On the linearization stability of the conformally (anti-) self-dual Einstein equations

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The Einstein equations with a cosmological constant, when restricted to Euclidean space-times with anti-self-dual Weyl tensor, can be replaced by a quadratic condition on the curvature of an $SU(2)$ (spin) connection. As has been shown elsewhere, when the cosmological constant is positive and the space-time is compact, the moduli space of gauge-inequivalent solutions to this equation is discrete, i.e., zero dimensional; when the cosmological constant is negative, the dimension of the moduli space is essentially controlled by the Atiyah–Singer index theorem provided the field equations are linearization stable. It is shown that linearization instability occurs whenever the unperturbed geometry possesses a Killing vector and/or a “harmonic Weyl spinor.” It is then proven that while there are no Killing vectors on compact conformally anti-self-dual Einstein spaces with a negative cosmological constant, it is possible to have harmonic Weyl spinors. Therefore, the conformally anti-self-dual Einstein equations on a compact Euclidean manifold are linearization stable when the cosmological constant is negative provided the unperturbed geometry admits no harmonic Weyl spinors.

I. INTRODUCTION

Recent work of Samuel¹ as well as Capovilla, Dell, and Jacobson² has shown that the Euclidean signature Einstein equations with a nonzero cosmological constant, when restricted to geometries with an anti-self-dual Weyl tensor, can be replaced by five quadratic conditions on the curvature of an $SU(2)$ spin connection on the space-time manifold M :

$$\frac{1}{4} F_{[ab} {}^{(AB} F_{cd]} {}^{CD)} = 0. \quad (1.1)$$

In the equation above, F_{ab} is the curvature of the left-handed spin connection; lower-case Latin indices are abstract space-time indices while capital Latin indices are abstract $SU(2)$ spinor indices that are raised and lowered with the antisymmetric spinor ϵ^{AB} and its inverse. As shown in Ref. 2, (1.1) is equivalent to the statement that

$$F_{ab} {}^{AB} = -\frac{1}{6} \lambda \Sigma_{ab} {}^{AB}, \quad (1.2)$$

where

$$\Sigma_{ab} {}^{AB} = 2\gamma_{[a} {}^{AA'} \gamma_{b]A'} {}^B, \quad (1.3)$$

and λ is the cosmological constant.³ In (1.3) γ_a is an $SU(2) \times SU(2)$ soldering form which defines the metric⁴ via

$$g_{ab} = \gamma_a {}^{AA'} \gamma_{bAA'}; \quad (1.4)$$

the metric in turn defines a Hodge duality operation with respect to which Σ_{ab} —and, from (1.2), F_{ab} —are self-dual. In a solution to (1.1), the curvature F_{ab} corresponds to the self-dual part of the Riemann tensor; (1.2) then implies that the Weyl tensor is anti-self-dual. Conversely, every conformally anti-self-dual Einstein space arises as a solution to (1.1).⁵

In Ref. 6 we began a study of the space of solutions to the conformally anti-self-dual Einstein equations by analyzing

the linearized version of (1.1), which is given by

$$D_1 C = F_{[ab} {}^{(AB} D_c C_d] {}^{CD)} = 0. \quad (1.5)$$

Here, C_a is the perturbation of the left-handed spin connection and D_a is the corresponding (unperturbed) derivative operator with curvature F_{ab} ; Eq. (1.5) is obtained only if the unperturbed curvature satisfies (1.1). The linearized equation (1.5) admits an infinite number of solutions that are generated by the action of the “gauge group” of general relativity which, in the formalism being used here, is a semidirect product of the diffeomorphism and local $SU(2)$ groups. Infinitesimal gauge transformations correspond to perturbations of the form (see Ref. 6 for details)

$$C_a = D_0(f, M, N) = (\nabla^b f) F_{ba} + [D^b M, F_{ba}] + D_a N, \quad (1.6)$$

where N and M are $\mathfrak{su}(2)$ -valued functions⁷ and f is a real-valued function. Here, and in what follows, space-time indices are lowered and raised by the metric associated with the unperturbed solution to (1.1). Notice that we are using an $\mathfrak{su}(2)$ matrix notation which suppresses spinor indices, e.g., the bracket in (1.6) is an $\mathfrak{su}(2)$ commutator. As all perturbations of the form (1.6) solve (1.5), one is naturally lead to study gauge-inequivalent solutions of the linearized equations; they are the equivalence classes

$$[C] = \text{kernel } D_1 / \text{image } D_0. \quad (1.7)$$

One of the central results of Ref. 6 was that these equivalence classes arise as the kernel of an elliptic operator D :

$$[C] = \text{kernel } D, \quad (1.8)$$

where

$$D = (D_1, D_0^*), \quad (1.9)$$

$$D_0^* C = (\text{tr } F^{ab} D_a C_b; [D_a C_b, F^{ab}]; -D^a C_a). \quad (1.10)$$

The operator in Eq. (1.10) corresponds to the L^2 adjoint of D_0 ; here and in what follows we extend the action of D_a to

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include tensor indices via the unique torsion-free derivative operator compatible with the metric obtained from the solution to (1.1). Because D is an elliptic operator on a compact manifold without boundary (we shall only work with such manifolds) it is clear that the gauge-inequivalent solutions to (1.5) form a finite-dimensional subspace of all possible perturbations. Furthermore, it was shown⁶ that when the cosmological constant is positive (1.5) admits only trivial (pure gauge) solutions; thus the moduli space of left-handed spin connections on conformally anti-self-dual Einstein spaces is discrete, i.e., zero dimensional, in this case. It was pointed out that the dimension of $[C]$ in the $\lambda < 0$ case could be determined via the Atiyah–Singer index theorem, however, the utility of this result for a determination of the dimension of the corresponding moduli space of gauge-inequivalent solutions to (1.1) depends on the linearization stability of (1.1), i.e., on whether every solution of (1.5) is an approximation to a solution of (1.1) modulo gauge transformations. As shown in Ref. 6, Eq. (1.1) is linearization stable if the adjoint of the operator D has a trivial kernel. Here, the adjoint is defined as

$$D^* = (D_1^*, D_0), \quad (1.11)$$

where D_1^* , which is the L^2 adjoint of D_1 , acts on totally symmetric, valence-four spinor-valued four-forms ω via

$$D_1^* \omega = F^{cd} D_{cd} D^b \omega_{abcd}{}^{ABCD}. \quad (1.12)$$

We have used the unperturbed equations, in particular (1.2), to simplify (1.12).

Our purpose in this paper is to analyze the issue of linearization stability by studying

$$\text{kernel } D^* = \text{kernel } D_0 \cup \text{kernel } D_1^*; \quad (1.13)$$

Eq. (1.1) is linearization stable if (1.13) is trivial. The existence of a kernel for D_0 corresponds to the existence of infinitesimal automorphisms of the $SU(2)$ bundle over M that leave the connection invariant, i.e., elements of kernel D_0 are gauge symmetries of the spin connection. The correlation of linearization stability and the absence of symmetries is familiar from the Cauchy problem in (Lorentzian) general relativity.⁸ However, unlike the situation which arises with the full Einstein equations, linearization stability of the restriction to the self-dual sector also involves the kernel of D_1^* , which turns out to correspond to “harmonic Weyl spinors” on M . We will treat each of these cases in the following two sections.

II. INFINITESIMAL GAUGE SYMMETRIES

Infinitesimal gauge symmetries, corresponding to elements of kernel D_0 , can also be identified with the kernel of the elliptic operator⁶

$$\Delta_0 := D_0^* D_0, \quad (2.1)$$

which leads to a coupled set of differential equations for f , M , and N . The results we desire can be obtained more easily if we follow a somewhat more indirect route, so we begin by studying gauge transformations of the form

$$C_a = M^b F_{ba} + D_a N, \quad (2.2)$$

where M^a is the vector field corresponding to an infinitesimal

mal diffeomorphism. As shown in Ref. 6, (1.6) is equivalent to (2.2) on conformally anti-self-dual Einstein spaces with a positive cosmological constant; when λ is negative (1.6) fails to be equivalent to (2.2) if M_a has a harmonic contribution to its Hodge decomposition. We will see below that this disparity is irrelevant for the characterization of infinitesimal symmetries.

Using (1.2) in (2.2), and then solving

$$-\lambda M^b \Sigma_{ba} + D_a N = 0 \quad (2.3)$$

for M_a , we find a necessary condition for the existence of an infinitesimal symmetry to be

$$M_a = -(1/\lambda) \text{tr } \Sigma_a{}^b D_b N = -(1/\lambda) \nabla_b (\text{tr } \Sigma_a{}^b N), \quad (2.4)$$

which, in particular, implies that M_a is coexact. From (2.4) we see that if M^a generates the diffeomorphism part of a gauge symmetry it can have no harmonic contribution to its Hodge decomposition, hence the set of all solutions to (2.3) is equivalent to kernel D_0 . Furthermore, (2.4) implies that the term involving the function f in (1.6) must vanish since this term comes from the exact part of M_a (Ref. 6).

Substituting (2.4) into (2.3) leads to a necessary and sufficient condition for N to generate the local $SU(2)$ part of a gauge symmetry:

$$d_0 N := D_a N + \frac{1}{4} [\Sigma_a{}^b, D_b N] = 0. \quad (2.5)$$

The L^2 adjoint of d_0 is given by

$$d_0^* C = -D^a C_a + \frac{1}{4} [D_a C_b, \Sigma^{ab}], \quad (2.6)$$

and we have

$$N \in \text{kernel } d_0 = \text{kernel } d_0^* d_0 \Leftrightarrow -D^a D_a N - \frac{2}{3} \lambda N = 0. \quad (2.7)$$

From (2.7) we see that the operator $d_0^* d_0$ is positive definite when the cosmological constant is negative, thus in this case the only solution to (2.7) is $N = 0$; this forces M^a to vanish also, so there are no infinitesimal symmetries of the spin connection on compact conformally anti-self-dual Einstein spaces with a negative cosmological constant.

We can obtain a more transparent geometrical interpretation of this result if we look for the necessary and sufficient restriction on the vector field M^a to yield an infinitesimal symmetry. Beginning again with (2.3), we now view it as an equation to be solved for N with M^a treated as given; we find

$$N = \frac{1}{4} \Sigma^{ab} \nabla_a M_b, \quad (2.8)$$

which implies that N is the spinor representation of the self-dual part of the exterior derivative of M_a . After substituting (2.8) into (2.3) we obtain

$$-\nabla^b \nabla_b M^a + \nabla^a \nabla_b M^b - \lambda M^a = 0. \quad (2.9)$$

By taking the divergence of both sides of (2.9) we deduce

$$\nabla_a M^a = 0, \quad (2.10)$$

which is consistent with (2.4). In the presence of (2.10), Eq. (2.9) becomes

$$-\nabla^b \nabla_b M^a - \lambda M^a = 0, \quad (2.11)$$

which, as before, has no solutions if $\lambda < 0$. It is easily verified that (2.10) and (2.11) imply M^a satisfies the Killing equations on the compact Einstein space (M, g) .

The results we have obtained in this section are analogous to the situation arising with compact two-dimensional (Riemann) surfaces. All such surfaces are Einstein spaces; genus-0, the sphere, has a positive cosmological constant, genus-1, the torus, has $\lambda = 0$, while all higher genus surfaces have a negative cosmological constant. The relevant symmetry group here is the group of conformal isometries; it is well-known that the sphere admits conformal isometries connected to the identity and, in addition, the moduli space of the sphere is trivial. For surfaces of genus greater than one, there are no infinitesimal conformal isometries, i.e., any conformal isometries are not in the connected component of the identity, while the moduli space for these surfaces is nontrivial.

III. HARMONIC WEYL SPINORS

We now turn to a study of the kernel of D_1^* , i.e., we analyze the solution space of

$$D_1^* \omega = F^{cd}{}_{CD} D^b \omega_{abcd}^{ABCD} = 0. \quad (3.1)$$

Keeping in mind that the unperturbed SU(2) curvature satisfies (1.2), Eq. (3.1) is equivalent to

$$D^b \omega_{ab} = 0, \quad (3.2)$$

where the su(2)-valued two-form ω_{ab} is defined by

$$\omega_{ab}^{AB} = \sum_{cdmn} \epsilon^{lmnp} \omega_{lmnp}^{ABCD}. \quad (3.3)$$

From (3.3) it is clear that ω_{ab} is self-dual, thus (3.2) can be reexpressed as

$$D_{[a} \omega_{bc]} = 0. \quad (3.4)$$

Notice that Eqs. (3.2) and (3.4) are precisely the Yang–Mills equations (in the self-dual sector) for SU(2) gauge theory. Similar equations also arise in the linearization stability analysis of the self-duality condition in Yang–Mills theory,⁹ however, there are two important flaws in the analogy between the gravitational and gauge theoretic treatments: (1) ω_{ab} is not quite an SU(2) curvature—it has the symmetries of the (self-dual) Weyl spinor; in particular, ω_{ab} has only five independent components while a self-dual SU(2) curvature has nine independent components; (2) the linearization stability analysis of the self-dual Yang–Mills equations actually yields Eqs. (3.2) and (3.4) as conditions on an *anti*-self-dual su(2)-valued two-form. Still, we shall now show that these two differences, so to speak, cancel each other, and we arrive at a vanishing theorem quite analogous to that arising in Yang–Mills theory.

We proceed using what is by now a familiar strategy. Denote by d_1^* the differential operator appearing in (3.2):

$$d_1^* \omega = 2D^b \omega_{ab}. \quad (3.5)$$

The L^2 adjoint of d_1^* is given by

$$d_1 C = (\delta_c^{[a} \delta_d^{b]} + \frac{1}{2} \epsilon_{cd}{}^{ab}) D_a C_b, \quad (3.6)$$

and we have

$$\text{kernel } d_1^* = \text{kernel } d_1 d_1^*. \quad (3.7)$$

Explicit computation, which makes use of the instanton equation (1.2) and the definition (3.3), reveals

$$\omega_{ab} \in \text{kernel } d_1 d_1^* \Leftrightarrow -D^c D_c \omega_{ab} + 2\lambda \omega_{ab} = 0. \quad (3.8)$$

Equation (3.8), when rewritten in terms of the totally symmetric spinor

$$\omega_{ABCD} = \epsilon^{abcd} \omega_{abcd}^{ABCD}, \quad (3.9)$$

is equivalent to

$$-D^c D_c \omega_{ABCD} + 2\lambda \omega_{ABCD} = 0; \quad (3.10)$$

thus Eqs. (3.2) and (3.3) are satisfied when the manifold admits harmonic Weyl spinors.¹⁰ The terminology “harmonic Weyl spinor” is meant to be suggestive of (3.10) but is not to be taken too literally: the self-dual Weyl spinor, which appears in the spinor decomposition of the Riemann tensor, is required to vanish on the space-times we are studying here. We call ω_{ABCD} a Weyl spinor only because it possesses all the algebraic symmetries of the Weyl (conformal) curvature spinor. Similarly, the term “harmonic” is not to be interpreted in the usual sense of Hodge–de Rham theory, but simply implies that ω_{ABCD} satisfies the most natural elliptic differential equation compatible with its algebraic symmetries.

When the cosmological constant is positive, the operator on the left-hand side of (3.10) is positive definite, thus there are no harmonic (self-dual) Weyl spinors on conformally anti-self-dual Einstein spaces with a positive cosmological constant. Unfortunately, there is no general obstruction to solutions of (3.10) when $\lambda < 0$, i.e., linearization stability is not guaranteed for $\lambda < 0$.

IV. LINEARIZATION STABILITY

For the convenience of the reader we will now assemble the results of the preceding sections. It is natural to classify the results by the sign of the cosmological constant.

A. $\lambda > 0$

Linearization stability is not really an issue here because there are no nontrivial solutions to the linearized equations. Nevertheless, we have found that $\lambda > 0$ is compatible with the existence of gauge symmetries of the connection; these correspond to the existence of Killing vectors on M . In addition, there are no harmonic Weyl spinors in this case.

B. $\lambda < 0$

In this case the situation is the reverse of the previous results. There are no infinitesimal symmetries of the connection; this corresponds to the absence of Killing vectors on compact Einstein spaces with a negative cosmological constant. On the other hand, there is no general obstruction to the existence of harmonic Weyl spinors when $\lambda < 0$; we conclude that the conformally anti-self-dual Einstein equations are linearization stable whenever there exist no Weyl spinors obeying the (eigenvalue) condition (3.10).

V. DISCUSSION

The linearization stability analysis of the conformally anti-self-dual Einstein equations is remarkably similar to the situation occurring in SU(2) Yang–Mills theory.⁹ The gravitational results *do*, however, differ from those of gauge theory owing to the presence of the diffeomorphism group as a symmetry group as well as the (related) fact that the gravitational SU(2) curvature perturbation is required to be *anti*-

self-dual even though the unperturbed $SU(2)$ curvature is self-dual.

As pointed out in Ref. 6, the relationship between the space of gauge symmetries, gauge inequivalent perturbations, and harmonic Weyl spinors is controlled by the Atiyah–Singer index theorem. If we denote the topological index as I , then we now know that

$$I = \dim \text{kernel } D_0; \quad \lambda > 0, \quad (5.1a)$$

$$I = \dim \text{kernel } D_1^* - \dim(\text{kernel } D_1 \cap \text{kernel } D_0^*); \lambda < 0, \quad (5.1b)$$

where the second term in (5.1b) is the dimension of the space of gauge-inequivalent perturbations satisfying (1.5). Clearly, it will be necessary to investigate the question of how “generic” is the existence of Weyl spinors satisfying the eigenvalue condition (3.10). The type of result which one might be able to obtain could be analogous to the fact that “If M is a connected, compact orientable Einstein space of scalar curvature 1, then M admits an eigenfunction f with $\Delta f = -nf$ if and only if M is isometric to $S^n(1)$ [the n -sphere of unit radius].”¹¹ In any event, it is clear that (5.1) may provide more than just a way to calculate the dimension of the moduli space of left-handed spin connections; indeed, the sign of I can represent a topological obstruction to the existence of a conformally anti-self-dual Einstein metric. For example, if the topology of M is such that $I < 0$, then (5.1) cannot be satisfied for any such metric with $\lambda > 0$. Alternatively, if the space-time admits no solutions to (3.10) and $I > 0$, then (5.1) cannot be satisfied when $\lambda < 0$. Another

interesting corollary to (5.1) is that the topological index controls the dimension (of the Lie algebra) of the isometry group when $\lambda > 0$. The topological implications of (5.1) will be explored in a future publication.

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¹ J. Samuel, *Class. Quantum Gravit.* **5**, L123 (1988).

² R. Capovilla, J. Dell, and T. Jacobson, *Class. Quantum Gravit.* **7**, L1 (1990).

³ We have chosen our conventions such that the cosmological constant is — 6 times that of Ref. 1. Our conventions are consistent with the Einstein equations $R_{ab} = \lambda g_{ab}$.

⁴ Because (1.1) is independent of the metric, it is possible to have solutions corresponding to a degenerate metric. Throughout this paper we only consider solutions of (1.1) which correspond to nondegenerate metrics.

⁵ S. Koshti and N. Dadhich, *Class. Quantum Gravit.* **7**, L5 (1990). Identical results for conformally self-dual space-times can be obtained by working with the right-handed spin connection.

⁶ C. G. Torre, *Phys. Rev. D*, **41**, 3620 (1990).

⁷ Notice that we notationally distinguish the group $SU(2)$ from its Lie algebra $\mathfrak{su}(2)$.

⁸ V. Moncrief, *J. Math. Phys.* **16**, 493 (1975).

⁹ M. Atiyah, N. Hitchin, and I. Singer, *Proc. R. Soc. London Ser. A* **362**, 425 (1978); A. Schwarz, *Commun. Math. Phys.* **64**, 233 (1979).

¹⁰ We could, of course, have arrived at (3.10) directly, i.e., without the use of the $\mathfrak{su}(2)$ -valued two-form. The introduction of this two-form was intended to strengthen the analogy with the results of the self-duality analysis of $SU(2)$ gauge theory.

¹¹ S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1969), Vol. II.