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## Mathematica Program to Compute Klein Gordon Equation for Generic Black Holes

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**Mathematica Program to Compute  
Klein Gordon Equation for Generic  
Black Holes**

**By Brant Smith**

**Fall 2018**

**PHYS 4900**

**Mentor Help**

**Dr. Maria Rodriquez**

## **Abstract**

The goal of this project is to develop a program that will compute the Klein Gordon equation for generic Black Holes through the program Mathematica. This program will be available on Utah State websites for public usage. This project focuses on an understanding of General Relativity and more concretely on theoretical aspects of Black Holes. Developing the program begins with computing the Laplace equation in flat space to understand what it means to have empty space without a Black Hole. The Klein Gordon equation for a Schwarzschild Black Hole is then solved to show what happens once a static, non-rotating Black Hole is added into empty space. The final portion of this project is solving the Klein Gordon equation for a non-static, rotating Kerr Black Hole and writing this equation in Mathematica. After completion, the program will be uploaded to the Utah State University website.

## **Motivation**

Many will argue that there is no real important information that an undergraduate physics student could offer to further the area of theoretical physics knowledge, let alone black holes. As an undergrad, one can do as much research as they want with reading research papers to reading your typical book that can be found at home by Neil deGrasse Tyson. But, even with all of this knowledge available, there is a disconnect with the simplicity of the books you buy from amazon to the research papers published by theoretical physicists, such as my Mentor, Dr. Maria Rodriquez. The motivation behind my project is to create a Mathematica project that helps everybody simplify the calculation of the Klein Gordon equation for everyone.

Many ask why I did not use MATLAB or Maple instead of Mathematica. Well, Mathematica was a no brainer for completing this project. With all of the support that Mathematica has received from the developers and the continued support makes it one of the top scripting programming languages to use for mathematical models. It has a great support system online, unlike any other coding language. A quick google search will answer any question you have about the coding.

Also, with black holes getting more popular in the science community and in the media, this is the perfect time for such a program to be written. This program will be available on the Utah State University website for all to use, to help all calculate the Klein Gordon equation for generic black holes. Attached to this paper in their own appendices are the code for the program and scans of my notebook where I solved the equations by hand. The reason for the hand solving is that others can interpret what exactly is going on when solving for the Klein Gordon Equation and get a general understanding for what the program is doing.

## Background

The Klein Gordon equation is a relativistic wave equation, related to the Schrodinger wave equation where it is a second order differential equation in space and time. This equation is named after Oskar Klein and Walter Gordon, who discovered the equation in 1926. The form of the Klein Gordon equation takes many different forms, but for the purpose of this project the equation took the form of:

$$\left[ \frac{1}{\sqrt{-g}} \partial_k (g^{kp} \sqrt{-g} \partial_p) \right] \Phi = 0$$

where  $g$  is the determinant of the matrix and the partial derivatives in the equations are their respectable parts of the metric.

In this project we use this Klein Gordon equation for a generic black hole called a Schwarzschild black hole. The code that is attached is for a Schwarzschild black hole, but the code can be applied to many more types of black holes. Such as the Kerr black hole. Below is outlined the process on how the program was made.

## Minkowski Flat Space

The first part to the project was being understanding what flat space meant. In the terms of black holes, Minkowski flat space is where you have a space that is three dimensional in space and has a time dimension. The flat space metric in cartesian is:

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next was converting to spherical coordinates of the flat space metric, which takes the form of:

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix}$$

Above all of this was solved by hand. The next part was to generate a Mathematica file to compute the Laplacian in spherical coordinates from this Minkowski flat space solved above. This was done from a tensor calculus equation of the form:

$$\nabla^2 = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial \xi^i} (\sqrt{\det g} g^{ij} \frac{\partial}{\partial \xi^j})$$

where g is the metric and i/j are the respective components of the metric. This result of the Laplacian is used later to prove that the Klein Gordon equation in spherical coordinates with no mass, in other words flat-space, is equal to the Laplacian in spherical coordinates. This paper will touch on this more later.

## Schwarzschild Part

Schwarzschild black hole is the first type of black hole that this program was applied to. This is good for reason! A Schwarzschild black hole is arguably the simplest type of black hole due to it being a static, non-rotating black hole. This is a solution to Einstein's equations, so we know that it is a possible for these types of black holes to exist. The metric for a Schwarzschild black hole is:

$$g^{\mu\nu} = \begin{bmatrix} \frac{r}{2M-r} & 0 & 0 & 0 \\ 0 & \frac{r-2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2(\theta)} \end{bmatrix}$$

The Klein Gordon solution to Schwarzschild black hole is:

$$\frac{\partial}{\partial r} \left( (r^2 - 2Mr) \frac{\partial \psi}{\partial r} \right) - \frac{r^4}{r(r-2M)} \frac{\partial^2 \psi}{\partial t^2} + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 \psi}{\partial \phi^2} = 0$$

A lot of interesting things can be obtained when looking at this solution. When we set the mass to be zero in the equation, we can prove that it is equal to flat space. The Laplacian in spherical coordinates is equal to the Klein Gordon equation for a Schwarzschild black hole with no mass. No surprise, when there is no mass in a space than it should be flat-space. The great thing about realizing this is that we have proven that our program works for computing the Klein Gordon equation! With proving that this equation works for basic and generic black holes, we now know that it can be applied to other more complicated black holes.

## Summary

The goal of this project was to write a Mathematica program that would compute the Klein Gordon equation for generic black holes. The project focused mainly on Minkowski flat-space and Schwarzschild black holes, but the program can be applied to other types of black holes. Such as Kerr and other generic black holes. We were able to prove that the program worked perfectly by using the program to solve the Klein Gordon equation for a Schwarzschild black hole and setting the mass equal to zero. This produces flat space all over again. The hope is that researchers will use the program to compute the Klein Gordon equation for their research needs in black holes, simplifying the work that they have to do. This program will be available on a Utah State Website for everybody to use.

## **Acknowledgements**

I would like to thank Dr. Maria Rodriguez for all of the help that she offered. Thank you for being willing to help even when you are out of the country doing your own research projects.

## **References**

Kazazakis, Vassilis. "Quasi-Normal Modes of Scalar Fields in Kerr Background." *Universiteit van Amsterdam*. March 2012. Web.

Lehn, Chabysheva, & Hiller. "Klein-Gordon equation in curved space-time." *Department of Physics and Astronomy, University of Minnesota-Duluth*. November 2017. Web.

## Appendix 1

### Code for Program

# Klein Gordon equation

Authors : Brant Smith and Maria J . Rodriguez

Date : Nov 18, 2018

```
(* Input needed: *)
(* This program generates the Kelin Gordon equation (KG) for any space-time metric *)
(* n --> is the number of variables in the metric = the dimension of the space-time*)
(* coordinates --> x1, x2, x3, x4 *)
(* gUU[i,j][x1, x2, x3, x4] --> components of the inverse metric *)
(* deto[x1, x2, x3, x4] --> the metric determinant *)
equationGordonKlein
```

Authors : and Brant Maria Smith J . Rodriguez

## General form of the KG - equation

```
In[3]:= n = 4;
coordinates = {x1 -> t, x2 -> r, x3 -> θ, x4 -> ϕ};
KGequation = 
$$\frac{1}{\sqrt{-\text{deto}[x_1, x_2, x_3, x_4]}} \sum_{j=1}^n \sum_{i=1}^n \partial \left[ \sqrt{-\text{deto}[x_1, x_2, x_3, x_4]} g_{UU}[i, j][x_1, x_2, x_3, x_4] \partial [\Phi[x_1, x_2, x_3, x_4], x_i], x_j \right] /. \text{coordinates}$$

- 
$$\frac{g_{UU}[4, 4][t, r, \theta, \phi] \text{deto}^{(0, 0, 0, 1)}[t, r, \theta, \phi] \Phi^{(0, 0, 0, 1)}[t, r, \theta, \phi] + \sqrt{-\text{deto}[t, r, \theta, \phi]} \Phi^{(0, 0, 0, 1)}[t, r, \theta, \phi] g_{UU}[4, 4]^{(0, 0, 0, 1)}[t, r, \theta, \phi]}{2 \sqrt{-\text{deto}[t, r, \theta, \phi]}}$$

Out[5]=
```

## KG - equation for flat space-time

```
(* This program generates the Klein Gordon Equation in Schwarzschild space time *)

(* This is the KG equation for four dimensional (n=4) Schwarzschild space-time in spherical coordinates *)
KGequationSchwarzschild = KGequation /. deto -> Function[{t, r, θ, ϕ}, ((-r^4)((Sin[θ])^2))]/. gUU[1, 1] -> Function[{t, r, θ, ϕ},

$$\frac{r}{2M-r}] /. gUU[1, 2] -> Function[{t, r, θ, ϕ}, 0] /. gUU[1, 3] -> Function[{t, r, θ, ϕ}, 0] /. gUU[1, 4] -> Function[{t, r, θ, ϕ}, 0] /. gUU[2, 1] -> Function[{t, r, θ, ϕ}, 0] /. gUU[2, 2] -> Function[{t, r, θ, ϕ},  $\frac{-2M+r}{r}$ ] /. gUU[2, 3] -> Function[{t, r, θ, ϕ}, 0] /. gUU[2, 4] -> Function[{t, r, θ, ϕ}, 0] /. gUU[3, 1] -> Function[{t, r, θ, ϕ}, 0] /. gUU[3, 2] -> Function[{t, r, θ, ϕ}, 0] /. gUU[3, 3] -> Function[{t, r, θ, ϕ}, (1/(r^2))] /. gUU[3, 4] -> Function[{t, r, θ, ϕ}, 0] /. gUU[4, 1] -> Function[{t, r, θ, ϕ}, 0] /. gUU[4, 2] -> Function[{t, r, θ, ϕ}, 0] /. gUU[4, 3] -> Function[{t, r, θ, ϕ}, 0] /. gUU[4, 4] -> Function[{t, r, θ, ϕ}, (1/(r^2)((Sin[θ])^2)))] // Simplify

In[8]:= (* We simplify and check each term individually to make sense of the derivate *)
In[9]:= (*First part of output*)

ϕ2ndD = (((Csc[θ]^2)/r^2) * ϕ^{(0, 0, 0, 2)}[t, r, θ, ϕ])
Out[9]= 
$$\frac{\text{Csc}[\theta]^2 \Phi^{(0, 0, 0, 2)}[t, r, \theta, \phi]}{r^2}$$$$

```



```

In[10]:= (*Second part of output*)
θ1stD = ((Cot[θ] / r^2) * Φ(0, 0, 1, 0)[t, r, θ, φ])
Out[10]= 
$$\frac{\text{Cot}[\theta] \Phi^{(0, 0, 1, 0)}[t, r, \theta, \phi]}{r^2}$$


In[11]:= (*Third part of output*)
θ2ndD = ((1 / (r^2)) * Φ(0, 0, 2, 0)[t, r, θ, φ])
Out[11]= 
$$\frac{\Phi^{(0, 0, 2, 0)}[t, r, \theta, \phi]}{r^2}$$


In[12]:= (*Fourth part of output*)
r1stD = ((2 * r / (r^2)) * Φ(0, 1, 0, 0)[t, r, θ, φ])
Out[12]= 
$$\frac{2 \Phi^{(0, 1, 0, 0)}[t, r, \theta, \phi]}{r}$$


In[13]:= (*Fifth part of output*)
r2ndD = ((r^2 / (r^2)) * Φ(0, 2, 0, 0)[t, r, θ, φ])
Out[13]= 
$$\Phi^{(0, 2, 0, 0)}[t, r, \theta, \phi]$$


In[14]:= (*Sixth part of output*)
t2ndD = -((r^2 / (r^2)) * Φ(2, 0, 0, 0)[t, r, θ, φ])
Out[14]= 
$$-\Phi^{(2, 0, 0, 0)}[t, r, \theta, \phi]$$


In[15]:= equation = φ2ndD + θ1stD + θ2ndD + r1stD + r2ndD + t2ndD
(* This equation is the same as the Laplacian for spherical coordinates? *)
Simplify[equation - KGequationfinal == 0]
Out[15]= 
$$\frac{\text{Csc}[\theta]^2 \Phi^{(0, 0, 0, 2)}[t, r, \theta, \phi] \text{Cot}[\theta] \Phi^{(0, 0, 1, 0)}[t, r, \theta, \phi] + \Phi^{(0, 0, 2, 0)}[t, r, \theta, \phi] + \frac{2 \Phi^{(0, 1, 0, 0)}[t, r, \theta, \phi]}{r} + \Phi^{(0, 2, 0, 0)}[t, r, \theta, \phi] - \Phi^{(2, 0, 0, 0)}[t, r, \theta, \phi]}{r^2} - \frac{\text{KGequationfinal} r^2 + \text{Csc}[\theta]^2 \Phi^{(0, 0, 0, 2)}[t, r, \theta, \phi] + \text{Cot}[\theta] \Phi^{(0, 0, 1, 0)}[t, r, \theta, \phi] + \Phi^{(0, 0, 2, 0)}[t, r, \theta, \phi] + 2 r \Phi^{(0, 1, 0, 0)}[t, r, \theta, \phi] + r^2 \Phi^{(0, 2, 0, 0)}[t, r, \theta, \phi] - \Phi^{(2, 0, 0, 0)}[t, r, \theta, \phi]}{r}$$

True

In[17]:= (* This equation is the same as the Laplacian for spherical coordinates*)

```

## KG - equation for Schwarzschild (black hole) space-time

```

In[18]:=
(* This program generates the Klein Gordon Equation in Schwarzschild space time *)

In[19]:=
(* This is the KG equation for four dimensional (n=4) Schwarzschild space-time in spherical coordinates
*)
KGequationSchwarzschild = KGequation /. deto → Function[{t, r, θ, φ}, ((-r^4)((Sin[θ])^2))]/. gUU[1, 1] → Function[{t, r, θ, φ},
r/(2M - r)]/. gUU[1, 2] → Function[{t, r, θ, φ}, 0]/. gUU[1, 3] → Function[{t, r, θ, φ}, 0]/. gUU[1, 4] → Function[{t, r, θ, φ}, 0]/. gUU[2,
1] → Function[{t, r, θ, φ}, 0]/. gUU[2, 2] → Function[{t, r, θ, φ}, -2M/r]/. gUU[2, 3] → Function[{t, r, θ, φ}, 0]/. gUU[2, 4] →
Function[{t, r, θ, φ}, 0]/. gUU[3, 1] → Function[{t, r, θ, φ}, 0]/. gUU[3, 2] → Function[{t, r, θ, φ}, 0]/. gUU[3, 3] → Function[{t, r, θ,
φ}, (1/(r^2))]/. gUU[3, 4] → Function[{t, r, θ, φ}, 0]/. gUU[4, 1] → Function[{t, r, θ, φ}, 0]/. gUU[4, 2] → Function[{t, r, θ, φ},
0]/. gUU[4, 3] → Function[{t, r, θ, φ}, 0]/. gUU[4, 4] → Function[{t, r, θ, φ}, (1 / ((r^2)((Sin[θ])^2)))]//Simplify
Out[19]= 
$$\frac{-(2M - r) \text{Csc}[\theta]^2 \Phi^{(0, 0, 0, 2)}[t, r, \theta, \phi] + (-2M + r) \text{Cot}[\theta] \Phi^{(0, 0, 1, 0)}[t, r, \theta, \phi] - 2M \Phi^{(0, 0, 2, 0)}[t, r, \theta, \phi] + r \Phi^{(0, 0, 2, 0)}[t, r, \theta, \phi] + 4M^2 \Phi^{(0, 1, 0, 0)}[t, r, \theta, \phi] - (-2M - r) \text{Csc}[\theta]^2 \Phi^{(0, 0, 0, 2)}[t, r, \theta, \phi] + (-2M + r) \text{Cot}[\theta] \Phi^{(0, 0, 1, 0)}[t, r, \theta, \phi] - 2M \Phi^{(0, 0, 2, 0)}[t, r, \theta, \phi] + r \Phi^{(0, 0, 2, 0)}[t, r, \theta, \phi] + 4M^2 \Phi^{(0, 1, 0, 0)}[t, r, \theta, \phi] - \Phi^{(2, 0, 0, 0)}[t, r, \theta, \phi]}{r^3}$$


In[20]:= KGequationlimit = KGequationSchwarzschild /. M → 0
(* This equation is the same as the KG-eq for flat space-time in spherical coordinates when the
black hole's mass vanishes (M=0) ? *)
Simplify[equation - KGequationlimit == 0]
Out[20]= 
$$\frac{r \text{Csc}[\theta]^2 \Phi^{(0, 0, 0, 2)}[t, r, \theta, \phi] + r \text{Cot}[\theta] \Phi^{(0, 0, 1, 0)}[t, r, \theta, \phi] + r \Phi^{(0, 0, 2, 0)}[t, r, \theta, \phi] + 2r^2 \Phi^{(0, 1, 0, 0)}[t, r, \theta, \phi] + r^3 \Phi^{(0, 2, 0, 0)}[t, r, \theta, \phi] - \Phi^{(2, 0, 0, 0)}[t, r, \theta, \phi]}{r^3}$$


```

Out[21]= **True**

In[22]:= (\* Yes, this equation is the same as the KG-eq for flat space-time in spherical coordinates when the  
black hole's mass vanishes (M=0)\*)

## Appendix 1

### Notebook Pages for Solving Klein Gordon Equation

$$\frac{1}{\sqrt{-g}} \nabla_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \Psi) = 0$$

Expanded:  $\frac{1}{\sqrt{-g}} \left[ \frac{\partial}{\partial t} (g^{tt} \sqrt{-g} \partial_t \Psi) + g^{tr} \sqrt{-g} (\partial_r \Psi) + g^{tr} \sqrt{-g} (\partial_\theta \Psi) + g^{t\phi} \sqrt{-g} (\partial_\phi \Psi) \right]$

part solved:

$$\frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial t} \left( \frac{r}{2M-r} \right) (r^2 \sin \theta) \frac{\partial \Psi}{\partial t} + 0 + 0 + 0 \right]$$

$$\left[ \frac{\partial}{\partial t} \left( \frac{r}{2M-r} \right) \frac{\partial \Psi}{\partial t} \right] = \frac{r}{2M-r} \frac{\partial^2 \Psi}{\partial t^2}$$

part expanded:

$$\frac{1}{\sqrt{-g}} \left[ \frac{\partial}{\partial r} (g^{rr} \sqrt{-g} \partial_r \Psi) + g^{r\theta} \sqrt{-g} (\partial_\theta \Psi) + g^{r\phi} \sqrt{-g} (\partial_\phi \Psi) + g^{\theta\theta} \sqrt{-g} (\partial_\theta \Psi) \right]$$

part solved:  $\frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} \left( 0 + \left( \frac{r-2M}{r} \right) (r^2 \sin \theta) (\partial_r \Psi) \right) + 0 + 0 \right]$

$$\frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} \left( (r-2M) (r \sin \theta) \right) (\partial_r \Psi) \right] = \frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( (r^2 - 2Mr) \frac{\partial \Psi}{\partial r} \right) \right]$$

$$= \frac{1}{r^2} \left[ (2r - 2M) \frac{\partial^2 \Psi}{\partial r^2} \right] = \left( \frac{2}{r} - \frac{2M}{r^2} \right) \frac{\partial^2 \Psi}{\partial r^2}$$

$$\theta_{part}: \frac{1}{\sqrt{g}} \left[ \partial_\theta (g^{\theta\theta} \sqrt{-g} \partial_\theta \psi) \right]$$

$$\theta_{part} \text{ solves: } \frac{1}{r^2 \sin \theta} \left[ \partial_\theta \left( \frac{1}{r^2} (r^2 \sin \theta) (\partial_\theta \psi) \right) \right] = \boxed{\frac{1}{r^2 \sin \theta} \left[ \partial_\theta \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) \right]}$$

$$\phi_{part}: \frac{1}{\sqrt{g}} \left[ \partial_\phi (g^{\phi\phi} \sqrt{-g} \partial_\phi \psi) \right]$$

$$\phi_{part} \text{ solves: } \frac{1}{r^2 \sin \theta} \left[ \partial_\phi \left( \frac{1}{r^2 \sin \theta} (r^2 \sin \theta) (\partial_\phi \psi) \right) \right]$$

$$= \frac{1}{r^2 \sin \theta} \left( \frac{1}{\sin \theta} \right) \left[ \frac{\partial^2 \psi}{\partial \phi^2} \right] = \boxed{\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}}$$

Answer: 
$$\left( \frac{r}{2M-r} \frac{\partial^2 \Psi}{\partial t^2} \right) + \left( \frac{2}{r} - \frac{2M}{r^2} \right) \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) \right] + \left( \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right) = 0$$

or when rearranging and multiplying by  $r^2$  and stepping back to derivatives

$$\frac{d}{dr} \left( (r^2 - 2Mr) \frac{\partial \Psi}{\partial r} \right) - \frac{r^4}{r(r-2M)} \frac{\partial^2 \Psi}{\partial t^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} = 0$$

When we set  $M=0$ , we get back Minkowski space, flat space.