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# Thermodynamic Properties of Black Holes

Geoffrey Schulthess
*Utah State University*

Geoffrey Schulthess

Mentor: Dr. Rodriguez

# Thermodynamic Properties of Black Holes

Black Holes are some of the most mysterious objects in the known universe. In 1975, Stephen Hawking stated that Black Holes can behave as thermodynamical objects with a finite mass, spin, angular velocity, temperature, and entropy. This has been one of the most fascinating yet perplexing breakthroughs in our understanding of these strongly gravitating objects. In this context, the purpose of this research was to use Wolfram Mathematica to create a program that would calculate the thermodynamic properties of a black hole, given a certain metric.

To start the research, Dr. Rodriguez asked me to study the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2$$

in Lecture Notes on Black Holes (5) to understand what a metric is and what all the variables represent. I then used Wolfram Mathematica to take the limit of the $g_{tt}$ term at r = infinity to find the expression for the mass (as the subleading term in the expansion) of a Schwarzschild black hole:

$$g_{tt=} - 1 + \frac{2M}{r} + \ldots$$

When taking the limits at r = infinity on Wolfram Mathemtica, it was easiest to set r = 1/x and take the limit at x = 0. Once I had found the correct expression, I studied the Kerr metric,

$$ds^2 = -dt^2 + (\frac{\rho^2}{\Delta})dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)sin^2\theta d\varphi^2 + 2Mr\rho^2(a\, sin^2\theta d\varphi - dt)^2$$

$$\Delta(r) = r^2 - 2GMr + a^2 \qquad \rho^2(r,\theta) = r^2 + a^2 cos^2\theta$$

and used Wolfram Mathematica to make the same calculation to find the mass. A similar calculation was made for the $g_{rr}$, $g_{\theta\theta}$, $g_{\phi\phi}$ and $g_{t\varphi}$ terms. To find the event horizon, we set the Δ expression to zero and solved it for r. This gave us two expressions for r=rh, which shows that there are two event horizons for a Kerr black hole, but we are only interested in the largest root,

$$r_h = M + \sqrt{-a^2 + M^2}$$

where M is the mass and a is the angular momentum parameter. We can then compute the angular velocity ($\omega$), at the event horizon, which gave us,

$$\omega = -\, g_{t\varphi} / \, g_{\phi\phi}$$

$$\omega = \frac{M + \sqrt{-a^2 + M^2}}{2aM}$$

To solve for the area of a black hole, we used,

$$A = \iint d\theta \, d\varphi \sqrt{g\theta\theta * g\varphi\varphi}\Big|_{rh}$$

$$A = 8\pi \, M \, (M + \sqrt{-a^2 + M^2})$$

And finally, to find the temperature evaluated at the event horizon, we used

$$T = \frac{(N^2)^`}{4\pi\sqrt{grr * N^2}}$$

where,

$$N^2 = -gtt + g\varphi\varphi(\frac{gt\varphi}{g\varphi\varphi})^2$$

In conclusion, we were able to write a basic program in Wolfram Mathematica that correctly computed all the thermodynamic quantities of a static (Schwarzschild), and rotating, (Kerr), black holes. We would like in the future to enable the program to accept user input and find these thermodynamic properties for general black hole metrics.

Bibliography


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# Schwarzschild Metric

In[•]:=
```
metric = {{(1 - 2 / r) ^ (-1), 0, 0, 0}, {0, r ^ 2, 0, 0}, {0, 0,
    r ^ 2 Sin[θ] ^ 2, 0}, {0, 0, 0, -(1 - 2 / r)}}
```

Out[•]= $\left\{\left\{\frac{1}{1-\frac{2}{r}}, 0, 0, 0\right\}, \{0, r^2, 0, 0\}, \{0, 0, r^2 \operatorname{Sin}[\theta]^2, 0\}, \left\{0, 0, 0, -1+\frac{2}{r}\right\}\right\}$

# Kerr Metric

ds^2 = -dt^ 2 + (ρ^2 /Δ) dr^ 2 + ρ^2 dθ^2(r^2 + a^2 ) sin^2 dφ^2 + 2Mr ρ^2 (a sin^2 θ dφ - dt)^2

Δ(r) = r^2 - 2GMr + a^2

ρ^2 (r, θ) = r^2 + a^2 cos^2 θ

(* Wherer = 1/x *)

## Gtt - Gives us the Mass

In[•]:=
```
Gtt := (((2 * m * r) / (r ^ 2 + a ^ 2 * Cos[θ] ^ 2) - 1))
Series[(((2 * m * x ^ -1) / (x ^ -2 + a ^ 2 * Cos[θ] ^ 2) - 1)), {x, 0, 2}]
```

Out[•]= $-1 + 2\, m\, x + O[x]^3$

## Grr

In[•]:=
```
Grr := ((r ^ 2 + a ^ 2 * Cos[θ] ^ 2) / (r ^ 2 - 2 * m * r + a ^ 2))
Series[((x ^ -2 + a ^ 2 * Cos[θ] ^ 2) / (x ^ -2 - 2 * m / x + a ^ 2)), {x, 0, 2}]
```

Out[•]= $1 + 2\, m\, x + (-a^2 + 4\, m^2 + a^2 \operatorname{Cos}[\theta]^2)\, x^2 + O[x]^3$

## Gθθ

In[•]:=
```
Gthth = r ^ 2 + a ^ 2 * Cos[θ] ^ 2
```

Out[•]= $r^2 + a^2 \operatorname{Cos}[\theta]^2$

In[•]:=
```
Series[(x ^ -2 + a ^ 2 * Cos[θ] ^ 2), {x, 0, 5}]
```

Out[•]= $\frac{1}{x^2} + a^2 \operatorname{Cos}[\theta]^2 + O[x]^6$

## Gϕϕ

In[•]:=
```
Gpp = ((r ^ 2 + a ^ 2 + ((2 * m * r * a ^ 2 * Sin[θ] ^ 2) / (r ^ 2 + a ^ 2 * Cos[θ] ^ 2))) * Sin[θ] ^ 2)
Series[((x ^ -2 + a ^ 2 + ((2 * m * x ^ -1 * a ^ 2 * Sin[θ] ^ 2) / (x ^ -2 + a ^ 2 * Cos[θ]))) * Sin[θ] ^ 2), {x, 0, 2}]
```

Out[•]= $\operatorname{Sin}[\theta]^2 \left(a^2 + r^2 + \frac{2\, a^2\, m\, r \operatorname{Sin}[\theta]^2}{r^2 + a^2 \operatorname{Cos}[\theta]^2}\right)$

Out[•]= $\frac{\operatorname{Sin}[\theta]^2}{x^2} + a^2 \operatorname{Sin}[\theta]^2 + 2\, a^2\, m \operatorname{Sin}[\theta]^4\, x + O[x]^3$

## Gt$\phi$

In[•]:=
```
Gtphi = (-2*m*r*a*Sin[θ]^2)/(r^2 + a^2*Cos[θ]^2)
```

Out[•]= $-\frac{2\,a\,m\,r\,\mathrm{Sin}[\theta]^2}{r^2 + a^2\,\mathrm{Cos}[\theta]^2}$

In[•]:=
```
Series[((-4*(x^-1)*m*a*Sin[θ]^2)/(x^-2 + a^2*Cos[θ]^2)), {x, 0, 2}]
```

Out[•]= $-4\left(a\,m\,\mathrm{Sin}[\theta]^2\right)x + O[x]^3$

## Event Horizon

In[•]:=
```
(* Where delta=0 *)
EvHorz = Solve[r^2 - 2*m*r + a^2 == 0, r]
```

Out[•]= $\left\{\left\{r \to m - \sqrt{-a^2 + m^2}\right\}, \left\{r \to m + \sqrt{-a^2 + m^2}\right\}\right\}$

In[•]:=
```
(r /. EvHorz[[1]]) (r /. EvHorz[[2]]) // Simplify
```

Out[•]= $a^2$

## Omega (Angular Velocity) at Event Horizon

In[•]:=
```
Rh := m + Sqrt[m^2 - a^2]

Gtphi2 := -((2*r*m*a*Sin[θ]^2)/(r^2 + a^2*Cos[θ]^2))
```

In[•]:=
```
Gpp := ((r^2 + a^2 + ((2*m*r*a^2*Sin[θ]^2) / (r^2 + a^2*Cos[θ]^2))) * Sin[θ]^2)
```

In[•]:=
```
Omeg := -Gtphi / Gpp
```

In[•]:=
```
FullSimplify[Omeg /. r -> Rh]
```

Out[•]= $\frac{m - \sqrt{-a^2 + m^2}}{2\,a\,m}$

## Area

In[•]:= `cons = FullSimplify[Sqrt[Gthth * Gpp] /. r → Rh /. (Sqrt[-a^2 + m^2]) → (Rh - m) /. (-a^2 + 2*m*Rh) → Rh^2, {π/2 > θ > 0}]`

Out[19]= $$2\sqrt{m^2\left(-a^2+2m\left(m+\sqrt{-a^2+m^2}\right)\right)}\,\mathrm{Sin}[\theta]$$

In[•]:= `FullSimplify[Integrate[cons /. (Sqrt[-a^2 + m^2]) → (Rh - m) /. (-a^2 + 2*m*Rh) → Rh^2, {θ, 0, π}] * 2π]`

Out[20]= $$8\sqrt{m^2\left(m+\sqrt{-a^2+m^2}\right)^2}\,\pi$$

## Temperature

In[•]:= `n2 = -Gtt + (Gtphi2^2/Gpp) // FullSimplify`

Out[21]= $$\frac{\left(a^2+r\,(-2m+r)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)}{2\left(a^2+r^2\right)\left(r^2+a^2\mathrm{Cos}[\theta]^2\right)+4a^2mr\,\mathrm{Sin}[\theta]^2}$$

In[22]:= `dn2 = D[n2, r]`

Out[22]= $$-\frac{\left(a^2+r\,(-2m+r)\right)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)\left(4r\left(a^2+r^2\right)+4r\left(r^2+a^2\mathrm{Cos}[\theta]^2\right)+4a^2m\,\mathrm{Sin}[\theta]^2\right)}{\left(2\left(a^2+r^2\right)\left(r^2+a^2\mathrm{Cos}[\theta]^2\right)+4a^2mr\,\mathrm{Sin}[\theta]^2\right)^2}+\frac{4r\left(a^2+r\,(-2m+r)\right)}{2\left(a^2+r^2\right)\left(r^2+a^2\mathrm{Cos}[\theta]^2\right)+4a^2mr\,\mathrm{Sin}[\theta]^2}+\frac{(-2m+2r)\left(a^2+2r^2+a^2\mathrm{Cos}[2\theta]\right)}{2\left(a^2+r^2\right)\left(r^2+a^2\mathrm{Cos}[\theta]^2\right)+4a^2mr\,\mathrm{Sin}[\theta]^2}$$

In[38]:= `top = FullSimplify[(Series[dn2, {r, Rh, 0}] // FullSimplify) /. a^2 → m^2 - (Rh - m)^2, Rh > m > 0]`

Out[38]= $$\frac{\sqrt{-a^2+m^2}\left(4m\left(m+\sqrt{-a^2+m^2}\right)-2a^2\mathrm{Sin}[\theta]^2\right)}{4m^2\left(-a^2+2m\left(m+\sqrt{-a^2+m^2}\right)\right)}+O\left[r-m-\sqrt{-a^2+m^2}\right]^1$$

In[39]:= `bottom = FullSimplify[(Sqrt[(4π * Series[Grr * n2, {r, Rh, 0}])] // FullSimplify) /. a^2 → m^2 - (Rh - m)^2, {2*m + Rh + (2*m - Rh)*Cos(2*θ) > 0, Rh > m > 0}]`

Out[39]= $$\frac{\sqrt{\pi}\sqrt{2m\left(m+\sqrt{-a^2+m^2}\right)+2m\left(m-\sqrt{-a^2+m^2}\right)\mathrm{Cos}[\theta]^4-a^2\mathrm{Sin}[\theta]^4}}{m}+O\left[r-m-\sqrt{-a^2+m^2}\right]^1$$

In[41]:= `temp = Normal[top / bottom] // FullSimplify`

Out[41]= $$\frac{\sqrt{-a^2+m^2}\left(4m\left(m+\sqrt{-a^2+m^2}\right)-2a^2\mathrm{Sin}[\theta]^2\right)}{4m\left(-a^2+2m\left(m+\sqrt{-a^2+m^2}\right)\right)\sqrt{\pi}\sqrt{2m\left(m+\sqrt{-a^2+m^2}\right)+2m\left(m-\sqrt{-a^2+m^2}\right)\mathrm{Cos}[\theta]^4-a^2\mathrm{Sin}[\theta]^4}}$$