ESTIMATED INSTABILITY OF INTERNAL WAVES DUE TO TIME-DEPENDENT SHEAR

Leonardo A. Latorre and Julie C. Vanderhoff
Brigham Young University

1 ABSTRACT

The ocean and atmosphere are characterized mainly by stable stratification which sustains propagation of particular occurrences called internal gravity waves. These internal waves are generated as long as a perturbation to the stratification occurs at a frequency lower than the buoyancy (natural) frequency of the medium. These waves, once generated, propagate with wavelengths which can vary from a few meters to hundreds of meters in the vertical and thousands of meters in the horizontal. These waves propagate through the ocean and atmosphere interacting with other flow phenomena and eventually overturn and break, dissipating their energy. This energy dissipation affects circulation, heat transport, nutrient distribution and biological activity in the oceans and the atmosphere. The scales at which this energy transfer occurs are relatively small for overall oceanic models, hence the interest in finding the connection between the energy these waves dissipate and where the overall oceanic and atmospheric systems. We have focused our attention to interactions of an internal gravity wave with a time dependent shear flow in the form of a near-inertial wave, which is common in the ocean. These interactions are studied using ray theory, which is a linear analysis and fully non-linear numerical simulations. The aim of this analysis is to compare the instability estimates found from both methods and to define the accuracy of ray tracing in approximating these wave-wave interactions.

2 Introduction

Internal gravity waves propagate through the oceans carrying energy that eventually dissipates across the ocean. This dissipation of energy occurs through interactions of internal waves with other flow phenomena. One of these is the interaction of small-scale internal waves with steady shears. Another type of interaction is between internal waves with vortices. In this paper we are interested in smaller-scale internal waves approaching larger-scale, near-inertial waves. During all these processes, the smaller scale waves can eventually steepen and break, resulting in biological mixing, heat transport and pollutant distribution. These mixing characteristics are important because they are one of the mechanisms that dynamically distribute nutrients and balance the overall energy across the oceans. However, locations where large amounts of energy dissipation occur are not well defined, nor are the mechanisms leading to these occurrences. Vanderhoff, Nomura, Rottman and Macaskill (2008) [1] found that wave-wave interactions, where the small wave approaches the inertial wave at a group speed slower than the phase speed of the large wave, exhibit locations of large amplitude changes during the refraction through the time-dependent background. During small-scale interactions with near-inertial waves, when the small wave group speed \( c_g \) and the inertial wave phase speed \( C \) propagate in the same direction, three types of encounters are defined. These encounters are described in detail in Broutman and Young (1986) [2]: first kind encounters, where the small internal waves approach the
time-dependent shear with group velocities faster than the phase speed of the near-inertial waves. Usually these encounters will exit the background shear with a group speed similar to the approaching group speed. Therefore, in this case the small internal gravity wave does not exchange significant amounts of energy with the time-dependent shear flow. Second kind encounters, where the small internal waves approach the time-dependent shear with group velocities approximately equal to the phase speed of the large-scale wave, these encounters normally exit the interaction with a group speed very close to the speed the wave exhibited before the encounter. Thus as in first kind encounters, the small internal waves do not demonstrate significant changes of energy. Third kind encounters, where the small internal waves approach the time-dependent shear with group velocities smaller than the phase speed of the large-scale wave. In this scenario the small waves leave the interaction with a group speed faster than the incident group speed of the small wave. In this particular case the small wave reveals a notable change of energy as it is observed in Figure 2(d), where the amplitude decreases drastically after the wave leaves the interaction. Section 2 of this paper will talk about the models used to represent these wave-wave interactions. Section 3 will present some background on instabilities and how they will be parameterized in this research, as well as the results obtained for instability estimates using the non-linear models and ray methods. Section 4 will discuss the results and conclude with a summary and comments on further work for this study.

3 Setup

3.1 The Mathematical Model

Before defining the ray tracing model and the numerical model, it is important to understand the mathematical model that we are trying to simulate. In this study a packet of waves is used to limit both the small-scale wave and the near-inertial wave individually. We will consider a spatially localized wave packet that propagates freely until it encounters a time-dependent shear which is also spatially localized but only its phases are propagating as seen in Figure 1. Initially the small-wave packet and the near-inertial wave packet are spatially separated and a coordinate system (x, y, z) is selected. We assumed the buoyancy frequency \( N \) and the Coriolis frequency \( f \) (frequency due to local rotation rate of the earth) to be constants.

The waves in the inertial-wave packet propagate purely vertically with \( M = 2\pi/\lambda_c \) where \( \lambda_c \) is the vertical wavelength. The velocity field for this inertial wave is infinite in the horizontal direction, but it is bounded vertically by a Gaussian envelope described by (1):

\[
U + iV = U_0 e^{-z/L^2} e^{i(Mz - ft)}
\]

Where \( L \) is the length-scale of the envelope and \( U_0 \) is the mean velocity of the wave packet. Although the envelope of the inertial wave does not translate, the waves inside the envelope move with a phase speed defined as \( C = f/M \) as labeled in Figure 1.

The short wave packet propagates in the vertical and horizontal directions and the small-scale waves inside the packet exhibit wavenumbers \( k \) and \( m \) in the horizontal and vertical directions respectively. In our case \( k \) is assumed constant, and the wave dispersion relation looks like equation (2):

\[
\omega_f^2 = \frac{N^2 k^2 + f^2 m^2}{k^2 + m^2}
\]

where \( \omega_f \) is the intrinsic frequency of the small wave. From this dispersion relation one can define the vertical group
speed of the small waves as \( c_g = \frac{\partial \omega}{\partial m} \), where \( c_g \) is negative if \( m \) is positive and \( c_g \) is positive if \( m \) is negative.

The numerical simulations integrate the fully nonlinear equations of motion under the Boussinesq approximation and assuming incompressible fluid. These equations are initialized at \( t = 0 \) with a short-wave packet, separated spatially from an inertial wave packet, which exhibits an initial vertical displacement field of the form:

\[
\zeta(x, z, 0) = \zeta_0 e^{-\left[\frac{z - z_0}{2l}\right]^2} e^{i(kx + mz)}
\]  

(3)

The real part of this equation is used to describe the vertical displacement, where \( l \) and \( z_0 \) are constants and \( \zeta_0 \) is a complex constant. The ray tracing model solves the ray equations under the assumptions that the fluid is inviscid and incompressible. The Boussinesq approximation is employed and it is assumed that the characteristics of the wave are slowly varying and that the amplitude perturbation is small. The last two assumptions are used to linearize the problem.

### 3.2 Numerical Simulations

The numerical simulations will be our base of comparison to ray theory. The two-dimensional fully non-linear numerical model solves the equations of motion in their vorticity-stream function form, assuming the fluid is incompressible and employing the Boussinesq approximation, the equations are in the following form:

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = -q
\]  

(4)

\[
\frac{\partial q}{\partial t} + J(\psi, q) - \frac{\partial \sigma}{\partial x} - f \frac{\partial v}{\partial z} = 0
\]  

(5)

\[
\frac{\partial v}{\partial t} + J(\psi, v) + fu = 0
\]  

(6)

\[
\frac{\partial \sigma}{\partial t} + J(\psi, \sigma) - N^2 w = 0
\]  

(7)

where \( q \) is the y-component of vorticity and \( J(\psi, q) \) is the Jacobian with respect to \((x, z)\). In these equations the fluid velocity \( u = (u, 0, w) \), and the stream function is defined as follows:

\[
u = -\frac{\partial \psi}{\partial z}
\]  

(8)

\[
w = \frac{\partial \psi}{\partial x}
\]  

(9)

\[
q = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}
\]  

(10)

The variable \( \sigma = \frac{g \rho'}{\rho_0} \) is the term in the equations of motion that includes the influence of the density perturbations due to the propagating internal waves. Notice that \( \rho' \) is normalized by the gravitational term \( g \) and the mean density profile \( \rho_0(z) \), hence the total density of the field can be defined as \( \rho = \rho' + \rho_0 \).

The equations of motion (4), (5), (6) and (7) are solved using Runge-Kutta techniques inside a time stepping loop. The boundary conditions imposed on the solver are periodic in the \( x \) and \( z \) directions. The simulation’s grid is defined by 512 points in the vertical direction and the computational domain is specified as follows: The vertical limit of the domain is inputed into the initial conditions of the numerical solver, and it is chosen arbitrarily. The horizontal direction is specified by the length of the horizontal wavelength of the inertial wave. A similar model description can be found in [1].

### 3.3 Ray tracing model

Using ray-tracing models, one can approximate the propagation of the short wave as it encounters a time-dependent shear. In this setup, it is assumed that the inertial wave is unaffected by the short wave interaction and initially sufficient scale separation is fixed between the small wave and the larger inertial wave. The ray equations have the following form in this particular scenario:
\[
\frac{dz}{dt} = c_g, \quad \frac{dm}{dt} = -k \frac{\partial U}{\partial z}
\] (11)

These equations were used to output the ray tracing representations of the interactions included in this paper. One more set of equations named the Hayes equations, taken from [3], were used to calculate the steepness of the small wave as it propagates through the inertial wave packet. These equations are:

\[
\frac{d\mathcal{V}}{dt} = \nabla G_{mm} \frac{\partial m}{\partial z}
\] (12)

\[
\frac{d}{dt} \left( \frac{\partial z}{\partial z_0} \right) = G_{mm} \frac{\partial m}{\partial z_0}
\] (13)

\[
\frac{d}{dt} \left( \frac{\partial m}{\partial z_0} \right) = -G_{zz} \frac{\partial z}{\partial z_0}
\] (14)

where \( G(m,z,t) = \omega(z,t) \). The notation \( G_{ii} \) refers to the second partial derivative of \( G \) with respect to \( i \), where \( i \) could be \( m, z \) or \( t \). These equations relate the volume of the ray tubes (tubular surface made up of rays) \( \mathcal{V} \), to the wave action density. The volume of these ray tubes goes to zero at caustics and yields singularities to the wave action solutions. This subject is covered in more detail in the next section.

### 3.4 Caustics

Caustics are regions of strong refraction where rays intercept their paths and the volume of the ray tube approaches zero. When rays intercept each other, the slowly varying assumption, made for the ray tracing modeling breaks down generating infinite amplitudes that do not occur in the fully non-linear numerical simulations. In addition, since the wave action density is defined as \( A = 1/|\mathcal{V}| \), a volume of zero would yield singularities at these locations. Because the amplitude and the steepness are defined in terms of wave action density as seen in equation (15), this term is also affected by the singularities and behave similar to the wave action density at caustics. In the setup described in this paper, caustics occur. An approach employed by Broutman in (1986) [4] was used in this paper to estimate the correct wave action density at places where singularities occurred. The process corrects the wave action density at singularities by using the Airy function. This function shows remarkable similarities to the wave action density at singularities. A more detailed explanation of the estimation process is found in [4]. The authors considered the short wave steepness, defined as \( (\zeta_z = \partial \zeta / \partial z) \) as a measure of the vertical displacement of the short waves, and as a parameter to estimate overturning when it approaches unity.

For intrinsic frequencies much less than the buoyancy frequency and much greater than the Coriolis frequency the steepness is defined as

\[
|\zeta_z| = k \left( \frac{2}{\rho_0} \right)^{1/2} A^{1/2} \omega_I^{-1/2}
\] (15)

This equation is derived using the dispersion relation and the wave energy density \( E \) of the form

\[
E = \frac{1}{2} \rho_0 \zeta_0^2 N^2 \left[ 1 + \left( \frac{\mathcal{F}_m}{Nk} \right)^2 \right]
\] (16)

Where \( \rho_0 \) is the mean density of the fluid, \( N \) is the natural frequency and \( \zeta_0 \) is the initial vertical displacement.

The general process of the Airy function technique is to use information about the wave action density in the vicinity of the caustic to approximate the corrected wave action density at the caustic. Then a correction factor for \( A \) will have the following form:

\[
A_{\text{max}} \approx 1.8 \text{Ri}_c^{1/6} A
\] (17)

In this equation, \( \text{max} \) denotes the maximum corrected value near the caustic, \( \text{Ri}_c = N^2 / U_z^2 \) and the subscript \( c \) denotes the value of \( \text{Ri} \) at the caustic. The shear \( U_z \) from the inertial background, which is the partial derivative of the velocity \( U \) with respect to \( z \), is used to compute the
Richardson’s number at the caustic and $A_\star$ is the value of $A$, the wave action density, away from the caustic in the direction along the short wave ray of decreasing vertical group speed $c_g$. The reason to use the Airy function to correct $A$ near caustics is because at locations where $c_g = C$ this function represents the action density function very well right before the caustic and it does not diverge as the linear solution does. Once wave action density is corrected, the magnitudes for the amplitude and the steepness at caustics can also be corrected. Figures 2(c) and 2(d) shows the corrected amplitude for encounters of the first and third kind. Note that caustic regions in Figures 2(a) and 2(b) are characterized by locations where rays turn from a vertical direction to a horizontal direction or vice versa.

4 Results

There are two types of instabilities that are expected to occur while the small-scale internal waves propagate through an inertial gravity wave: convective instabilities and shear instabilities. Convective instabilities would occur when the propagating waves develop large steepness equal to 1 or larger. These large vertical displacements induce isopycnals, lines of constant density, to become vertical until an unstable density distribution leads the isopycnals to overturn and break (Thorpe, 2005) [5]. Shear instabilities would occur because the inertial wave packet acts as a time-dependent background shear. As small waves interact with the inertial wave they will be subjected to the straining imposed by the shear. This straining will translate into increments to the small wave amplitudes, which can lead the waves to statically unstable regions conducive to breaking. To parameterize the regions where the small waves exhibit shear instabilities or convective instabilities, a gradient Richardson’s number is defined as follows

$$\text{Richardson's number} = \frac{N^2 (1 - \frac{\partial \zeta}{\partial z})}{(\frac{\partial U}{\partial z})^2} < \frac{1}{4} \quad (18)$$

Where $\frac{\partial U}{\partial z} = \frac{\partial U}{\partial z} + (\frac{\partial U}{\partial t})(1/C)$. This non-dimensional parameter is subject to the inequality $< 1/4$ because this is the critical value at which the necessary condition for instability is encountered.

4.1 Instabilities in Numerical Simulations

A wave steepness plot for the numerical simulation of a small-scale wave, approaching an inertial wave packet from below and propagating slowly, relative to the inertial wave, is shown in Figure 3(b). For this simulation the initial wave steepness is $\zeta_e = 0.06$ and it never reaches a magnitude of 1. To the left, Figure 3(a), the corresponding background wave shear field is represented. The background shear does not appear in the steepness plot because there are no vertical displacements that occur to the background shear. In this plot the inertial wave is propagating downward, the length-scale of the inertial envelope is chosen to be 50 meters and the initial amplitude of the velocity shear $U_0 = 0.02$. The envelope is centered in the middle of the computational domain and the vertical wavelength chosen was 75 meters.

The small wave packet was centered at a distance far enough from the inertial packet as to prevent any interactions between the small wave and the inertial shear prematurely in the numerical simulation. The steepness plot is also a good representation of the small wave propagating through the shear but we want to explain some of the characteristics of this figure. The steepness plot presented in Figure 3(b) is obtained from taking the difference of the maximum steepness found at each time step as seen in Figure 4. This was performed to obtain a more comparable plot to the ray tracing outputs. This plot corresponds to the steepness evolution of a small wave experiencing a third kind encounter. Notice the spreading occurs as the wave propagates and interacts with the inertial wave. This spreading makes the analysis of the results problematic because it is difficult to determine at later times what part of the spreading is due to the propagation of the small wave and what fraction is due to the interaction with the large-scale inertial wave. During the interaction the steepness $\zeta_e$
Figure 2. (a) Raypath representing a first kind interaction with the inertial shear. The ellipses display regions where the condition $C = c_n$, also defined in this paper as caustics, occur. In (b) the raypath shows a second kind interaction. Subfigures (c) and (d) present the corrected values of the amplitude for the 1st and 3rd kind encounters. These corrections are performed at the corresponding times that caustic are observed in the ray paths.

Figure 3. In (a) the inertial shear that the small-scale wave encountered is presented. Subfigure (b) the steepness of the small wave in time never reaches a value of one or larger. This is important because when the steepness is greater than one, it is estimated that waves have overturned and breaking is very likely.

4.2 Instabilities in Ray Tracing models

In this section we will try to replicate the interaction presented earlier with the numerical simulation, but using ray tracing techniques. To do this we set the initial loca-
Figure 4. This is the maximum steepness calculated for each time slice from the steepness plot shown in Figure 3(b).

Figure 5. This is the ray tracing representation of the interaction found from the non-linear numerical simulations.

Figure 6 presents the steepness in time of one of the rays from ray tracing (b) compared to the maximum steepness found from numerical simulations (a). One notices that the overall decaying trends are comparable and some of the highs and lows are perceived in both plots. The peak at inertial period 2.1 in Figure 6(b) is similar to the peak shown at inertial period 1.6 in Figure 6(a). Also at inertial period 2.1 of Figure 6(a) one notices a relatively large dip, which relates to the strong refraction occurring in the steepness plot, shown in Figure 3(b), at the same inertial time. This low is also detected in the ray tracing results, but at an inertial period of 2.2, as seen in Figure 6(b). This dip connects to a strong refraction observed at the same inertial period in Figure 5. The peaks noted in Figure 6(b) at inertial periods 2.3 and 3 correspond to the inertial periods in Figure 5, delimiting the region where the ray is propagating vertically. These peaks are comparable to those spotted in Figure 6(a) at inertial periods 2.3 and 3. In Figure 6(a), other peaks are seen at inertial periods 3.4, 3.7, 3.9, 4.7 and 5, which are not present in Figure 6(b). This is because the particular ray chosen does not interact with the inertial wave at those inertial periods. At the present time a more quantitative comparison cannot be drawn due to the spreading detected in the numerical simulations, which is not exhibited in the ray models.

The information of the steepness and shear in time can now be used to define the non-dimensional gradient Richardson number. However, we chose to use the parameters from the gradient Richardson number to define an instability parameter $\theta$ such that

$$\theta = \arctan \left( \frac{\left( \frac{DU}{Dz} \right)^2}{N^2 \left( 1 - \frac{2\zeta}{\partial \zeta} \right)} \right)$$

At each time slice $\theta$ is calculated and the wave is classified into a stable state, a state of shear instability, or purely convective instability as seen in Figure 7.

Notice that the instability chart does not show the wave becoming unstable due to the shear, which is attributed to the small magnitude of the shear. Since the initial steepness is relatively small, the steepness ratio represented in Figure 6 never surpasses the critical value of $1/\zeta_{0}$, when convective instabilities would be expected in Figure 7.

5 Discussion

The main goal of this study is to demonstrate how well the linear model estimates the outcomes (i.e. instabilities
Figure 6. Figure 6(a) shows the maximum steepness at each time slice normalized by the initial steepness. Figure 6(b) is the normalized steepness of the small wave through time, obtained from the top ray in Figure 5.

Figure 7. This is the format of the instability chart. Note how it is divided into three sections. Below 45 degrees, it is a region of stable waves, from 45 to 90 degrees the waves experience shear instabilities and from 90 to 180 the instabilities are purely convective.

and wave breaking) of the interactions in regions that exhibit strong refraction. The comparison is needed since at these locations interactions are strongly non-linear and slowly varying assumptions break down. To do this we have used a 2-dimensional fully non-linear solver which is not subject to the limitations of ray theory and is a good base of comparison. To achieve the goal of this research a few things need to be analyzed further. First, as mentioned earlier the spreading observed in the numerical simulations has to be accounted for or reduced by choosing initial conditions that will allow us to capture the characteristics of the first two or three refractions before significant spreading occurs. Second, an instability chart needs to be developed for the results of the numerical simulations. Third, find a better method to represent the steepness through time from the ray tracing model. Fourth, determine an approach to
use the instability charts in the ray theory to classify each of the rays in time as stable, shear unstable or convectively unstable, to consequently estimate which waves would be expected to break.

REFERENCES