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Physics 4900

David Maughan
Utah State University

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1 Introduction

More than a century has passed since Albert Einstein published his general theory of relativity. The theory has been tested many times experimentally, primarily in the relatively weak gravitational fields of the solar system [1,2]. More recently the first experimental results from the strong gravitational fields of two black holes have been measured in the form of gravitational waves, which are another prediction of general relativity. The 2017 Nobel prize in physics was awarded to Kip Thorne, Rainer Weiss, and Barry Barish for their role in the detection of gravitational waves. This year we have seen the first image of a black hole from a team of over 200 scientists, further confirming the predictions of general relativity.

General relativity is one of the pillars of modern physics and explains how mass and energy curve spacetime (a Lorentzian manifold), and how the curvature of spacetime affects matter. As presented in 1915, general relativity is a system of ten nonlinear coupled partial differential equations. This means that it is seldom possible to find analytic solutions to his equations. In fact, Einstein did not believe that an exact solution would ever be found to his theory. Over the course of the last century, numerous exact solutions have been found. The technique used in the derivation of these solutions is to assume certain symmetries and other properties of a solution. These symmetries greatly reduce the complexity of the system and allow for a metric (the solution) to be calculated.

In this manuscript we examine a set of symmetries of a spacetime known as affine collineations, which are transformations of the spacetime that preserve the connection. We show how these symmetries give rise to conserved quantities in the geodesic equation. In the case that the metric possesses certain symmetries such that the metric is ”homogeneous”, we explicitly calculate these symmetries and quantities. By homogeneous, we mean that every point of space has the same geometrical properties as every other point. (Two examples of homogeneous manifolds would be the infinite plane $\mathbb{R}^2$ or the surface of a smooth sphere $\mathbb{S}^2$). We finish with a discussion on potential applications of affine collineations in astrophysics and numerical relativity.

2 Affine Collineations

In differential geometry, there is a coordinate free method for calculating the change in a tensor field along the flow defined by a vector field. This method is known as the Lie derivative. It is a fundamental technique to discover the symmetries of a manifold. It can be thought of as a generalization of the directional derivative.

Consider a metric tensor $g_{\alpha\beta}$ defined on a Lorentian Manifold M. Let $\pounds_x$ denote the Lie derivative along a vector field x. If

$$\pounds_v g_{\alpha\beta} = 0$$

then we say that the vector field v is an isometry (or Killing Vector) of M. Since $g_{\alpha\beta}$ defines the measure of distance and angles, we see that v is a transformation that preserves distance and angles.

We could also have a transformation u such that distances are rescaled, but the measure of angles remains the same. Such a transformation as u is known as homothety, and they exist when u satisfies the following equation.

$$\pounds_u g_{\alpha\beta} = c g_{\alpha\beta}$$

Note that when $c = 0$ the homothety is also an isometry.

From the metric we can build the Christoffel symbols, which is a metric connection. The Christoffel symbols will be useful in many practical calculations, including building the Riemann curvature tensor, calculating covariant derivatives, and building the geodesic equations. We can write the Christoffel symbols entirely in terms of the metric tensor as follows,

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha})$$

where a comma denotes a partial derivative with respect to the index that follows.

From the above equation, it is easy to see that any isometry will also preserve the metric connection. However, there may also be other symmetries that preserve the connection. Any vector field y satisfying the following equation is known as an affine collineation.

$$\pounds_y \Gamma^{\gamma}_{\alpha\beta} = 0$$

If the Lie derivative of the metric along an affine collineation is not a real constant times the metric (i.e a homothety) then we call it a proper affine collineation.

3 The Geodesic Equation

In general relativity, the idea of a straight line in flat space is generalized to a geodesic on a curved manifold. One reason geodesics are of such interest is that free falling

particles move along geodesics. The equation of motion for a particle acting under no force is given by the geodesic equation, which is written as follows

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

where the x's are coordinates and τ denotes proper time.

Since β indexes over four coordinates, we see that the geodesic equation is a set of four coupled nonlinear ordinary differential equations. Due to their nonlinear nature, it is often impossible to write down an analytic solution. As we shall see, affine collineations can help us in the integration of the geodesic equation, since they define first integrals (or simply, conserved quantities) of the equation.

It is not difficult to see that since the connection appears in the equation, that affine collineations define symmetries of the geodesic equation. In fact, affine collineations map geodesics to geodesics which is where the word collineation comes from. Two points which are collinear before the transformation (on the same line, or in our case geodesic) are collinear after the transformation. It is important that we now state a very important theorem in dynamical systems and calculus of variations known as Noether's Theorem.

Fact: (Noether's Theorem) Every differentiable symmetry of the action of a physical system has a corresponding conservation law.

The action is the time integral of the Lagrangian. Since we can derive the geodesic equation from a Lagrangian using calculus of variations, there is an action corresponding to geodesic motion. Since we have these affine symmetries, by Noether's Theorem we expect that there are corresponding conservation laws.

In the literature, it was stated that affine symmetries give rise to "non-Noetherian" symmetries of geodesic motion. However, the viewpoint the authors took was too narrow since they only made use of point symmetries (which works for homotheties), but did not allow for generalized symmetries (which are needed for proper affine symmetries). In [3] we have given a proof that these affine symmetries correspond to conserved quantities by a direct application of Noether's Theorem.

4 Maple

Calculations in general relativity can be long and tedious. Errors during calculations can be unfortunately common, even in the published literature. To combat these errors, improve the quality of the research, and speed up calculations, a package named *DifferentialGeometry* was created by researchers at Utah State University. This package is included in every copy of Maple, a computer algebra software, and includes the USU Library of Solutions to the Einstein Equations. This library contains nearly all known solutions to the Einstein Equations, almost 1000, and includes metadata on many of these solutions, such as the number of isometries and homotheties.

Inside the library is a set of metrics describing all possible homogenous solutions to the Einstein Equations. It is on this set of spacetimes that we calculated all affine collineations and their conservation laws.

Below is a worksheet showing the simplicity of the *DifferentialGeometry* commands that were used to find the affine collineations.

```
g,d := eval(op(Retrieve("Stephani", 1, [12, 26, 1], manifoldname = P, output = ["Metric","IsometryDimension"])),_a=a);
```

$$g, d := -a^2\, dt \otimes dt - a^2 e^x\, dt \otimes dz + a^2\, dx \otimes dx + a^2\, dy \otimes dy - a^2 e^x\, dz \otimes dt - \frac{a^2 e^{2x}}{2}\, dz \otimes dz, 5 \tag{1}$$

```
> C:=Christoffel(g):
> K:=InfinitesimalSymmetriesOfGeometricObjectFields(C, output = "list");
```

$$K := \left[-2 e^{-x} \partial_t + z \partial_x - \left(\frac{z^2}{2} - e^{-2x} \right) \partial_z, -\partial_x + z \partial_z, \partial_z, y \partial_y, \partial_y, \partial_t \right] \tag{2}$$

```
> nops(K);
```

$$6 \tag{3}$$

```
> h:=LieDerivative(K, g);
```

$$h := \left[0\, dt \otimes dt, 0\, dt \otimes dt, 0\, dt \otimes dt, 2 a^2\, dy \otimes dy, 0\, dt \otimes dt, 0\, dt \otimes dt \right] \tag{4}$$

```
> GetComponents(h[1],[g]);
```

$$[0] \tag{5}$$

```
> GetComponents(h[2], [g]);
```

$$[0] \tag{6}$$

```
> GetComponents(h[3], [g]);
```

$$[0] \tag{7}$$

```
> GetComponents(h[4], [g]);
```

$$[\ ] \tag{8}$$

```
> GetComponents(h[5], [g]);
```

$$[0] \tag{9}$$

```
> GetComponents(h[6], [g]);
```

$$[0] \tag{10}$$

In the first line we import the metric (in the case the Godel spacetime) from the USU Library of Solutions to the Einstein Equations. In the second line we built the Christoffel symbols. This command usually executes almost instantaneously. However, to do the calculation by hand requires solving forty different scalar values and 120 derivatives. The computer does not make errors either, which greatly increases the value of the *DifferentialGeometry* software. In line three we calculate all of the affine symmetries in the spacetime. In this case we found six, which is what we are verifying in the fourth line. In the remaining part of the worksheet, we are verify which symmetries are isometries, homotheties, and proper affine collineations. We see that the symmetry stored in $h[4]$ is a proper affine symmetry, while the others are all isometries. One can note that they are isometries by the zero value, since the Lie derivative of the metric along an isometry is zero.

5 Applications and Utility

Now that we have calculated these symmetries, it is natural to wonder what utility affine collineations have in our study of general relativity. Initially, we pursued these calculations as a means to classify solutions to the Einstein Equations. With almost 1000 known solutions, each derived in its own coordinate system, it is difficult to know whether two solutions represent the same physical spacetime, but are represented in different coordinates. However, two spacetimes with differing numbers of affine collineations cannot be the same. Thus, affine collineations aid us in knowing when two solutions are actually the same.

Another application of affine collineations lies with the conserved quantities to which affine collineations give rise. For the homogeneous spacetimes (of which we examined all) they have between 4 and 7 affine collineations, which gives us between 4 and 7 conserved quantities. If one can find 4 quantities that are conserved along geodesics, have vanishing Poisson brackets, and have differentials with a nonzero wedge product, then affine collineations guarantee that the geodesic equation is in-

tegrable. This is a very special property because it means that it is possible to write down an analytic solution to the geodesic equation (at least up to quadrature).

Affine collineations could also play a role in simulations of astrophysical systems. In numerical analysis it is well known that symplectic integrators bound the error on conserved quantities (with the technical detail that the conserved quantity does not depend on the angle variables in action-angle coordinates). In numerical relativity, test particles are released during the evolution of binary black hole systems to measure the location of the event horizon. Thus affine collineations are also of use in numerical analysis and numerical relativity. If during a long term simulation of an astrophysical system changes are observed in the conserved quantities, then there is a bug in the code.

Those familiar with numerical analysis may complain that symplectic methods are implicit and slow. In recent years a new explicit method for symplectic integration has been developed [4]. I implemented this new technique in Python in an attempt to recover the results of the paper (and hoping to find a use for this technique in integrating the geodesic equation numerically). My results and a quick introduction to symplectic integrators is in the appendix.

The next step in my research would be to find a spacetime in which the geodesic equations are integrable and demonstrate the utility of the new explicit symplectic method in numerical relativity. I carefully performed these calculations for the Godel spacetime. However, it is not possible using only affine collineations to guarantee the integrability of geodesic equation. It is possible this could be achieved using the Killing Tensor, but my current goal is to demonstrate applications of affine collineations.

It is certainly possible that there are other uses of affine collineations in general relativity. However, the geodesic equation is defined on manifolds with a metric that have nothing to do with relativity, so there could also be applications outside of general relativity, or even outside of physics.

6 Conclusions

We have presented a technique to calculate affine collineations and then used the affine collineations to generate conserved quantities. We have discussed their utility in distinguishing spacetimes, guaranteeing the intregrability of the geodesic equation, and in astrophysical simulations. Further work in this field would be explicit examples of affine collineations distinguishing two 'similar' spacetimes, or seeing these quantities conserved in an astrophysical simulation. I hope to push forward on the latter of the two this summer and present my results at a conference at some point in the future.

7 Acknowledgements

I cannot thank Dr. Charles Torre enough for his mentorship during this project. He contributed to, and supervised, this work from October 2016 till May 2019. I am very grateful for all the effort he has spent on instructing me. I contribute much of my future success to his mentoring. I would also like to thank the Howard L. Blood family for

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Last, but not least, I would be remiss to not express my gratitude to all involved in the Utah State University Department of Physics. The faculty have been extraordinary, and it was Karalee who pulled me out of my engineering degree, thankfully.

References

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Appendix One: Introduction to Symplectic Integrators

A symplectic integrator is a numerical scheme for integrating Hamiltonian systems. The time evolution of Hamilton's equations is a symplectomorphism, meaning that it conserves the symplectic two-form $dp \wedge dq$. A numerical scheme is a symplectic integrator if it also conserves this two-form. This is equivalent to saying the Jacobian matrix of the mapping $\Phi : R^{2n} \to R^{2n}$ is in the symplectic group.

Recall that Hamilton's Equations are

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

We will call a Hamiltonian separable if it is written as $H(q,p) = T(p)+V(q)$. Previous to the new integration technique in [4] Hamilton's equation could only be solved using an explicit symplectic method if the Hamiltonian was separable. Otherwise, a slower implicit method was required.

The most simple symplectic technique is known as the semi-implicit Euler method. It is only slightly different the Euler method which is used to numerically solve ordinary differential equations. However, the small difference is what makes the method symplectic, and thus energy conserving.

Theorem: The semi-implicit Euler scheme is a symplectic integrator for a separable Hamiltonian with one degree of freedom.

Proof: The transformation is given by $q_{n+1} = q_n + h\dot{q}_n$ and $p_{n+1} = p_n + h\dot{p}_{n+1}$ where h is some small step size. By the fact that we have separable Hamiltonian, we can use Hamilton's equations to write

$$q_{n+1} = q_n + h\frac{\partial H}{\partial p}\bigg|_n = q_n + h\frac{dT}{dp}\bigg|_n$$
$$p_{n+1} = p_n - h\frac{\partial H}{\partial q}\bigg|_{n+1} = p_n - h\frac{dV}{dq}\bigg|_{n+1}$$

We can then rewrite these equations again as follows.

$$q_{n+1} = q_n + hT'(p_n)$$
$$p_{n+1} = p_n - hV'(q_{n+1})$$

Now we can write the Jacobian of this transformation in the usual form, $J = \begin{bmatrix} \frac{dq_{n+1}}{dq_n} & \frac{dq_{n+1}}{dp_n} \\ \frac{dp_{n+1}}{dq_n} & \frac{dp_{n+1}}{dp_n} \end{bmatrix}$.

$$J = \begin{bmatrix} 1 & hT''(p_n) \\ -hV''(q_{n+1})\frac{dq_{n+1}}{dq_n} & 1 - h^2V''(q_{n+1})\frac{dq_{n+1}}{dp_n} \end{bmatrix}$$

J has a determinant of 1. Thus, it is area preserving. We can also see that

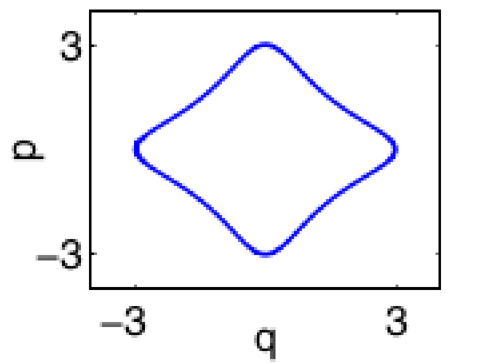
$$J^T \Omega J = \Omega$$

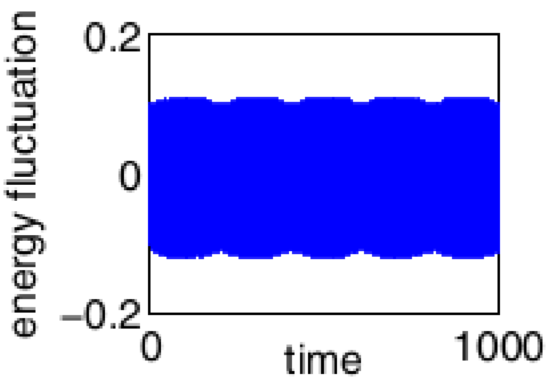
where $\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Thus the Jacobian matrix of the transformation is a member of the symplectic group, which implies that this method (the semi-implicit Euler method) is a symplectic method.

Appendix Two: Explicit Symplectic Integration

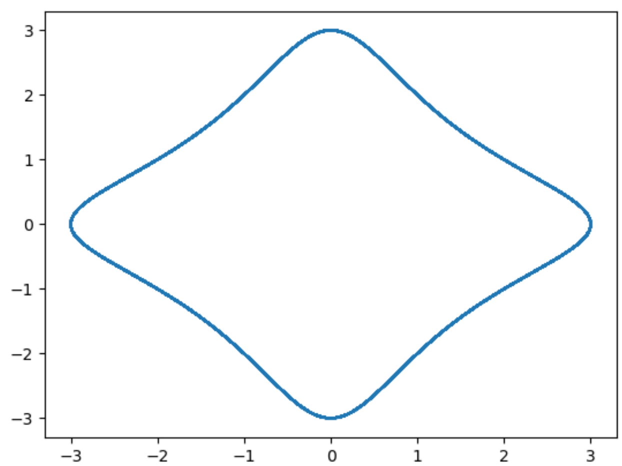
In 2016 Molei Tao published "Explicit symplectic approximation of nonseparable Hamiltonians: Algorithm and long time performance". In this paper he gives an explicit symplectic method to integrate nonseperable Hamiltonians with pleasant long time behavior. In general, explicit methods are much faster than implicit methods. Previous to this publication, nonseperable Hamiltonians required implicit integration. In the paper, Tao considers the system governed by the Hamiltonian $H(Q, P) = (Q^2 + 1)(P^2 + 1)/2$. He obtains the following plots.





Qualitatively, these plots show that over a long simulation, the deviation in the energy is bounded and that the solution in phase-space is not accruing many small errors.

Using the method outlined in the paper, I attempted to recreate the second order method. Below are my plots.



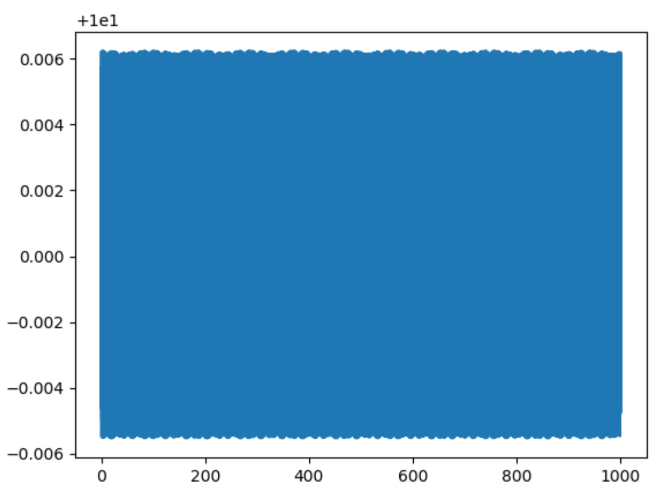


Figure 1: Plot of the energy deviation over time

We see that my implementation of the method produced the same desirable qualitative behaviors as those published in [4]. This gives us hope that we can use these techniques to integrate the geodesic equations in a faster and more accurate way than scientists have done previously.