ on $\mathscr{D}$, then $\mathfrak{r} \leftarrow \doteqdot_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow$ "". The relation $\leftarrow \doteqdot_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow$ is symmetric.*

We prove

**Theorem 2.** *Let $\mathbb{A} \Leftarrow\between\Rightarrow_{\mathscr{A},\mathscr{L},\mathscr{D}} t_{\diamondsuit} \in \mathscr{A}^{+}_{\mathscr{L}}$ and let $\mathscr{D}$ be a FMD with $\texttt{CCAP}(\mathscr{D}) = k \in \mathbb{Z}^+$. Let*

$$\mathbf{S}_{\mathscr{A},\mathscr{L},\mathscr{D}} = \{(r_1, r_2, \ldots, r_l) | r_i \in \mathbf{R}_{\mathscr{A},\mathscr{L},\mathscr{D}}\}, 1 \le i \le l, l > 1.$$

*Let $\mathfrak{r} = (r_1, r_2, \ldots, r_l) \in \mathbf{S}_{\mathscr{A},\mathscr{L},\mathscr{D}}$ and $r_i \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow s_i \in \mathscr{A}^{+}_{\mathscr{L}}$, $1 \le i \le l$. Then,*

$$\mathfrak{r} \leftarrow\!\div_{\mathscr{A},\mathscr{L},\mathscr{D}}\!\rightarrow \bigoplus\nolimits_{\mathscr{A},\mathscr{L}}\Big(t_\diamondsuit, \trianglelefteq, s_1, \mp, s_2, \mp, \ldots, \mp, s_l, \trianglerighteq\Big)$$

*if and only if*

$$\Big|\bigoplus\nolimits_{\mathscr{A},\mathscr{L}}\Big(t_\diamondsuit, \trianglelefteq, s_1, \mp, s_2, \mp, \ldots, \mp, s_l, \mp, \trianglerighteq\Big)\Big| \leq k.$$

**Proof.** Let $\mathfrak{r} = (r_1, r_2, \ldots, r_l) \in \mathbf{S}_{\mathscr{A},\mathscr{L},\mathscr{D}}$ and $s = \oplus_{\mathscr{A},\mathscr{L}}\Big(t_\diamondsuit, \trianglelefteq, s_1, \mp, s_2, \mp, \ldots, \mp, s_l, \trianglerighteq\Big)$, where $r_i \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow s_i \in \mathscr{A}^+_{\mathscr{L}}$, $1 \leq i \leq l$. If $\mathfrak{r} \leftarrow\!\div_{\mathscr{A},\mathscr{L},\mathscr{D}}\!\rightarrow s$, then $|s| \leq \mathtt{CCAP}(\mathscr{D}) = k$. Conversely, if $|s| \leq k$, then $\mathfrak{r} \leftarrow\!\div_{\mathscr{A},\mathscr{L},\mathscr{D}}\!\rightarrow s$, because $r_i \leftarrow (=)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow s_i$. □

## 6. Signification of Standard Data Structures

The next theorem is a technical step toward extending signification to data structures representable as multidimensional matrices of discretely finite numbers.

**Theorem 3.** *Let $M_{nm}$ be an $n \times m$ matrix of discretely finite $r_{ij} \in \mathbb{R}$, $1 \leq i \leq n \in \mathbb{Z}^+$, $n > 1$, $1 \leq j \leq m \in \mathbb{Z}^+$, $m > 1$, such that $r_{ij} \leftarrow \{=\}_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow S_{r_{ij}} \neq \emptyset$, and let $\mathscr{D}$ be a FMD with $\mathtt{CCAP}(\mathscr{D}) = k \in \mathbb{Z}^+$. Then, $M_{nm}$ is signifiable on $\mathscr{D}$ in $\mathscr{L}$ on $\mathscr{A}$ if and only if*

$$\begin{array}{ll} \mathbb{A} \Leftarrow\!\between\!\Rightarrow_{\mathscr{A},\mathscr{L},\mathscr{D}} t_\diamondsuit \in \mathscr{A}^+_{\mathscr{L}} & \wedge \\ \mathfrak{r}_i = (r_{i1}, \ldots, r_{im}) \leftarrow \div_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow \mathfrak{t}_i \in \mathscr{A}^+_{\mathscr{L}} & \wedge \\ (\mathfrak{r}_1, \ldots, \ldots, \mathfrak{r}_n) \leftarrow \div_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow \mathfrak{t}_M \in \mathscr{A}^+_{\mathscr{L}}. & \end{array}$$

**Proof.** We let

$$M_{nm} = \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1m} \\ r_{21} & r_{22} & \ldots & r_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ r_{n1} & r_{n2} & \ldots & r_{nm} \end{bmatrix}$$

and omit the subscripts $\mathscr{A}, \mathscr{L}, \mathscr{D}$ for brevity. If the three components of the predicate in the statement of the theorem hold, then

$$\mathfrak{r}_i = (r_{i1}, \ldots, r_{im}) \leftarrow (\div) \rightarrow \mathfrak{t}_i = \bigoplus\Big(t_\diamondsuit, \trianglelefteq, s_{i1}, \mp, \ldots, \mp, s_{im}, \mp, \trianglerighteq\Big),$$

where $r_{ij} \leftarrow (=) \rightarrow s_{ij}$, and we have

$$M_{nm} \leftarrow (\div) \rightarrow \bigoplus\Big(t_\diamondsuit, \mathfrak{t}_1, \mp, \ldots, \mp, \mathfrak{t}_n\Big) = \mathfrak{t}_M \in \mathscr{A}^+_{\mathscr{L}}.$$

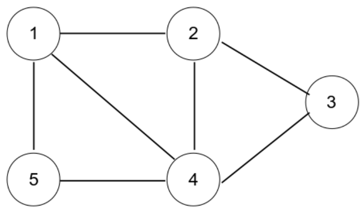
Conversely, if $M_{nm} \leftarrow (\div) \rightarrow_{\mathscr{A},\mathscr{L},\mathscr{D}}\!\rightarrow \mathfrak{t}_M$, then, by Definition (10) and Theorem (2), $\mathfrak{t}_M$ signifies the $n$-tuple of $m$-tuples of all numbers in $M_{nm}$. Consequently, $\mathfrak{t}_M$ must be the concatenation of $t_\diamondsuit$ with the concatenations, properly separated by $\mp$, of the texts signifying each individual $m$-tuple. □

An induction on $m$ furnishes us two corollaries.

**Corollary 1 of Theorem 3.** *Let $M_{d_1 d_2 \ldots d_m}$ be a $d_1 \times d_2 \times \ldots \times d_m$ matrix of discretely finite real numbers, $d_j \in \mathbb{Z}^+$, $1 \leq j \leq m$, $m \geq 2$. Then, $M_{d_1 d_2 \ldots d_m} \leftarrow \div_{\mathscr{A},\mathscr{L}} \rightarrow \mathfrak{t}_M \in \mathscr{A}^+_{\mathscr{L}}$.*

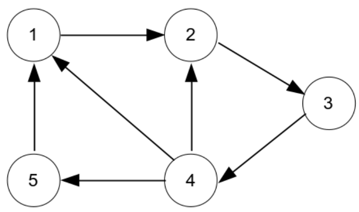
**Corollary 2 of Theorem 3.** *Let $M_{d_1 d_2 \ldots d_m}$ be a $d_1 \times d_2 \times \ldots \times d_m$ matrix of discretely finite real numbers, $d_j \in \mathbb{Z}^+$, $1 \leq j \leq m$, $m \geq 2$. Then, $M_{d_1 d_2 \ldots d_m} \leftarrow \div_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow \mathfrak{t}_M \in \mathscr{A}^+_{\mathscr{L}}$ if and only if $0 < |\mathfrak{t}_M| \leq k = \mathtt{CCAP}(\mathscr{D})$.*

E.g., consider the graph $G_1$ and $G_2$ and their respective matrix representations $M_1$ and $M_2$ in Figures 1 and 2, where, by Theorem (3) and its corollaries, $M_1 \leftarrow (\div)_{\mathscr{A},\mathscr{L}} \rightarrow \mathfrak{t}_{M_1} \in \mathscr{A}^+_{\mathscr{L}}$ and $M_2 \leftarrow (\div)_{\mathscr{A},\mathscr{L}} \rightarrow \mathfrak{t}_{M_1} \in \mathscr{A}^+_{\mathscr{L}}$. Furthermore, $M_1 \leftarrow (\div)_{\mathscr{D},\mathscr{A},\mathscr{L}} \rightarrow \mathfrak{t}_{M_1} \in \mathscr{A}^+_{\mathscr{L}}$ for a FMD $\mathscr{D}$ such that $|\mathfrak{t}_{M_1}| \leq \mathtt{CCAP}(\mathscr{D})$, and $M_2 \leftarrow (\div)_{\mathscr{A},\mathscr{L},\mathscr{D}} \rightarrow \mathfrak{t}_{M_2} \in \mathscr{A}^+_{\mathscr{L}}$, for a FMD $\mathscr{D}$ such that $|\mathfrak{t}_{M_2}| \leq \mathtt{CCAP}(\mathscr{D})$.




| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 1 |
| 2 | 1 | 0 | 1 | 1 | 0 |
| 3 | 0 | 1 | 0 | 1 | 0 |
| 4 | 1 | 1 | 1 | 0 | 1 |
| 5 | 1 | 0 | 0 | 1 | 0 |

**Figure 1.** (**Top**): An undirected unweighted graph $G_1$. (**Bottom**): matrix $M_1$ representing $G_1$ so that $M_1[i, j] = 1, 1 \leq i, j \leq 5$, if and only if $G_1$ has an edge between the nodes $i$ and $j$; otherwise, $M_1[i, j] = 0$.




| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 1 | 0 | 0 |
| 3 | 0 | 0 | 0 | 1 | 0 |
| 4 | 1 | 1 | 0 | 0 | 1 |
| 5 | 1 | 0 | 0 | 0 | 0 |

**Figure 2.** (**Top**): Directed unweighted graph $G_2$. (**Bottom**): a matrix $M_2$ representing $G_2$ so that $M_2[i, j] = 1, 1 \leq i, j \leq 5$ if and only if $G_2$ has a edge from the node $i$ to the node $j$; otherwise, $M_2[i, j] = 0$.

## 7. A Gödel Numbering of Texts

Let $\gamma_{\mathscr{A}} : \mathscr{A} \mapsto \mathbb{Z}^+$ be a 1–1 function that maps each sign of $\mathscr{A}$ to a unique odd prime number, i.e.,

$$\gamma_{\mathscr{A}}(s_i) \quad = \quad \pi(i+1), 1 \leq i \leq l, l > 0 \tag{28}$$

where $\pi(\cdot)$ is defined in Appendix A Definition (A1). E.g., for the alphabet $\mathscr{B}$ in (25), we have

$$\begin{array}{llllllllllll}
\gamma_{\mathscr{B}}(\texttt{"0"}) & = & 3; & \gamma_{\mathscr{B}}(\texttt{"1"}) & = & 5; & \gamma_{\mathscr{B}}(\texttt{"2"}) & = & 7; & \gamma_{\mathscr{B}}(\texttt{"3"}) & = & 11; \\
\gamma_{\mathscr{B}}(\texttt{"4"}) & = & 13; & \gamma_{\mathscr{B}}(\texttt{"5"}) & = & 17; & \gamma_{\mathscr{B}}(\texttt{"6"}) & = & 19; & \gamma_{\mathscr{B}}(\texttt{"7"}) & = & 23; \\
\gamma_{\mathscr{B}}(\texttt{"8"}) & = & 29; & \gamma_{\mathscr{B}}(\texttt{"9"}) & = & 31; & \gamma_{\mathscr{B}}(\texttt{"+"}) & = & 37; & \gamma_{\mathscr{B}}(\texttt{"-"}) & = & 41; \\
\gamma_{\mathscr{B}}(\texttt{"."}) & = & 43; & \gamma_{\mathscr{B}}(\texttt{"}\diamondsuit\texttt{"}) & = & 47; & \gamma_{\mathscr{B}}(\texttt{"}|_a\texttt{"}) & = & 53; & \gamma_{\mathscr{B}}(\texttt{"}_a|\texttt{"}) & = & 59; \\
\gamma_{\mathscr{B}}(\texttt{";"}) & = & 61; & \gamma_{\mathscr{B}}(\texttt{"}\triangledown\texttt{"}) & = & 67; & \gamma_{\mathscr{B}}(\texttt{"}\square\texttt{"}) & = & 71; & \gamma_{\mathscr{B}}(\texttt{"}\blacktriangleleft\texttt{"}) & = & 73; \\
\gamma_{\mathscr{B}}(\texttt{"}\circledast\texttt{"}) & = & 79. & & & & & & & & &
\end{array}$$

The 1–1 function $g_{\mathscr{A},\mathscr{L}} : \{\texttt{""}\} \cup \mathscr{A}^+_{\mathscr{L}} \mapsto \mathbb{N}$

$$g_{\mathscr{A},\mathscr{L}}(s) = \begin{cases} [\gamma_{\mathscr{A}}(s_1), \ldots, \gamma_{\mathscr{A}}(s_{|s|})]_{\Delta=1} & \text{if } s \in \mathscr{A}^+_{\mathscr{L}}, \\ 0 & \text{if } |s| = 0 \end{cases} \tag{29}$$

maps a text $s$ in $\mathscr{L}$ on $\mathscr{A}$ to a unique positive natural number through the shifted Gödel numbering (cf. Appendix A Definition (A3)) and maps the empty sign to 0. E.g., if $\mathscr{L}$ is tuple sufficient on $\mathscr{B}$ in (25), then

$$
\begin{array}{lcl}
g_{\mathscr{B},\mathscr{L}}(\texttt{"+12.7"}) & = & [37,5,7,43,23]_{\Delta=1}; \\
g_{\mathscr{B},\mathscr{L}}(\texttt{"03.134"}) & = & [3,11,43,5,11,13]_{\Delta=1}; \\
g_{\mathscr{B},\mathscr{L}}(\texttt{"-93.134"}) & = & [41,31,11,43,5,11,13]_{\Delta=1}; \\
g_{\mathscr{B},\mathscr{L}}(\texttt{"}|_a 3;5_a|\texttt{"}) & = & [53,11,61,17,59]_{\Delta=1}; \\
g_{\mathscr{B},\mathscr{L}}(\texttt{"}\diamondsuit|_a 3;5;7_a|\texttt{"}) & = & [47,53,11,61,17,61,23,59]_{\Delta=1}.
\end{array}
$$

Let $\mathscr{A}$ be an alphabet and

$$
\mathbf{B}_{\mathscr{A}} = \Big\{0\Big\} \cup \Big\{z \in \mathbb{Z}^+ \Big| \gamma_{\mathscr{A}}(s) = (z)_{\Delta=1,i}, s \in \mathscr{A}, 1 \leq i \leq Lt_{\Delta=1}(z)\Big\},
$$

where $(x)_{\Delta=j,i}$ and $Lt_{\Delta=j}(x)$ are defined in Appendix A Definitions (A5) and (A7), respectively. Then, the 1–1 function $g^{-1}_{\mathscr{A},\mathscr{L}} : \mathbf{B}_{\mathscr{A}} \mapsto \{\texttt{""}\} \cup \mathscr{A}^+_{\mathscr{L}}$ is defined as

$$
\begin{array}{lcl}
g^{-1}_{\mathscr{A},\mathscr{L}}(0) & = & \texttt{""}; \\
g^{-1}_{\mathscr{A},\mathscr{L}}(z) & = & \bigoplus_{\mathscr{A},\mathscr{L}}\Big(\gamma^{-1}_{\mathscr{A}}\Big((z)_{\Delta=1,1}\Big), \ldots, \gamma^{-1}_{\mathscr{A}}\Big((z)_{\Delta=1,Lt_{\Delta}(z)}\Big)\Big).
\end{array}
$$

E.g.,

$$
\begin{array}{lcl}
g^{-1}_{\mathscr{B},\mathscr{L}}([37,5,7,43,23]_{\Delta=1}) & = & \texttt{"+12.7"}; \\
g^{-1}_{\mathscr{B},\mathscr{L}}([3,11,43,5,11,13]_{\Delta=1}) & = & \texttt{"03.134"}; \\
g^{-1}_{\mathscr{B},\mathscr{L}}([41,31,11,43,5,11,13]_{\Delta=1}) & = & \texttt{"-93.134"}; \\
g^{-1}_{\mathscr{B},\mathscr{L}}([53,11,61,17,59]_{\Delta=1}) & = & \texttt{"}|_a 3;5_a|\texttt{"}; \\
g^{-1}_{\mathscr{B},\mathscr{L}}([47,53,11,61,17,61,23,59]_{\Delta=1}) & = & \texttt{"}\diamondsuit|_a 3;5;7_a|\texttt{"}.
\end{array}
$$

We prove

**Theorem 4.** *Let* $\mathbf{R}$ *be a set of discretely finite real numbers and let*

$$
\begin{array}{lcl}
\mathbf{T}_{\mathscr{A},\mathscr{L}} & = & \Big\{\mathfrak{t} \in \mathscr{A}^+_{\mathscr{L}} \Big| \mathfrak{t} \leftarrow (=)_{\mathscr{A},\mathscr{L}} \rightarrow r \in \mathbf{R}\Big\}; \\
\mathbf{S}_{\mathscr{A},\mathscr{L}} & = & \Big\{\mathfrak{s} \in \mathscr{A}^+_{\mathscr{L}} \Big| \mathfrak{s} \leftarrow (\doteqdot)_{\mathscr{A},\mathscr{L}} \rightarrow (r_1, r_2, \ldots, r_k), r_i \in \mathbf{R}, 1 \leq i \leq k, k > 1\Big\}; \\
\mathbf{M}_{\mathscr{A},\mathscr{L}} & = & \Big\{\mathfrak{m} \in \mathscr{A}^+_{\mathscr{L}} \Big| \mathfrak{m} \leftarrow (\doteqdot)_{\mathscr{A},\mathscr{L}} \rightarrow M_{d_1 \ldots d_m}, m > 1\Big\}; \\
\mathbf{D}_{\mathscr{A},\mathscr{L}} & = & \mathbf{T}_{\mathscr{A},\mathscr{L}} \cup \mathbf{S}_{\mathscr{A},\mathscr{L}} \cup \mathbf{M}_{\mathscr{A},\mathscr{L}}.
\end{array}
$$

*Then, there exists a 1–1 function* $\mathfrak{g}_{\mathscr{A},\mathscr{L}} : \{\texttt{""}\} \cup \mathbf{D}_{\mathscr{A},\mathscr{L}} \mapsto \mathbb{N}$ *that maps each text in* $\mathbf{D}_{\mathscr{A},\mathscr{L}}$ *to a unique positive integer and maps the empty sign to 0.*

**Proof.** If $t \in \mathscr{A}^+_{\mathscr{L}}$, then $t = \bigoplus_{\mathscr{A},\mathscr{L}}(s_1, \ldots, s_n)$, $s_i \in \mathscr{A}$, $1 \leq i \leq n$, and $|t| = n > 0$. Otherwise, $t = \texttt{""}$ and $|t| = 0$. Let

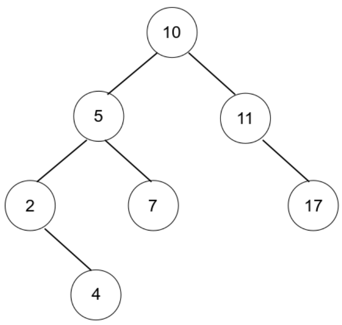
$$
\mathfrak{g}_{\mathscr{A},\mathscr{L}}(t) = \begin{cases} [\gamma_{\mathscr{A}}(s_1), \ldots, \gamma_{\mathscr{A}}(s_n)]_{\Delta=1} & \text{if } |t| > 0, \\ 0 & \text{if } |t| = 0. \end{cases} \tag{30}
$$

□

## 8. Discussion

If a real number can be written as a text on a sufficiently expressive alphabet, the individual elements of the number can be enumerated. E.g., the number's characteristic, the period that separates the characteristic from the mantissa in the standard decimal notation, and the individual elements of the mantissa can be enumerated by defining a 1–1 correspondence between these elements of the number and a finite subset of natural numbers. In this sense, some real numbers are discretely finite, while others are not. Discretely finite numbers can be signified completely by texts on sufficiently expressive alphabets in such a way that the signs of the designating texts completely coincide with the elements of the numbers they signify, whereas no such coincidence is possible with discretely infinite numbers, e.g., $\pi$, $e$, $\sqrt{2}$, on a sufficiently expressive alphabet. E.g., the texts "2.7182818284" and "3.1415926535" in the standard decimal notation either signify two concrete discretely finite numbers (i.e., the number 2.7182818284 and the number 3.1415926535) or, if they are elements of $S_e$ or $S_\pi$ (cf. Equation (14)), only reference the discretely infinite numbers $e$ and $\pi$, respectively. The concepts of signifiability and referenceability were introduced to distinguish designating and referencing texts. A consequence of the set of real numbers signifiable on a FMD on a sufficiently expressive alphabet being enumerably finite is that the quantity of real numbers so signifiable is a natural number. The set of real numbers referenceable on a FMD on a sufficiently expressive alphabet is also enumerably finite.

We extended the concepts of signifiability to real number tuples, data types, and instances and showed (cf. Theorem 2) that a discretely finite data structure representable as a tuple of discretely finite numbers is signifiable on a FMD on a sufficiently expressive alphabet so long the finite amount of memory of the FMD suffices to hold a designating text. Consequently, standard data structures such as lists, arrays, tuples, queues, stacks, hash tables, priority queues, and heaps (cf., e.g., Cormen et al. 1990 [10], Chapters 11–15, 19–22, 23) whose elements are discretely finite numbers are signifiable on a sufficiently expressive alphabet. Furthermore, only a finite quantity of these data structures are signifiable on a FMD on that alphabet. Theorem 3 shows that a discretely finite data structure representable as a multidimensional matrix of discretely finite numbers is signifiable on a sufficiently expressive alphabet and signifiable on a FMD on the same alphabet so long as the designating text fits into the FMD's memory units. Since any text can be uniquely mapped to a positive whole number, an instance of a data type representable as a text on some alphabet corresponds to a unique positive integer. E.g., since a tree data structure of discretely finite numbers is a graph (cf., e.g., a binary tree $T$ in Figure 3), it is also representable as a matrix. Hence, it is signifiable in a sufficiently expressive formalism and FM-signifiable on a FMD capable of holding the designating text in its finite memory. The same argument would hold if, instead of 0's and 1's, the matrices in Figure 3 had discretely finite real numbers in their cells indicating the weights of the corresponding edges or if the labels of the nodes were changed to discretely finite real numbers.




| | 2 | 4 | 5 | 7 | 10 | 11 | 17 |
|---|---|---|---|---|---|---|---|
| 2 | 0 | 1 | −1 | 0 | 0 | 0 | 0 |
| 4 | −2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 0 | 0 | 1 | −1 | 0 | 0 |
| 7 | 0 | 0 | −2 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 11 | 0 | 0 | 0 | 0 | −2 | 0 | 1 |
| 17 | 0 | 0 | 0 | 0 | 0 | −2 | 0 |

**Figure 3.** (**Top**): A binary tree $T$. (**Bottom**): A matrix representation $M_T$ of $T$. $M_T[i,j] = 1$ if and only if $i$ is the parent $j$; $M_T[i,j] = -1$ if $i$ is the left child of $j$; $M_T[i,j] = -2$ if $i$ is the right child of $j$; $M_T[i,j] = 0$ otherwise.

Our approach to types as data structures may be seen as partially based on the approach proposed by Church (1940) [11]. However, our aim is not a complete or partial integration of $\lambda$-calculus (or any other formalism) into the hierarchical theory of logical types offered by Russell (1908) [12], Whitehead and Russell (1910, 1912, 1913) [13–15] (cf. more recent summaries in, e.g., Kammareddine et al. (2002) [16] or Urquhart (2003) [17]). Rather, our aim in formulating a theory of signifiable computability is to characterize, in a formal way, what is computable in principle and what is computable with performable processes on computational devices with finite amounts of memory available for computation. In our previous investigation (Kulyukin, 2023 [18]), we formulated this aim as the separation of computability into two intersecting categories—general and actual. The former puts no limitations on memory, whereas the latter does.

In using logically quantified variables in our definitions, lemmas, and theorems, we make no ontological commitment that assertions containing variables carry with them the ontological commitment that the ranges of the variables exist (Church, 1958 [19]) or that designating texts, whenever they are quantified, can serve as senses of names (Church, 1993 [20]). The statements in this article or implications thereof should not be construed as arguments for or against replacing the Zermelo–Fraenkel set theory with the axiom of choice with the modern type theory as the foundation of mathematics (cf., e.g., Altenkirch, 2023 [21]). Alternative foundations of mathematics are beyond the scope of our article.

## 9. Conclusions

In Part I of our investigation, we defined the concepts of signification and reference of real numbers and showed that a discretely finite data structure representable as a tuple of discretely finite numbers is signifiable on a FMD on a sufficiently expressive alphabet. Such standard data structures of computer science as lists, arrays, tuples, queues, stacks, hash tables, priority queues, and heaps whose elements are discretely finite numbers are signifiable on any sufficiently expressive alphabet. However, only a finite quantity of these data structures are signifiable on the same alphabet on a FMD. In Part II of our investigation, which we intend to cover in our next article, we plan to use the results of this article to axiomatize some aspects of signifiable computability.

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## Abbreviations

| | |
|---|---|
| FM | finite memory |
| FMD | finite memory device |
| CCAP | cell capacity |

## Appendix A

Let $p_n$ be the $n$-th prime so that $p_0 = 0$, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. We define this function as

$$\pi(i) = p_i. \tag{A1}$$

Let $(n_1, ..., n_k)$ be an $k$-tuple such that $n_i \in \mathbb{N}, 1 \leq i \leq k$. The Gödel number (G-number) of this tuple is defined as

$$[n_1, ..., n_k] = \prod_{i=1}^{k} \pi(i)^{n_i}. \tag{A2}$$

Let $(z_1, ..., z_k)$ be a $k$-tuple such that $z_i \in \mathbb{Z}^+, 1 \leq i \leq k$. The $j$-shifted Gödel number $G_{\Delta=j}$, $j \in \mathbb{N}$, of this tuple is defined as

$$[z_1, ..., z_k]_{\Delta=j} = \prod_{i=1}^{k} \pi(i+j)^{z_i}. \tag{A3}$$

The G- and $G_{\Delta=j}$-numbers of the empty number sequence $()$ are defined to be 1. E.g., if the sequence is $(5, 103, 1009, 47, 49)$, then $[5, 103, 1009, 47, 49] = 2^5 3^{103} 5^{1009} 7^{47} 11^{49}$; $[5, 103, 1009, 47, 49]_{\Delta=0} = 2^5 3^{103} 5^{1009} 7^{47} 11^{49}$; $[5, 103, 1009, 47, 49]_{\Delta=1} = 3^5 5^{103} 7^{1009} 11^{47} 13^{49}$; $[5, 103, 1009, 47, 49]_{\Delta=2} = 5^5 7^{103} 11^{1009} 13^{47} 17^{49}$.

The accessor function $(x)_i = n_i$ in (A4) returns the $i$-th element of a G-number, i.e.,

$$(x)_i = \begin{cases} \min\limits_{t \leq x}\{\neg\{\pi(i)^{t+1} | x\}\} & \text{if } x > 0 \wedge i > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{A4}$$

E.g., if $x = [3, 5, 19] = 2^3 3^5 5^{19}$, then $(x)_1 = 3$; $(x)_2 = 5$; $(x)_3 = 19$; $(x)_i = 0$, if $i = 0 \vee i > 3$.

If $x$ is a $G_{\Delta=j}$, $j \in \mathbb{N}$, then the accessor function is defined as

$$(x)_{\Delta=j,i} = \begin{cases} \min\limits_{t \leq x}\{\neg\{\pi(i+j)^{t+1} | x\}\} & \text{if } x > 0 \wedge i > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{A5}$$

E.g., if $x = [3, 5, 19]_{\Delta=1} = 3^3 5^5 7^{19}$, then $(x)_{\Delta=1,1} = 3$; $(x)_{\Delta=1,2} = 5$; $(x)_{\Delta=1,3} = 19$; $(x)_{\Delta=1,i} = 0$, if $i = 0 \vee i > 3$.

All these functions are primitive recursive, because $\pi()$ is primitive recursive.

If $x = [n_1, n_2, ..., n_k]$, its length is the position of the last non-zero prime power in $x$ and is computed by the primitive recursive function $\texttt{Lt}(\cdot)$ in (A6).

$$\texttt{Lt}(x) = \min_{i \leq x}\left\{(x)_i \neq 0 \wedge (\forall l)_{\leq x}\left\{l \leq i \vee (x)_l = 0\right\}\right\} \tag{A6}$$

Thus,

$$\texttt{Lt}(540) = \texttt{Lt}([2, 3, 1]) = 3.$$

If $x = [z_1, z_2, ..., z_k]_{\Delta=j}$, $j \in \mathbb{N}$, then the length of $x$ is the position of the last non-zero power of an odd prime in $x$ and is computed by the primitive recursive function $\texttt{Lt}_{\Delta=j}(\cdot)$ in (A7).

$$\texttt{Lt}_{\Delta=j}(x) = \min_{i \leq x}\left\{(x)_{\Delta=j,i} \neq 0 \wedge (\forall l)_{\leq x}\left\{l \leq i \vee (x)_{\Delta=j,l} = 0\right\}\right\} \tag{A7}$$

All these functions are primitive recursive, because $\pi()$ and $[n_1, \ldots, n_k]$, $n_i \in \mathbb{N}$, $1 \leq i \leq k$, are primitive recursive and the bounded minimalization of a predicate (cf., e.g., Definition (A4) above) belongs to the same primitive recursively closed class and the class of primitive recursive functions is primitive recursively closed (cf., e.g., Davis et al., 1994 [2], Chapter 3).

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