8-12-2019

# Yang-Mills Sources for Biconformal Gravity

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August 12, 2019

**Abstract**

We present a gauge formulation of Yang-Mills matter sources for Biconformal gravity. Biconformal gravity is a 2n-dimensional conformal gauge theory with a curvature linear action that has been shown to reproduce scale invariant general relativity on the cotangent bundle of n-dimensional space time. We present a generalization of Yang-Mills theories in biconformal space and show that the field equations with sources reduce the Yang-Mills sector to n-dimensional Yang-Mills theory in curved spacetime. We compute the restrictive conditions on the energy-momentum tensor required by the gravitational field equations.

## Contents



# 1 Introduction

Over a century ago, Albert Einstein proposed a way by which the curvature of spacetime and the matter fields that reside in that space-time interact. In particular, the coupling of matter sources with gravity in general relativity was achieved by adding a coordinate invariant form of the matter action to the Einstein-Hilbert action. We will explore the case of Yang-Mills fields as sources for Biconformal gravity.

Biconformal gravity is a gauge theory and is therefore to be partly understood as a formulation of physical laws based on known symmetries of nature. In particular, since laboratory measurements are in fact relative and not absolute, the ability to choose local units constitutes a symmetry of nature. More precisely, the action functional that encodes the laws of nature ought to be invariant under local scale changes $i.e$ invariant under $SO(1,1)$ transformations.

As a gauge theory, not only is biconformal gravity related to the ripples in space-time, but also to a wondrously beautiful mathematical object known as a *principle fiber bundle*. The principle fiber bundle is constructed by use of the quotient method, a technique DUE TO CARTAN that was first developed and employed for gravity by Ne'eman and Regge. We will briefly introduce the reader to the quotient technique in the next section.

### 1.1 Sources in biconformal gravity

The incontestable utility of the conformal group in modern physics has spawned a great deal of interest in symmetric homogeneous spaces. Among homogeneous spaces constructed from the conformal group, biconformal space is unique for not only resulting in a Kähler manifold of doubled dimension, but also in providing myriad rich structures including a non-degenerate Killing form and orthogonal Lagrangian submanifolds.

Biconformal gravity arises by taking the quotient of the conformal group of an $SO(p,q)$-symmetric space $(p+q=n)$ by its homogeneous Weyl subgroup, $\mathcal{W}_{p,q} \equiv SO(p,q) \times SO(1,1)$. This leads to a $2n$-dimensional homogeneous space with local $\mathcal{W}_{p,q}$ symmetry. This homogeneous space, discussed in [10] and [11] and studied extensively in [24, 46, 25], is found to have compatible symplectic, metric and complex structures, making it Kähler [24]. In addition, the Killing form is nondegenerate and scale invariant, and the volume form of the base manifold is scale invariant. The homogeneous space, and its curved generalizations are called biconformal spaces. All of these results apply immediately to the conformal group $SO(p+1,q+1)$ of any $SO(p,q)$-symmetric space.

The most general action linear in the biconformal curvature is given by

$$S = \lambda \int e_{ac\cdots d}{}^{be\cdots f} \left(\alpha \mathbf{\Omega}^{a}{}_{b} + \beta \delta^{a}_{b} \mathbf{\Omega} + \gamma \mathbf{e}^{a} \wedge \mathbf{f}_{b}\right) \wedge \mathbf{e}^{c} \wedge \cdots \wedge \mathbf{e}^{d} \wedge \mathbf{f}_{e} \wedge \cdots \wedge \mathbf{f}_{f} \tag{1}$$

where $\mathbf{\Omega}^{a}{}_{b}$ is the curvature of the spin connection and $\mathbf{\Omega}$ is the dilatational curvature. Here $\lambda = \frac{(-1)^n}{(n-1)!(n-1)!}$ is a convenient constant, chosen to eliminate a combinatoric factor and to make our sign conventions agree with [1]. The cotangent space is spanned by the pair, $(\mathbf{e}^a, \mathbf{f}_b)$, called the solder form and the co-solder form, respectively.

Remarkably, even though the full gravity theory initially depends on all $2n$ independent coordinates, it has been shown to reduce to $n$-dimensional gravity [11, 13, 40, 32, 33, 34, 46, 24, 25, 1]. Here we show that the same reduction occurs for Yang-Mills matter sources. The central issue is to show that a completely general $SU(N)$ gauge theory over a $2n$-dimensional biconformal space reduces to the expected $n$-dim gravitational source.

## 2 Constructing the Yang-Mills Action

A Yang-Mills theory follows from an action functional of the form

$$S_{\mathrm{YM}} = -\frac{\kappa}{2} \int tr \mathcal{F} \wedge^{*} \bar{\mathcal{F}} \tag{2}$$

where $*$ is the usual Hodge dual, $\mathcal{F}$ is a curvature 2-form, $\bar{\mathcal{F}}$ is a conjugate curvature formed using the complex structure, and the trace is over the $SU(N)$ generators. Several issues arise when we expand this expression in biconformal spaces.

Because the basis forms are distinguishable by their conformal weights, each biconformal 2-form may be uniquely expanded in three parts,

$$\mathcal{F}^i = \frac{1}{2}\mathcal{F}^i{}_{ab}\mathbf{e}^a \wedge \mathbf{e}^b + \mathcal{F}^{ia}{}_b\mathbf{f}_a \wedge \mathbf{e}^b + \frac{1}{2}\mathcal{F}^{iab}\mathbf{f}_a \wedge \mathbf{f}_b$$

where $i$ is an index of the internal Lie algebra. This index can be suppressed without loss of generality in most of the following, and it proves more transparent to give the three coefficients distinct names, leading to,

$$\mathcal{F} = \frac{1}{2}F_{ab}\mathbf{e}^a \wedge \mathbf{e}^b + G^a{}_b\mathbf{f}_a \wedge \mathbf{e}^b + \frac{1}{2}H^{ab}\mathbf{f}_a \wedge \mathbf{f}_b$$

Note that $F_{ab}$ and $H^{ab}$ are antisymmetric.

We observe here that in defining the dual operation for the above 2 -form a number of ambiguities need to be clarified. We establish the following conventions:

1. All factors of $\mathbf{f}_a$ are written first, followed by all of the $\mathbf{e}^b$.
2. The Levi-Civita tensor is written as $e^{ab\cdots c}{}_{de\cdots f}$, with all $n$ up indices first.
3. $(-1)^k$ to correct any index on the on the coefficient that is not summed on the first index of the $\varepsilon$.
4. An $m$-form is a polynomial with each term having different numbers of $\mathbf{e}$'s and $\mathbf{f}$'s. We write the terms in order of increasing number of $\mathbf{f}$'s.

It has been noted elsewhere [57] that there are alternative duals in biconformal space. For instance, we may use the symplectic form instead of the metric to connect indices. The difference resides in the relativie signs between the $\mathbf{e}^a$, $\mathbf{f}_a$, and mixed terms. Care must be exercised to keep the correct signs.

Another issue in finding the biconformal dual is the surprising invisibility of the metric, which in the usual "null" basis takes the form

$$K_{AB} = \begin{pmatrix} 0 & \delta^a{}_b \\ \delta_a{}^b & 0 \end{pmatrix}$$

where the raised or lowered position indicates the conformal weight. Since $K_{AB}$ is built from Kronecker deltas, its presence in an expression is often masked. In order to vary the metric, it is necessary to write it in an explicitly general form, as

$$K_{AB} = \begin{pmatrix} K_{ab} & K^a{}_b \\ K_a{}^b & K^{ab} \end{pmatrix}$$

and only restore the null form after variation.With these factors in mind, we find the dual of the Yang-Mills field is

$$\begin{aligned}
{}^*\mathcal{F} = {} & \frac{1}{2}F_{ab}\left(\frac{K^{am}K^{bn}}{n!\,(n-2)!}\varepsilon^{c\cdots d}{}_{mne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} + \frac{(-1)^{n-1}K^a{}_m K^{bn}}{(n-1)!\,(n-1)!}\varepsilon^{mc\cdots d}{}_{ne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\right) \\
& +\frac{1}{2}F_{ab}\left(\frac{(-1)^n K^{am}K^b{}_n}{(n-1)!\,(n-1)!}\varepsilon^{nc\cdots d}{}_{me\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} + \frac{K^a{}_m K^b{}_n}{n!\,(n-2)!}\varepsilon^{mnc\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\right) \\
& +G^a{}_b\left(\frac{K_a{}^m K^{bn}}{n!\,(n-2)!}\varepsilon^{c\cdots d}{}_{mne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} + \frac{(-1)^{n-1}K_{am}K^{bn}}{(n-1)!\,(n-1)!}\varepsilon^{mc\cdots d}{}_{ne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\right) \\
& +G^a{}_b\left(\frac{(-1)^n K_a{}^m K^b{}_n}{(n-1)!\,(n-1)!}\varepsilon^{nc\cdots d}{}_{me\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} + \frac{K_{am}K^b{}_n}{n!\,(n-2)!}\varepsilon^{mnc\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\right) \\
& +\frac{1}{2}H^{ab}\left(\frac{K_a{}^m K_b{}^n}{n!\,(n-2)!}\varepsilon^{c\cdots d}{}_{mne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} + \frac{(-1)^{n-1}K_{am}K_b{}^n}{(n-1)!\,(n-1)!}\varepsilon^{mc\cdots d}{}_{ne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\right) \\
& +\frac{1}{2}H^{ab}\left(\frac{(-1)^n K_a{}^m K_{bn}}{(n-1)!\,(n-1)!}\varepsilon^{nc\cdots d}{}_{me\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} + \frac{K_{am}K_{bn}}{n!\,(n-2)!}\varepsilon^{mnc\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\right)
\end{aligned}$$

where for brevity we omit the wedge between forms. Thus, for example, a $2n-2$ basis form becomes

$$\mathbf{f}_a \wedge \mathbf{f}_b \wedge \cdots \wedge \mathbf{f}_c \wedge \mathbf{e}^d \wedge \mathbf{e}^e \cdots \mathbf{e}^f \Rightarrow \mathbf{f}_{ab\cdots c}\mathbf{e}^{de\cdots f}$$

Since the Levi-Civita tensor always has $n$ up and $n$ down indices, the number of basis forms is unambiguous.

It is convenient to define a volume form as

$$\begin{aligned}
\mathbf{\Phi} &\equiv {}^*1 \\
&= \frac{1}{n!n!} e^{c\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f} \\
&= \frac{1}{n!n!}\sqrt{K}\varepsilon^{c\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f}
\end{aligned}$$

and consequently,

$$\begin{aligned}
\mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f} &= \frac{1}{\sqrt{K}}\varepsilon^{e\cdots f}{}_{c\cdots d}\mathbf{\Phi} \\
&= \bar{e}_{c\cdots d}{}^{e\cdots f}\mathbf{\Phi}
\end{aligned}$$

where the overbar denotes the contravariant form of the Levi-Civita tensor. (See Appendix A for the definition of the contravariant form of the Levi-Civita tensor, $\bar{e}_{c\cdots d}{}^{e\cdots f}$).

We also need the reduction formulas,

$$\begin{aligned}
\varepsilon^{e\cdots f}{}_{mnc\cdots d}\varepsilon^{ghc\cdots d}{}_{e\cdots f} &= n!\,(n-2)!\left(\delta^g_m\delta^h_n - \delta^h_m\delta^g_n\right) \\
\varepsilon^{mc\cdots d}{}_{ne\cdots f}\varepsilon^{he\cdots f}{}_{gc\cdots d} &= (n-1)!\,(n-1)!\delta^h_n\delta^m_g
\end{aligned}$$

Then, forming the wedge product, $\mathcal{F} \wedge {}^*\mathcal{F}$, we may eliminate the basis forms in favor of the volume form $\mathbf{\Phi}$. After a bit of algebra, we have

$$\begin{aligned}
\mathcal{F}^*\mathcal{F} &= \left(\frac{1}{2}F_{mn}K^{am}K^{bn} + G^m{}_nK^a{}_mK^{bn} + \frac{1}{2}H^{mn}K^a{}_mK^b{}_n\right)F_{ab}\mathbf{\Phi} \\
&\quad + \left(F_{mn}K_a{}^mK^{bn} + G^m{}_n\left(K_{am}K^{bn} - K_a{}^nK^b{}_m\right) + H^{mn}K_{am}K^b{}_n\right)G^a{}_b\mathbf{\Phi} \\
&\quad + \left(\frac{1}{2}F_{mn}K_a{}^mK_b{}^n + G^m{}_nK_{am}K_b{}^n + \frac{1}{2}H^{mn}K_{am}K_{bn}\right)H^{ab}\mathbf{\Phi} \qquad (3)
\end{aligned}$$

and the matter action is given by Ea.(2). The full action is the combination of Eq.(1) and Eq.(2),

$$S = S_G + S_{YM}$$

# 3 Variation

## 3.1 Gravity variation

The gravity action is given by Eq.(1) where the curvature components are given in terms of the connection by

$$\begin{aligned}
\mathbf{\Omega}^a{}_b &= \mathbf{d}\boldsymbol{\omega}^a{}_b - \boldsymbol{\omega}^c{}_b\boldsymbol{\omega}^a{}_c - 2\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d & (4) \\
\mathbf{T}^a &= \mathbf{d}\mathbf{e}^a - \mathbf{e}^c\boldsymbol{\omega}^a{}_c - \boldsymbol{\omega}\mathbf{e}^a & (5) \\
\mathbf{S}_a &= \mathbf{d}\mathbf{f}_a - \boldsymbol{\omega}^c{}_a\mathbf{f}_c - \mathbf{f}_a\boldsymbol{\omega} & (6) \\
\mathbf{\Omega} &= \mathbf{d}\boldsymbol{\omega} - \mathbf{e}^c\mathbf{f}_c & (7)
\end{aligned}$$

The variation is discussed in detail in [1], so we simply state the result. The variation of the spin connnection and Weyl vector give

$$\begin{aligned} T^{ae}{}_{e} - T^{ea}{}_{e} - S_{e}{}^{ae} &= 0 \\ T^{a}{}_{ca} + S_{c}{}^{a}{}_{a} - S_{a}{}^{a}{}_{c} &= 0 \\ \alpha\Delta^{ar}_{sb}\left(T^{mb}{}_{a} - \delta^{m}_{a}T^{eb}{}_{e} - \delta^{m}_{a}S_{c}{}^{bc}\right) &= 0 \\ \alpha\Delta^{ar}_{sb}\left(\delta^{b}_{c}T^{d}{}_{ad} + S_{c}{}^{b}{}_{a} - \delta^{b}_{c}S_{d}{}^{d}{}_{a}\right) &= 0 \end{aligned} \tag{8}$$

and these acquire no sources since the Yang-Mills action is independent of the these connection forms. The variation of the solder and co-solder forms lead to

$$\begin{aligned} &\left[\alpha\left(\Omega^{n}{}_{b}{}^{b}{}_{m} - \Omega^{a}{}_{b}{}^{b}{}_{a}\delta^{n}_{m}\right) + \beta\left(\Omega^{n}{}_{m} - \Omega^{a}{}_{a}\delta^{n}_{m}\right) + \Lambda\delta^{n}_{m}\right] A^{m}{}_{n} \\ &\left[\alpha\left(\Omega^{a}{}_{n}{}^{m}{}_{a} - \Omega^{a}{}_{b}{}^{b}{}_{a}\delta^{m}_{n}\right) + \beta\left(\Omega^{m}{}_{n} - \Omega^{a}{}_{a}\delta^{m}_{n}\right) + \Lambda\delta^{m}_{n}\right] D_{n}{}^{m} \\ &\qquad - \left[\alpha\Omega^{a}{}_{nam} + \beta\Omega_{nm}\right] B^{mn} \\ &\qquad \left[\alpha\Omega^{n}{}_{b}{}^{bm} + \beta\Omega^{nm}\right] C_{mn} \end{aligned}$$

where we define the coefficients of the variation as

$$\begin{aligned} \delta\mathbf{e}^{c} &= A^{c}{}_{f}\mathbf{e}^{f} + B^{cf}\mathbf{f}_{f} \\ \delta\mathbf{f}_{a} &= C_{af}\mathbf{e}^{f} + D_{a}{}^{f}\mathbf{f}_{f} \end{aligned}$$

In [1] these are equated to zero, but they now acquire sources. It therefore becomes important to keep track of overall factors.

## 3.2 Yang-Mills variation

Unlike the gravity sector, the Yang-Mills action requires the metric. Since the gravity variation requires the variation of all the gauge fields, $(\boldsymbol{\omega}^{a}{}_{b}, \mathbf{e}^{a}, \mathbf{f}_{b}, \boldsymbol{\omega})$, we need to express the variation of the metric in terms of the variation of the solder and co-solder forms by expanding the null inner product as

$$\begin{aligned} \left\langle \mathbf{e}^{A}, \mathbf{e}^{B}\right\rangle &\equiv K^{AB} \\ \begin{pmatrix} \left\langle \mathbf{e}^{a}, \mathbf{e}^{b}\right\rangle & \left\langle \mathbf{e}^{a}, \mathbf{f}_{b}\right\rangle \\ \left\langle \mathbf{f}_{a}, \mathbf{e}^{b}\right\rangle & \left\langle \mathbf{f}_{a}, \mathbf{f}_{b}\right\rangle \end{pmatrix} &= \begin{pmatrix} K^{ab} & K^{a}{}_{b} \\ K_{a}{}^{b} & K_{ab} \end{pmatrix} \end{aligned}$$

as noted above, the variation of the solder and co-solder forms is given by

$$\begin{aligned} \delta\mathbf{e}^{c} &= A^{c}{}_{f}\mathbf{e}^{f} + B^{cf}\mathbf{f}_{f} \\ \delta\mathbf{f}_{a} &= C_{af}\mathbf{e}^{f} + D_{a}{}^{f}\mathbf{f}_{f} \end{aligned}$$

with $A^{c}{}_{f}, B^{cf}, C_{af}$ and $D_{a}{}^{f}$ arbitrary. Then we may expand each quadrant of the metric separately. For example,

$$\begin{aligned} K^{ab} &= \left\langle \mathbf{e}^{a}, \mathbf{e}^{b}\right\rangle \\ \delta K^{ab} &= \left\langle \delta\mathbf{e}^{a}, \mathbf{e}^{b}\right\rangle + \left\langle \mathbf{e}^{a}, \delta\mathbf{e}^{b}\right\rangle \\ &= \left\langle A^{a}{}_{c}\mathbf{e}^{c} + B^{ac}\mathbf{f}_{c}, \mathbf{e}^{b}\right\rangle + \left\langle \mathbf{e}^{a}, A^{b}{}_{c}\mathbf{e}^{c} + B^{bc}\mathbf{f}_{c}\right\rangle \\ &= A^{a}{}_{c}K^{cb} + A^{b}{}_{c}K^{ac} + B^{ac}K_{c}{}^{b} + B^{bc}K^{a}{}_{c} \end{aligned}$$

and similarly for the remaining quadrants, yielding

$$\begin{aligned} \delta K^{ab} &= A^{a}{}_{c}K^{cb} + A^{b}{}_{c}K^{ac} + B^{ac}K_{c}{}^{b} + B^{bc}K^{a}{}_{c} \\ \delta K^{a}{}_{b} &= B^{ac}K_{cb} + A^{a}{}_{c}K^{c}{}_{b} + D_{b}{}^{c}K^{a}{}_{c} + C_{bc}K^{ac} \\ \delta K_{a}{}^{b} &= C_{ac}K^{cb} + D_{a}{}^{c}K_{c}{}^{b} + A^{b}{}_{c}K_{a}{}^{c} + B^{bc}K_{ac} \\ \delta K_{ab} &= C_{ac}K^{c}{}_{b} + D_{a}{}^{c}K_{cb} + C_{bc}K_{a}{}^{c} + D_{b}{}^{c}K_{ac} \end{aligned}$$

Once the variation is accomplished, we may replace $K^{AB}$ by its null-orthonormal form,

$$\left(\begin{array}{cc} K^{ab} & K^{a}{}_{b} \\ K_{a}{}^{b} & K_{ab} \end{array}\right) = \left(\begin{array}{cc} 0 & \delta^{a}{}_{b} \\ \delta_{a}{}^{b} & 0 \end{array}\right) \tag{9}$$

This simplifies the metric variation to

$$\begin{aligned} \delta K^{ab} &= B^{ab} + B^{ba} = 2B^{(ab)} \\ \delta K^{a}{}_{b} &= A^{a}{}_{b} + D_{b}{}^{a} \\ \delta K_{b}{}^{a} &= A^{a}{}_{b} + D_{b}{}^{a} \\ \delta K_{ab} &= C_{ab} + C_{ba} = 2C_{(ab)} \end{aligned} \tag{10}$$

The variation of the action is now accomplished in two steps–the variation of the volume form and the variation of the remaining, explicit metric dependencies.

First, the variation of the volume form is

$$\begin{aligned} \delta\mathbf{\Phi} &= \frac{1}{n!n!}\delta\sqrt{K}\varepsilon^{c\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\wedge\mathbf{e}^{e\cdots f} \\ &= -\frac{1}{2}\frac{1}{n!n!}\bar{K}_{AB}\delta K^{AB}\sqrt{K}\varepsilon^{c\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\wedge\mathbf{e}^{e\cdots f} \end{aligned}$$

Notice that the variation of the $2n$-form part $\varepsilon^{c\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\wedge\mathbf{e}^{e\cdots f}$ vanishes, since

$$\varepsilon^{c\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\wedge\mathbf{e}^{e\cdots f} = \varepsilon^{\alpha\cdots\beta}{}_{\mu\cdots\nu}\mathbf{d}y_{\alpha\cdots\beta}\wedge\mathbf{d}x^{\mu\cdots\nu}$$

which is independent of the metric. Expanding $K_{AB}\delta K^{AB}$ in terms of the different quadrants using Eq.(10) and Eq.(9), and restoring the volume form, the variation of the volume form reduces to

$$\delta\mathbf{\Phi} = -\delta^{b}{}_{a}\left(A^{a}{}_{b} + D_{b}{}^{a}\right)\mathbf{\Phi}$$

Now we combine the volume form variation with the remaining direct metric variations. Varying the Yang-Mills action, Eq.(2), with the Lagrange density Eq.(3) and using the metric variation given by Eqs.(10), yields

$$\begin{aligned} -\frac{\kappa}{2}\delta\int\mathcal{F}^{*}\mathcal{F} &= -\kappa\int A^{n}{}_{a}\left(H^{ba}F_{bn} - G^{b}{}_{n}G^{a}{}_{b} - \frac{1}{2}\delta^{a}_{n}\left(F_{bc}H^{bc} - G^{b}{}_{c}G^{c}{}_{b}\right)\right)\mathbf{\Phi} \\ &\quad -\kappa\int B^{(bc)}\left(F_{ab}G^{a}{}_{c} + F_{ac}G^{a}{}_{b}\right)\mathbf{\Phi} \\ &\quad -\kappa\int C_{(ac)}\left(H^{cb}G^{a}{}_{b} + H^{ab}G^{c}{}_{b}\right)\mathbf{\Phi} \\ &\quad -\kappa\int D_{a}{}^{n}\left(H^{ba}F_{bn} - G^{b}{}_{n}G^{a}{}_{b} - \frac{1}{2}\delta^{a}_{n}\left(F_{bc}H^{bc} - G^{b}{}_{c}G^{c}{}_{b}\right)\right)\mathbf{\Phi} \end{aligned}$$

### 3.3 Combining the variations and final field equations

Combining both variations, we have the final field equations,

$$\alpha\left(\Omega^{n}{}_{b}{}^{b}{}_{m} - \Omega^{a}{}_{b}{}^{b}{}_{a}\delta^{n}_{m}\right) + \beta\left(\Omega^{n}{}_{m} - \Omega^{a}{}_{a}\delta^{n}_{m}\right) + \Lambda\delta^{n}_{m} = \kappa\left(H^{bn}F_{bm} - G^{b}{}_{m}G^{n}{}_{b} - \frac{1}{2}\delta^{n}_{m}\left(F_{bc}H^{bc} - G^{b}{}_{c}G^{c}{}_{b}\right)\right) \tag{11}$$

$$\alpha\left(\Omega^{a}{}_{n}{}^{m}{}_{a} - \Omega^{a}{}_{b}{}^{b}{}_{a}\delta^{m}_{n}\right) + \beta\left(\Omega^{m}{}_{n} - \Omega^{a}{}_{a}\delta^{m}_{n}\right) + \Lambda\delta^{m}_{n} = \kappa\left(H^{bm}F_{bn} - G^{b}{}_{n}G^{m}{}_{b} - \frac{1}{2}\delta^{m}_{n}\left(F_{bc}H^{bc} - G^{b}{}_{c}G^{c}{}_{b}\right)\right) \tag{12}$$

$$\alpha\Omega^{a}{}_{nam} + \beta\Omega_{nm} = \kappa\left(F_{am}G^{a}{}_{n} + F_{an}G^{a}{}_{m}\right) \tag{13}$$

$$\alpha\Omega^{n}{}_{b}{}^{bm} + \beta\Omega^{nm} = \kappa\left(H^{nb}G^{m}{}_{b} + H^{mb}G^{n}{}_{b}\right) \tag{14}$$

with sources, where we define

$$\Lambda \equiv \alpha(n-1) - \beta + n^2\gamma \tag{15}$$

and

$$\begin{aligned}
T^{ae}{}_{e} - T^{ea}{}_{e} - S_{e}{}^{ae} &= 0 && (16)\\
T^{a}{}_{ca} + S_{c}{}^{a}{}_{a} - S_{a}{}^{a}{}_{c} &= 0 && (17)\\
\alpha\Delta^{ar}_{sb}\left(T^{mb}{}_{a} - \delta^{m}_{a}T^{eb}{}_{e} - \delta^{m}_{a}S_{c}{}^{bc}\right) &= 0 && (18)\\
\alpha\Delta^{ar}_{sb}\left(\delta^{b}_{c}T^{d}{}_{ad} + S_{c}{}^{b}{}_{a} - \delta^{b}_{c}S_{d}{}^{d}{}_{a}\right) &= 0 && (19)
\end{aligned}$$

unchanged from the pure gravity case. We also have the structure equations, Eqs.(4)-(7), and their integrability conditions–generalized Bianchi identities–which follow by exterior differentiation,

$$\begin{aligned}
\mathbf{D}\mathbf{\Omega}^{a}{}_{b} &= 2\Delta^{ac}_{db}\mathbf{f}_{c}\mathbf{T}^{d} - 2\Delta^{ac}_{db}\mathbf{S}_{c}\mathbf{e}^{d} && (20)\\
\mathbf{D}\mathbf{T}^{a} &= \mathbf{e}^{c}\mathbf{\Omega}^{a}{}_{c} - \mathbf{\Omega}\mathbf{e}^{a} && (21)\\
\mathbf{D}\mathbf{S}_{a} &= -\mathbf{\Omega}^{c}{}_{a}\mathbf{f}_{c} + \mathbf{f}_{a}\mathbf{\Omega} && (22)\\
\mathbf{D}\mathbf{\Omega} &= -\mathbf{T}^{c}\mathbf{f}_{c} + \mathbf{e}^{c}\mathbf{S}_{c} && (23)
\end{aligned}$$

Thus far, we have introduced the reader to the basic algorithm for the development of field equations with Yang-Mills sources. Without these sources, the field equations for the torsion free solution reduce to the vacuum Einstein equation[1]. We will now pursue the question of whether or not the Yang-Mills portion also reduces to the usual energy-momentum source to the Einstein tensor.

# 4 Solving the field equations

Our solution follows many of the steps presented in detail in [1].

## 4.1 Vanishing torsion

With vanishing torsion, Eqs.(16)-(19) reduce to

$$\begin{aligned}
S_{e}{}^{ae} &= 0\\
S_{c}{}^{a}{}_{a} - S_{a}{}^{a}{}_{c} &= 0\\
\alpha\Delta^{ar}_{sb}\left(S_{c}{}^{b}{}_{a} - \delta^{b}_{c}S_{d}{}^{d}{}_{a}\right) &= 0
\end{aligned}$$

Together with these and vanishing torsion, three of the Bianchi identities, Eqs.(20),(21) and (23), simplify to

$$\begin{aligned}
\mathbf{D}\mathbf{\Omega}^{a}{}_{b} &= -2\Delta^{ac}_{db}\mathbf{S}_{c}\mathbf{e}^{d}\\
0 &= \mathbf{e}^{c}\mathbf{\Omega}^{a}{}_{c} - \mathbf{\Omega}\mathbf{e}^{a}\\
\mathbf{d}\mathbf{\Omega} &= \mathbf{e}^{c}\mathbf{S}_{c}
\end{aligned}$$

The algebraic condition $\mathbf{e}^{c}\mathbf{\Omega}^{a}{}_{c} = \mathbf{\Omega}\mathbf{e}^{c}$ expands to

$$\mathbf{e}^{b}\wedge\left(\frac{1}{2}\Omega^{a}{}_{bcd}\mathbf{e}^{c}\wedge\mathbf{e}^{d} + \Omega^{a}{}_{b}{}^{c}{}_{d}\mathbf{f}_{c}\wedge\mathbf{e}^{d} + \frac{1}{2}\Omega^{a}{}_{b}{}^{cd}\mathbf{f}_{c}\wedge\mathbf{f}_{d}\right) = \left(\frac{1}{2}\Omega_{cd}\mathbf{e}^{c}\wedge\mathbf{e}^{d} + \Omega^{c}{}_{d}\mathbf{f}_{c}\wedge\mathbf{e}^{d} + \frac{1}{2}\Omega^{cd}\mathbf{f}_{c}\wedge\mathbf{f}_{d}\right)\wedge\mathbf{e}^{a}$$

so that equating like components,

$$\begin{aligned}
\Omega^{a}{}_{[bcd]} &= \delta^{a}_{[b}\Omega_{cd]} && (24)\\
\Omega^{a}{}_{b}{}^{c}{}_{d} - \Omega^{a}{}_{d}{}^{c}{}_{b} &= \delta^{a}_{b}\Omega^{c}{}_{d} - \delta^{a}_{d}\Omega^{c}{}_{b} && (25)\\
\Omega^{a}{}_{b}{}^{cd} &= \delta^{a}_{b}\Omega^{cd} && (26)
\end{aligned}$$

For the third, take the $ab$ trace. Since $\eta_{ea}\Omega^{a}{}_{b}{}^{cd} = -\eta_{ba}\Omega^{a}{}_{e}{}^{cd}$ we are left with $\Omega^{cd} = 0$ and therefore both terms vanish separately,

$$\Omega^{a}{}_{b}{}^{cd} = 0 \tag{27}$$

$$\Omega^{cd} = 0 \tag{28}$$

From the second equation, the $ad$ trace gives

$$\Omega^{a}{}_{b}{}^{c}{}_{a} = -(n-1)\,\Omega^{c}{}_{b} \tag{29}$$

## 4.2 Curvature equations

We now combine the vanishing torsion simplifications with the curvature and dilatation field equations.

### 4.2.1 Momentum terms

Combining Eqs.(27) and (28) with Eq.(14) we immediately have

$$H^{nb}G^{m}{}_{b} + H^{mb}G^{n}{}_{b} = 0 \tag{30}$$

or, as matrix multiplication,

$$GH + (GH)^{t} = 0$$

### 4.2.2 Cross-terms

Moving to the cross-term equation, we formally lower an index in Eq.(25)

$$\eta_{ea}\Omega^{a}{}_{b}{}^{c}{}_{d} - \eta_{ea}\Omega^{a}{}_{d}{}^{c}{}_{b} = \eta_{eb}\Omega^{c}{}_{d} - \eta_{ed}\Omega^{c}{}_{b}$$

Now, cycling $ebd$, adding the first two and subtracting the third, and using the the antisymmetry of the curvature on the first two indices, $\eta_{ea}\Omega^{a}{}_{d}{}^{c}{}_{b} = -\eta_{da}\Omega^{a}{}_{e}{}^{c}{}_{b}$ we find

$$\begin{aligned}\Omega^{a}{}_{e}{}^{c}{}_{d} &= \eta^{ab}\eta_{ed}\Omega^{c}{}_{b} - \delta^{a}_{d}\delta^{b}_{e}\Omega^{c}{}_{b} \\ &= -2\Delta^{ab}_{de}\Omega^{c}{}_{b}\end{aligned} \tag{31}$$

We next observe that the difference between Eqs.(11) and (12) gives

$$\Omega^{n}{}_{b}{}^{b}{}_{m} = \Omega^{b}{}_{m}{}^{n}{}_{b} \tag{32}$$

Applying this to the two contractions of Eq.(31) lets us write

$$-\delta^{a}_{d}\Omega^{c}{}_{c} + \eta^{ae}\eta_{cd}\Omega^{c}{}_{e} = -(n-1)\,\Omega^{a}{}_{d}$$

Contracting with $\eta_{ba}$, we see that the antisymmetric part vanishes,

$$(n-2)\,\eta_{bc}\Omega^{c}{}_{d} - (n-2)\,\eta_{cd}\Omega^{c}{}_{b} = 0$$

in dimensions greater than 2. Therefore, the symmetric part, $\eta_{bc}\Omega^{c}{}_{d} + \eta_{dc}\Omega^{c}{}_{b} = \frac{2}{n}\eta_{bd}\Omega^{c}{}_{c}$ may be rewritten as a solution for the full cross dilatation in terms of the trace,

$$\Omega^{c}{}_{d} = \frac{1}{n}\delta^{a}_{b}\Omega^{c}{}_{c} \tag{33}$$

This, in turn, combines with Eq.(31) to give the cross curvature in terms of the trace of the dilatation,

$$\Omega^{a}{}_{b}{}^{c}{}_{d} = -\frac{2}{n}\Delta^{ac}_{db}\Omega^{e}{}_{e} \tag{34}$$

We have one remaining cross-curvature field equation, Eq.(11), which couples the results above to the Yang-Mills source. Using Eqs.(33) and (34) to replace the cross-curvature and the cross-dilatation, we find

$$\frac{n-1}{n}\left(\left(n-1\right)\alpha-\beta\right)\Omega^{a}{}_{a}\delta^{n}_{m}+\Lambda\delta^{n}_{m} = -\kappa\left(H^{bn}F_{bm}-G^{b}{}_{m}G^{n}{}_{b}+\frac{1}{2}\delta^{n}_{m}\left(F_{bc}H^{bc}-G^{b}{}_{c}G^{c}{}_{b}\right)\right) \quad (35)$$

Now taking the trace and collecting terms, we relate the trace of the source to the trace of the dilatation. Solving for the trace of the dilatation

$$\left(\left(n-1\right)\alpha-\beta\right)\Omega^{a}{}_{a}=-\frac{n+2}{2\left(n-1\right)}\kappa\left(F_{bc}H^{bc}-G^{b}{}_{c}G^{c}{}_{b}\right)-\frac{n}{n-1}\Lambda \quad (36)$$

The traced source terms on the right therefore drive the entire cross-curvature and cross-dilatation. It is striking that the only source dependence is the $U\left(1\right)$ Lagrangian density.

Returning to Eq.(35), and substituting for the trace of the dilatation from Eq.(36), we find

$$H^{bn}F_{bm}-G^{b}{}_{m}G^{n}{}_{b} = \frac{1}{n}\left(F_{bc}H^{bc}-G^{b}{}_{c}G^{c}{}_{b}\right)\delta^{n}_{m}$$

We note that the trace gives no further constraint.

#### 4.2.3 Spacetime terms

Finally, we combine the remaining field equation, Eq.(13),

$$\alpha\Omega^{a}{}_{nam}+\beta\Omega_{nm}=\kappa\left(F_{am}G^{a}{}_{n}+F_{an}G^{a}{}_{m}\right)$$

and the corresponding part of the vanishing torsion Bianchi, Eq.(24), which expanded becomes

$$\Omega^{a}{}_{bcd}+\Omega^{a}{}_{cdb}+\Omega^{a}{}_{dbc}=\delta^{a}_{b}\Omega_{cd}+\delta^{a}_{c}\Omega_{db}+\delta^{a}_{d}\Omega_{bc}$$

The $ac$ trace reduces this to

$$\Omega^{c}{}_{bcd}-\Omega^{c}{}_{dcb}=-\left(n-2\right)\Omega_{bd}$$

Combining this with the antisymmetric part of the field equation,

$$\alpha\left(\Omega^{a}{}_{nam}-\Omega^{a}{}_{man}\right) = -2\beta\Omega_{nm}$$

shows that

$$\left(\left(n-2\right)\alpha-2\beta\right)\Omega_{ab} = 0$$

so that generically (i.e., unless $\left(n-2\right)\alpha=2\beta$), the spacetime dilatation vanishes. Note that this is true for any symmetric source tensor. As a result,

$$\begin{aligned}\Omega^{a}{}_{nam} &= \frac{\kappa}{\alpha}\left(F_{am}G^{a}{}_{n}+F_{an}G^{a}{}_{m}\right)\\ \Omega_{nm} &= 0\end{aligned}$$

### 4.3 Dilatation

Having reduced the dilatational curvature to a single function, we can now use the dilatational structure equation Eq.(7) and its integrability condition Eq.(23) to press further. Substituting the reduced form of the dilatational curvature, $\mathbf{\Omega}=\Omega^{a}{}_{b}\mathbf{f}_{a}\wedge\mathbf{e}^{b}=\chi\mathbf{e}^{a}\wedge\mathbf{f}_{a}$, where

$$\chi\equiv-\frac{1}{n}\Omega^{a}{}_{a}$$

is given by Eq.(36), we expand Eq.(23),

$$
\begin{aligned}
0 &= \mathbf{d}\boldsymbol{\Omega} - \mathbf{e}^b \wedge \mathbf{S}_b \\
&= \mathbf{d}\left(-\chi \mathbf{f}_a \wedge \mathbf{e}^a\right) \wedge \boldsymbol{\Omega} - \mathbf{e}^b \wedge \mathbf{S}_b \\
&= -\mathbf{d}\chi \wedge \mathbf{f}_a \wedge \mathbf{e}^a - \chi \mathbf{D}\left(\mathbf{f}_a \wedge \mathbf{e}^a\right) - \mathbf{e}^b \wedge \mathbf{S}_b \\
&= \mathbf{d}\chi \wedge \mathbf{e}^a \wedge \mathbf{f}_a - (1+\chi)\, \mathbf{e}^a \wedge \mathbf{S}_a
\end{aligned}
$$

Setting $\mathbf{d}\chi = \chi_c \mathbf{e}^c + \chi^c \mathbf{f}_c$, expanding the co-torsion, and combining like forms,

$$
\begin{aligned}
0 &= \left(\chi_c \mathbf{e}^c + \chi^c \mathbf{f}_c\right) \wedge \mathbf{e}^a \wedge \mathbf{f}_a - (1+\chi)\, \mathbf{e}^a \wedge \left(\frac{1}{2} S_{acd} \mathbf{e}^c \wedge \mathbf{e}^d + S_a{}^c{}_d \mathbf{f}_c \wedge \mathbf{e}^d + \frac{1}{2} S_a{}^{cd} \mathbf{f}_c \wedge \mathbf{f}_d\right) \\
&= -\frac{1}{2}(1+\chi)\, S_{acd} \mathbf{e}^a \wedge \mathbf{e}^c \wedge \mathbf{e}^d + \left(\chi_c \delta^a_d + (1+\chi)\, S_c{}^a{}_d\right) \mathbf{e}^c \wedge \mathbf{e}^d \wedge \mathbf{f}_a \\
&\quad - \left(\chi^c \delta^d_a + \frac{1}{2}(1+\chi)\, S_a{}^{cd}\right) \mathbf{e}^a \wedge \mathbf{f}_c \wedge \mathbf{f}_d
\end{aligned}
$$

with independent parts

$$
\begin{aligned}
S_{[acd]} &= 0 \\
(1+\chi)\left(S_c{}^a{}_d - S_d{}^a{}_c\right) &= \chi_d \delta^a_c - \chi_c \delta^a_d \\
(1+\chi)\, S_a{}^{cd} &= \chi^d \delta^c_a - \chi^c \delta^d_a
\end{aligned}
$$

In components, the co-torsion field equations are

$$
\begin{aligned}
S_e{}^{ae} &= 0 \\
S_c{}^a{}_a - S_a{}^a{}_c &= 0 \\
\alpha \Delta^{ar}_{sb}\left(S_c{}^b{}_a - \delta^b_c S_d{}^d{}_a\right) &= 0
\end{aligned}
$$

Applying these to the Bianchi conditions by taking the appropriate traces,

$$
\begin{aligned}
0 &= S_c{}^a{}_a - S_a{}^a{}_c \\
&= -\frac{n-1}{1+\chi} \chi_c
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= S_b{}^{cb} \\
&= -\frac{n-1}{1+\chi} \chi^c
\end{aligned}
$$

and, except in the case when $1+\chi = 0$,

$$
\mathbf{d}\boldsymbol{\chi} = 0 \tag{37}
$$

The co-torsion field equations reduce to

$$
\begin{aligned}
S_{[bcd]} &= 0 \\
S_b{}^c{}_d - S_d{}^c{}_b &= 0 \\
S_b{}^{cd} &= 0
\end{aligned}
$$

and the dilatation takes the form

$$
\boldsymbol{\Omega} = \chi \mathbf{e}^a \wedge \mathbf{f}_a
$$

with $\chi$ constant.

This means that the Lagrangian density

$$\xi = F_{bc}H^{bc} - G^b{}_c G^c{}_b = \mathcal{L}$$

is constant, so (recalling Eq.(30)) we have a pair of strong constraints on the source fields:

$$H^{nb}G^m{}_b + H^{mb}G^n{}_b = 0 \quad (38)$$

$$H^{bn}F_{bm} - G^b{}_m G^n{}_b = \frac{1}{n}\xi\delta^n_m \quad (39)$$

The remaining combination is source to the spacetime curvture,

$$\Omega^a{}_{nam} = \frac{\kappa}{\alpha}\left(F_{am}G^a{}_n + F_{an}G^a{}_m\right)$$

It is encouraging to check degrees of freedom for a $U(1)$ field at this point. The full field strength, $F_{MN}$, has $\frac{2n(2n-1)}{2}$ degrees of freedom. Eq.(38) may be satisfied by setting $H^{ab} = 0$. This is

$$\frac{n(n-1)}{2}$$

constraints. The second constraint, Eq.(39), fixes a full $n \times n$ tensor, constraining $n^2$ functions. The number of remaining degrees of freedom is therefore

$$\begin{aligned} \frac{2n(2n-1)}{2} - \frac{n(n-1)}{2} - n^2 &= 2n^2 - n - \frac{1}{2}n^2 + \frac{1}{2}n - n^2 \\ &= \frac{1}{2}n(n-1) \end{aligned}$$

This is just the number of degrees of freedom of a $U(1)$ field in $n$-dimensions.

## 4.4 Collected results

So far, we have

$$\begin{aligned} \Omega^a{}_b{}^{cd} &= 0 \\ \Omega^a{}_b{}^c{}_d &= -\frac{2}{n}\Delta^{ac}_{db}\Omega^e{}_e \\ \Omega^a{}_{nam} &= \frac{\kappa}{\alpha}\left(F_{am}G^a{}_n + F_{an}G^a{}_m\right) \\ \\ \Omega^{cd} &= 0 \\ \Omega^c{}_d &= \frac{1}{n}\delta^a_b\Omega^c{}_c \\ \Omega_{nm} &= 0 \end{aligned}$$

where $\chi \equiv -\frac{1}{n}\Omega^a{}_a$ and

$$\Omega^a{}_a = -\frac{n-2}{2(n-1)((n-1)\alpha-\beta)}\kappa\left(F_{bc}H^{bc} - G^b{}_c G^c{}_b\right) - \frac{n\Lambda}{(n-1)((n-1)\alpha-\beta)}$$

is constant. We may summarize the curvatures as

$$\boldsymbol{\Omega}^a{}_b = \frac{1}{2}\Omega^a{}_{bcd}\mathbf{e}^c\mathbf{e}^d + 2\chi\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d \quad (40)$$

$$\boldsymbol{\Omega} = \chi\mathbf{e}^a \wedge \mathbf{f}_a \quad (41)$$

and the co-torsion satisfies

$$
\begin{aligned}
S_{[bcd]} &= 0 \\
S_b{}^c{}_d - S_d{}^c{}_b &= 0 \\
S_b{}^{cd} &= 0 \\
\alpha\Delta^{ar}_{sb}\left(S_c{}^b{}_a - \delta^b_c S_d{}^d{}_a\right) &= 0
\end{aligned}
\tag{42}
$$

We also have two constraints on the energy-momentum sources $F_{ab}, G^a{}_b, H^{ab}$, Eqs.(38) and (39).

### 4.5 Constraints on the source

Defining the source for the spacetime curvature as

$$
\begin{aligned}
\Omega^a{}_{nam} &= \frac{\kappa}{\alpha}\left(F_{am}G^a{}_n + F_{an}G^a{}_m\right) \\
&\equiv -\frac{\kappa}{\alpha}T_{ab}
\end{aligned}
$$

the constraints on the curvature may be written in matrix notation as

$$
\begin{aligned}
GH + (GH)^t &= 0 \\
HF + GG &= -\xi 1 \\
FG + (FG)^t &= T
\end{aligned}
$$

Using the antisymmetry of $H$ and $F$, these become

$$
\begin{aligned}
GH &= HG^t \\
HF + GG &= -\xi 1 \\
FG - G^t F &= T
\end{aligned}
$$

Now, multiply the middle equation by $G$,

$$
\begin{aligned}
GHF + GGG &= -\xi G \\
HG^t F + GGG &= -\xi G \\
HG^t F + GGG &= -\xi G \\
H\left(FG - T\right) + \left(-HF - \xi 1\right)G &= -\xi G \\
HFG - HT - HFG - \xi G &= -\xi G \\
HT &= 0
\end{aligned}
$$

and since the source is generically non-degenerate, we take

$$
H = 0
$$

This satisfies the first constraint completely, leaving

$$
\begin{aligned}
GG &= -\xi 1 \\
FG - G^t F &= T
\end{aligned}
$$

## 5 Discussion

Having worked out the constraints on the energy-momentum tensor $T$, we would like to know how the biconformal curvature is influenced by Yang-Mills matter sources. Beginning with a substitution of the reduced forms of the curvatures, Eqs.(40) and (41), into the structure equations in order to determine the connection, we find that with $H = 0$ , $T$ still satisfies the condition

$$G^{n}{}_{b}G^{b}{}_{m} = -\frac{1}{n}\,\xi\,\delta^{n}_{m}$$

However, the above condition implies that

$$G \neq F$$

In effect, the above condition yields an energy momentum tensor without the trace term. Furthermore, we expressed the curvature in the orthonormal basis and found that the sources for the Ricci tensor vanish identically. In this basis, the energy momentum tensor drives the cross-term of the curvature instead of being the source for biconformal gravity. We surmise that by taking torsion to zero, we unwittingly chose a gauge in which the momentum term that is supposed to drive gravity vanished. This suggests the need, supported by the findings of [57], that the biconformal Yang-Mills action differes from the usual $F * F$ form of Eq.(2). This possibility is under study.

# Appendix A: Covariant and contravariant forms of the Levi-Civita tensor

Here, we present the contra- and covariant forms:

Let

$$\begin{aligned}
e_{AB\cdots C} &= e^{a\cdots b}{}_{c\cdots d} \\
\bar{e}^{AB\cdots C} &= K^{AA'}K^{BB'}\cdots K^{CC'}e_{A'B'\cdots C'} \\
&= K^{AA'}K^{BB'}\cdots K^{CC'}\sqrt{K}\varepsilon_{A'B'\cdots C'} \\
&= \sqrt{K}\left(\frac{1}{K}\varepsilon^{AB\cdots C}\right) \\
&= \frac{1}{\sqrt{K}}\varepsilon^{AB\cdots C}
\end{aligned}$$

Now expand the indices,

$$\begin{aligned}
\bar{e}^{AB\cdots C} &= K^{AA'}K^{BB'}\cdots K^{CC'}e_{A'B'\cdots C'} \\
&= K^{A}{}_{a'}\cdots K^{B}{}_{b'}K^{c'}{}_{C}\cdots K^{d'}{}_{D}\sqrt{K}\varepsilon^{a'\cdots b'}{}_{c'\cdots d'} \\
&= \frac{1}{K}\delta^{a}{}_{a'}\cdots\delta^{b}{}_{b'}\delta^{c'}{}_{c}\cdots\delta^{d'}{}_{d}\sqrt{K}\varepsilon^{a'\cdots b'}{}_{c'\cdots d'} \\
&= \frac{1}{\sqrt{K}}\varepsilon^{a\cdots b}{}_{c\cdots d} \\
&= \bar{e}^{a\cdots b}{}_{c\cdots d}
\end{aligned}$$

This makes sense if we always define the antisymmetric symbol as $\varepsilon^{a\cdots b}{}_{c\cdots d}$, but see below where we would write the last step as

$$\frac{1}{\sqrt{K}}\varepsilon^{a\cdots b}{}_{c\cdots d} = \bar{\varepsilon}^{a\cdots b}{}_{c\cdots d}$$

Check the contraction. The (inverse) determinant may be defined as

$$\begin{aligned}
\frac{1}{K} &= \frac{1}{n!n!} K^{AA'} K^{BB'} \cdots K^{CC'} \varepsilon_{AB\cdots C} \varepsilon_{A'B'\cdots C'} \\
\frac{1}{\sqrt{K}} &= \frac{1}{n!n!} \left( K^{AA'} K^{BB'} \cdots K^{CC'} \varepsilon_{A'B'\cdots C'} \right) e_{AB\cdots C} \\
\frac{1}{\sqrt{K}} &= \frac{1}{n!n!} \left( \varepsilon^{AB\cdots C} \right) e_{AB\cdots C}
\end{aligned}$$

$$\begin{aligned}
\bar{e}^{AB\cdots C} e_{AB\cdots C} &= K^{AA'} K^{BB'} \cdots K^{CC'} e_{AB\cdots C} e_{A'B'\cdots C'} \\
\bar{e}^{AB\cdots C} e_{AB\cdots C} &= (2n)! \\
\bar{e}^{A\cdots BC\cdots D} e^{a\cdots b}{}_{c\cdots d} &= n!n! \\
\bar{e}_{a\cdots b}{}^{c\cdots d} e^{a\cdots b}{}_{c\cdots d} &= n!n!
\end{aligned}$$

If we had the diagonal form of the metric it would be clear:

$$\begin{aligned}
K^{AA'} K^{BB'} \cdots K^{CC'} e_{A'B'\cdots C'} &= \bar{e}^{AB\cdots C} \\
K_{aa'} \cdots K_{bb'} K^{cc'} \cdots K^{dd'} e^{a'\cdots b'}{}_{c'\cdots d'} &= \bar{e}_{a\cdots b}{}^{c\cdots d}
\end{aligned}$$

Now we must worry how to define the antisymmetric symbol. It should always be the same, so from antisymmetry we have

$$\begin{aligned}
\bar{e}_{a\cdots b}{}^{c\cdots d} &= K_{aa'} \cdots K_{bb'} K^{cc'} \cdots K^{dd'} e^{a'\cdots b'}{}_{c'\cdots d'} \\
&= \lambda \varepsilon^{a\cdots b}{}_{c\cdots d} \\
\bar{e}_{a\cdots b}{}^{c\cdots d} e^{a\cdots b}{}_{c\cdots d} &= n!n! \\
\lambda \varepsilon^{c\cdots d}{}_{a\cdots b} \sqrt{K} \varepsilon^{a\cdots b}{}_{c\cdots d} &= n!n!
\end{aligned}$$

so that

$$\lambda = \frac{1}{\sqrt{K}}$$

and

$$\bar{e}_{a\cdots b}{}^{c\cdots d} = \frac{1}{\sqrt{K}} \varepsilon^{a\cdots b}{}_{c\cdots d}$$

# Appendix B : Variation of the Yang-Mills Action

The variation of the volume form in the matter action now gives

$$\begin{aligned}
\kappa \delta_\Phi \int \mathcal{F}^* \mathcal{F} &= \kappa \int \left( \frac{1}{2} F_{mn} K^{am} K^{bn} + G^m{}_n K^a{}_m K^{bn} + \frac{1}{2} H^{mn} K^a{}_m K^b{}_n \right) F_{ab} \delta \mathbf{\Phi} \\
&+ \kappa \int \left( F_{mn} K_a{}^m K^{bn} + G^m{}_n \left( K_{am} K^{bn} - K_a{}^n K^b{}_m \right) + H^{mn} K_{am} K^b{}_n \right) G^a{}_b \delta \mathbf{\Phi} \\
&+ \kappa \int \left( \frac{1}{2} F_{mn} K_a{}^m K_b{}^n + G^m{}_n K_{am} K_b{}^n + \frac{1}{2} H^{mn} K_{am} K_{bn} \right) H^{ab} \delta \mathbf{\Phi} \\
&= \kappa \int \left( \frac{1}{2} F_{mn} K^{am} K^{bn} + G^m{}_n K^a{}_m K^{bn} + \frac{1}{2} H^{mn} K^a{}_m K^b{}_n \right) F_{ab} \left( A^c{}_d \delta^d_c \mathbf{\Phi} + D_d{}^c \delta^d_c \mathbf{\Phi} \right) \\
&+ \kappa \int \left( F_{mn} K_a{}^m K^{bn} + G^m{}_n \left( K_{am} K^{bn} - K_a{}^n K^b{}_m \right) + H^{mn} K_{am} K^b{}_n \right) G^a{}_b \left( A^c{}_d \delta^d_c \mathbf{\Phi} + D_d{}^c \delta^d_c \mathbf{\Phi} \right) \\
&+ \kappa \int \left( \frac{1}{2} F_{mn} K_a{}^m K_b{}^n + G^m{}_n K_{am} K_b{}^n + \frac{1}{2} H^{mn} K_{am} K_{bn} \right) H^{ab} \left( A^c{}_d \delta^d_c \mathbf{\Phi} + D_d{}^c \delta^d_c \mathbf{\Phi} \right)
\end{aligned}$$

and if we set the metric to null-orthonormal,

$$\kappa\delta_\Phi \int \mathcal{F}^*\mathcal{F} = \kappa \int \left(F_{ab}H^{ab} - G^b{}_a G^a{}_b\right)\left(A^c{}_d \delta^d_c + D_d{}^c \delta^d_c\right)\mathbf{\Phi}$$

The direct metric variation is more complicated, so we carry out the independent variations separately. Varying $K^{ab}$ gives

$$\begin{aligned}
\kappa\delta_B \int \mathcal{F}^*\mathcal{F} &= \kappa \int \left(\frac{1}{2}F_{mn}\delta K^{am}K^{bn} + \frac{1}{2}F_{mn}K^{am}\delta K^{bn} + G^m{}_n K^a{}_m \delta K^{bn}\right) F_{ab}\mathbf{\Phi} \\
&\quad + \kappa \int \left(F_{mn}K_a{}^m \delta K^{bn} + G^m{}_n K_{am}\delta K^{bn}\right) G^a{}_b \mathbf{\Phi} \\
&= 2\kappa \int \left(\frac{1}{2}F_{mn}B^{an}K^{bn} + \frac{1}{2}F_{mn}K^{am}B^{bn} + G^m{}_n K^a{}_m B^{bn}\right) F_{ab}\mathbf{\Phi} \\
&\quad + 2\kappa \int \left(F_{mn}K_a{}^m B^{bn} + G^m{}_n K_{am} B^{bn}\right) G^a{}_b \mathbf{\Phi}
\end{aligned}$$

Replacing the null-orthonormal form,

$$\frac{\kappa}{2}\delta_B \int \mathcal{F}^*\mathcal{F} = \kappa \int B^{bc}\left(F_{ab}G^a{}_c + F_{ac}G^a{}_b\right)\mathbf{\Phi}$$

The $C_{ab}$ variartion is similarly simple,

$$\begin{aligned}
\kappa\delta_C \int \mathcal{F}^*\mathcal{F} &= \kappa \int \left(G^m{}_n \delta K_{am}K^{bn} + H^{mn}\delta K_{am}K^b{}_n\right) G^a{}_b \mathbf{\Phi} \\
&\quad + \kappa \int \left(G^m{}_n \delta K_{am}K_b{}^n + \frac{1}{2}H^{mn}\delta K_{am}K_{bn} + \frac{1}{2}H^{mn}K_{am}\delta K_{bn}\right) H^{ab}\mathbf{\Phi} \\
&= \kappa \int \left(G^m{}_n 2C_{am}K^{bn} + H^{mn}2C_{am}K^b{}_n\right) G^a{}_b \mathbf{\Phi} \\
&\quad + \kappa \int \left(G^m{}_n 2C_{am}K_b{}^n + \frac{1}{2}H^{mn}2C_{am}K_{bn} + \frac{1}{2}H^{mn}K_{am}2C_{bn}\right) H^{ab}\mathbf{\Phi}
\end{aligned}$$

and therefore,

$$\kappa\delta_C \int \mathcal{F}^*\mathcal{F} = \kappa \int \left(H^{cb}G^a{}_b + H^{ab}G^c{}_b\right) 2C_{ac}\mathbf{\Phi}$$

Finally, we vary the metric cross terms, involving $A$ and $D$,

$$\begin{aligned}
\kappa\delta_{A,D}\int\mathcal{F}^*\mathcal{F} &= \kappa\int\left(G^m{}_n\delta K^a{}_m K^{bn}+\frac{1}{2}H^{mn}\delta K^a{}_m K^b{}_n+\frac{1}{2}H^{mn}K^a{}_m\delta K^b{}_n\right)F_{ab}\mathbf{\Phi}\\
&\quad+\kappa\int\left(F_{mn}\delta K_a{}^m K^{bn}+G^m{}_n\left(-\delta K_a{}^n K^b{}_m-K_a{}^n\delta K^b{}_m\right)+H^{mn}K_{am}\delta K^b{}_n\right)G^a{}_b\mathbf{\Phi}\\
&\quad+\kappa\int\left(\frac{1}{2}F_{mn}\delta K_a{}^m K_b{}^n+\frac{1}{2}F_{mn}K_a{}^m\delta K_b{}^n+G^m{}_n K_{am}\delta K_b{}^n\right)H^{ab}\mathbf{\Phi}\\
&= \kappa\int\left(G^m{}_n\left(A^a{}_m+D_m{}^a\right)K^{bn}+\frac{1}{2}H^{mn}\left(A^a{}_m+D_m{}^a\right)K^b{}_n+\frac{1}{2}H^{mn}K^a{}_m\left(A^b{}_n+D_n{}^b\right)\right)F_{ab}\mathbf{\Phi}\\
&\quad+\kappa\int\left(F_{mn}\left(A^a{}_m+D_m{}^a\right)K^{bn}+G^m{}_n\left(-\left(A^n{}_a+D_a{}^n\right)K^b{}_m-K_a{}^n\left(A^b{}_m+D_m{}^b\right)\right)+H^{mn}K_{am}\left(A^a{}_m\right.\right.\\
&\quad+\kappa\int\left(\frac{1}{2}F_{mn}\left(A^m{}_a+D_a{}^m\right)K_b{}^n+\frac{1}{2}F_{mn}K_a{}^m\left(A^n{}_b+D_b{}^n\right)+G^m{}_n K_{am}\delta K_b{}^n\left(A^n{}_b+D_b{}^n\right)\right)H^{ab}\mathbf{\Phi}\\
&= \kappa\int\left(\frac{1}{2}H^{mb}\left(A^a{}_m+D_m{}^a\right)+\frac{1}{2}H^{an}\left(A^b{}_n+D_n{}^b\right)\right)F_{ab}\mathbf{\Phi}\\
&\quad+\kappa\int\left(-G^b{}_n\left(A^n{}_a+D_a{}^n\right)-G^m{}_a\left(A^b{}_m+D_m{}^b\right)\right)G^a{}_b\mathbf{\Phi}\\
&\quad+\kappa\int\left(\frac{1}{2}F_{mb}\left(A^m{}_a+D_a{}^m\right)+\frac{1}{2}F_{an}\left(A^n{}_b+D_b{}^n\right)\right)H^{ab}\mathbf{\Phi}\\
&= \kappa\int\left(\left(A^n{}_a+D_a{}^n\right)H^{ba}F_{bn}-2\left(A^n{}_a+D_a{}^n\right)G^b{}_nG^a{}_b+\left(A^n{}_a+D_a{}^n\right)H^{ba}F_{bn}\right)\mathbf{\Phi}\\
&= 2\kappa\int\left(A^n{}_a+D_a{}^n\right)\left(H^{ba}F_{bn}-G^b{}_nG^a{}_b\right)\mathbf{\Phi}
\end{aligned}$$

Putting the various pieces together:

$$\begin{aligned}
\frac{\kappa}{2}\delta\int\mathcal{F}^*\mathcal{F} &= \frac{\kappa}{2}\int\left(A^n{}_a+D_a{}^n\right)\left(H^{ba}F_{bn}-G^b{}_nG^a{}_b\right)\mathbf{\Phi}\\
&\quad+\frac{\kappa}{2}\int\left(H^{cb}G^a{}_b+H^{ab}G^c{}_b\right)2C_{ac}\mathbf{\Phi}\\
&\quad+\kappa\int B^{bn}\left(G^a{}_nF_{ab}\mathbf{\Phi}+G^a{}_bF_{an}\right)\mathbf{\Phi}\\
&\quad+\frac{\kappa}{2}\int 2B^{bc}\left(F_{ab}G^a{}_c+F_{ac}G^a{}_b\right)\mathbf{\Phi}\\
&\quad+\frac{\kappa}{2}\int\left(F_{ab}H^{ab}-G^b{}_aG^a{}_b\right)\left(A^c{}_d\delta^d_c+D_d{}^c\delta^d_c\right)\mathbf{\Phi}\\
&= \kappa\int\left(A^n{}_a+D_a{}^n\right)\left(H^{ba}F_{bn}-G^b{}_nG^a{}_b+\frac{1}{2}\delta^a_n\left(F_{bc}H^{bc}-G^b{}_cG^c{}_b\right)\right)\mathbf{\Phi}\\
&\quad+\kappa\int\left(H^{cb}G^a{}_b+H^{ab}G^c{}_b\right)C_{(ac)}\mathbf{\Phi}\\
&\quad+\kappa\int\left(F_{ab}G^a{}_c+F_{ac}G^a{}_b\right)B^{(bc)}\mathbf{\Phi}
\end{aligned}$$

where we have replaced $C_{ac}=C_{(ac)}$ and $B_{ab}=B_{(ab)}$.

## Appendix C: Variation of the Co-solder form in the gravity action

Consider the $\mathbf{f}_a$ variation,

$$
\begin{aligned}
S_g &= \int \left(\alpha\mathbf{\Omega}^a{}_b + \beta\delta^a_b\mathbf{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\mathbf{f}_{cd\cdots e}\mathbf{e}^{fg\cdots h} \\
\delta_f S_g &= \int \left(\alpha\delta\mathbf{\Omega}^a{}_b + \beta\delta^a_b\delta\mathbf{\Omega} + \gamma\mathbf{e}^a\delta\mathbf{f}_b\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\mathbf{f}_{cd\cdots e}\mathbf{e}^{fg\cdots h} \\
&\quad + \int \left(\alpha\delta\mathbf{\Omega}^a{}_b + \beta\delta^a_b\delta\mathbf{\Omega} + \gamma\mathbf{e}^a\delta\mathbf{f}_b\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\left(n-1\right)\delta\mathbf{f}_c\mathbf{f}_{d\cdots e}\mathbf{e}^{fg\cdots h} \\
&= \int \left(\alpha\left(-2\Delta^{am}_{nb}\delta\mathbf{f}_m\mathbf{e}^n\right) + \beta\delta^a_b\left(-\mathbf{e}^m\delta\mathbf{f}_m\right) + \gamma\mathbf{e}^a\delta^m_b\delta\mathbf{f}_m\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\mathbf{f}_{cd\cdots e}\mathbf{e}^{fg\cdots h} \\
&\quad + \int \left(\alpha\delta\mathbf{\Omega}^a{}_b + \beta\delta^a_b\mathbf{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\left(n-1\right)\delta\mathbf{f}_c\mathbf{f}_{d\cdots e}\mathbf{e}^{fg\cdots h}
\end{aligned}
$$

Now let

$$
\delta\mathbf{f}_m = C_{ms}\mathbf{e}^s + D_m{}^s\mathbf{f}_s
$$

so that

$$
\begin{aligned}
\delta_f S_g &= \int \left(C_{ms}\mathbf{e}^s + D_m{}^s\mathbf{f}_s\right)\left(\alpha\left(-2\Delta^{am}_{nb}\mathbf{e}^n\right) + \beta\delta^a_b\mathbf{e}^m - \gamma\mathbf{e}^a\delta^m_b\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\mathbf{f}_{cd\cdots e}\mathbf{e}^{fg\cdots h} \\
&\quad + \int \left(\alpha\mathbf{\Omega}^a{}_b + \beta\delta^a_b\mathbf{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)\varepsilon^{bmd\ldots e}{}_{afg\ldots h}\left(n-1\right)\left(C_{ms}\mathbf{e}^s + D_m{}^s\mathbf{f}_s\right)\mathbf{f}_{d\cdots e}\mathbf{e}^{fg\cdots h} \\
&= \int C_{ms}\mathbf{e}^s\left(\alpha\left(-2\Delta^{am}_{nb}\mathbf{e}^n\right) + \beta\delta^a_b\mathbf{e}^m - \gamma\mathbf{e}^a\delta^m_b\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\mathbf{f}_{cd\cdots e}\mathbf{e}^{fg\cdots h} \\
&\quad + \int \left(n-1\right)C_{ms}\mathbf{e}^s\left(\alpha\mathbf{\Omega}^a{}_b + \beta\delta^a_b\mathbf{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)\varepsilon^{bmd\ldots e}{}_{afg\ldots h}\mathbf{f}_{d\cdots e}\mathbf{e}^{fg\cdots h} \\
&\quad + \int D_m{}^s\mathbf{f}_s\left(\alpha\left(-2\Delta^{am}_{nb}\mathbf{e}^n\right) + \beta\delta^a_b\mathbf{e}^m - \gamma\mathbf{e}^a\delta^m_b\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\mathbf{f}_{cd\cdots e}\mathbf{e}^{fg\cdots h} \\
&\quad + \int \left(n-1\right)D_m{}^s\mathbf{f}_s\left(\alpha\mathbf{\Omega}^a{}_b + \beta\delta^a_b\mathbf{\Omega} + \gamma\mathbf{e}^a\mathbf{f}_b\right)\varepsilon^{bmd\ldots e}{}_{afg\ldots h}\mathbf{f}_{d\cdots e}\mathbf{e}^{fg\cdots h} \\
&= \int \left(-1\right)^n\left(n-1\right)C_{ms}\frac{1}{2}\left(\alpha\Omega^a{}_b{}^{uv} + \beta\delta^a_b\Omega^{uv}\right)\varepsilon^{bmd\ldots e}{}_{afg\ldots h}\mathbf{f}_{uvd\cdots e}\mathbf{e}^{sfg\cdots h} \\
&\quad + \int \left(-1\right)^{n-1}D_m{}^s\left(-2\alpha\Delta^{am}_{nb} + \beta\delta^a_b\delta^m_n - \gamma\delta^a_n\delta^m_b\right)\varepsilon^{bcd\ldots e}{}_{afg\ldots h}\mathbf{f}_{scd\cdots e}\mathbf{e}^{nfg\cdots h} \\
&\quad + \int \left(-1\right)^n\left(n-1\right)D_m{}^s\left(\alpha\Omega^a{}_b{}^u{}_v + \beta\delta^a_b\Omega^u{}_v - \gamma\delta^u_b\delta^a_v\right)\varepsilon^{bmd\ldots e}{}_{afg\ldots h}\mathbf{f}_{sud\cdots e}\mathbf{e}^{vfg\cdots h}
\end{aligned}
$$

Now use

$$
\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} = \varepsilon^{e\cdots f}{}_{c\cdots d}\mathbf{\Phi}
$$

and

$$
\varepsilon^{a\cdots b}{}_{cd\cdots e}\varepsilon^{fd\cdots e}{}_{a\cdots b} = n!\left(n-1\right)!\delta^f_c\mathbf{\Phi}
$$

so that

$$
\begin{aligned}
\delta_f S_g &= (-1)^n (n-1) \int C_{ms} \frac{1}{2} \left(\alpha \Omega^a{}_b{}^{uv} + \beta \delta^a_b \Omega^{uv}\right) \varepsilon^{bmd\ldots e}{}_{afg\ldots h} \varepsilon^{sfg\cdots h}{}_{uvd\cdots e} \mathbf{\Phi} \\
&\quad + (-1)^{n-1} \int D_m{}^s \left(-2\alpha \Delta^{am}_{nb} + \beta \delta^a_b \delta^m_n - \gamma \delta^a_n \delta^m_b\right) \varepsilon^{bcd\ldots e}{}_{afg\ldots h} \varepsilon^{nfg\cdots h}{}_{scd\cdots e} \mathbf{\Phi} \\
&\quad + (-1)^n \int (n-1) D_m{}^s \left(\alpha \Omega^a{}_b{}^u{}_v + \beta \delta^a_b \Omega^u{}_v - \gamma \delta^u_b \delta^a_v\right) \varepsilon^{bmd\ldots e}{}_{afg\ldots h} \varepsilon^{vfg\cdots h}{}_{sud\cdots e} \mathbf{\Phi} \\
&= (-1)^n (n-1) \int C_{ms} \frac{1}{2} \left(\alpha \Omega^a{}_b{}^{uv} + \beta \delta^a_b \Omega^{uv}\right) (n-2)! (n-1)! \left(\delta^b_u \delta^m_v - \delta^m_u \delta^b_v\right) \delta^s_a \mathbf{\Phi} \\
&\quad + (-1)^{n-1} \int D_m{}^s \left(-2\alpha \Delta^{am}_{nb} + \beta \delta^a_b \delta^m_n - \gamma \delta^a_n \delta^m_b\right) (n-1)! (n-1)! \delta^b_s \delta^n_a \mathbf{\Phi} \\
&\quad + (-1)^n \int (n-1) D_m{}^s \left(\alpha \Omega^a{}_b{}^u{}_v + \beta \delta^a_b \Omega^u{}_v - \gamma \delta^u_b \delta^a_v\right) (n-2)! (n-1)! \left(\delta^b_s \delta^m_u - \delta^m_s \delta^b_u\right) \delta^v_a \mathbf{\Phi} \\
&= (-1)^n (n-1)! (n-1)! \int C_{ms} \left(\alpha \Omega^a{}_b{}^{bm} + \beta \delta^a_b \Omega^{bm}\right) \delta^s_a \mathbf{\Phi} \\
&\quad + (-1)^n (n-1)! (n-1)! \int D_m{}^s \left(\alpha (n-1) - \beta + n\gamma\right) \delta^m_s \mathbf{\Phi} \\
&\quad + (-1)^n (n-1)! (n-1)! \int D_m{}^s \left(\alpha \Omega^a{}_s{}^m{}_a - \alpha \Omega^a{}_b{}^b{}_a \delta^m_s + \beta \Omega^m{}_s - \beta \Omega^a{}_a \delta^m_s + n(n-1)\gamma \delta^m_s\right) \mathbf{\Phi} \\
&= (-1)^n (n-1)! (n-1)! \int C_{ms} \left(\alpha \Omega^a{}_b{}^{bm} + \beta \delta^a_b \Omega^{bm}\right) \delta^s_a \mathbf{\Phi} \\
&\quad + (-1)^n (n-1)! (n-1)! \int D_m{}^s \left(\alpha \Omega^a{}_s{}^m{}_a - \alpha \Omega^a{}_b{}^b{}_a \delta^m_s + \beta \Omega^m{}_s - \beta \Omega^a{}_a \delta^m_s + \left(\alpha (n-1) - \beta + n^2 \gamma\right) \delta^m_s\right) \mathbf{\Phi}
\end{aligned}
$$

## Appendix D: Curvature in the Orthonormal basis

We build an orthonormal basis. Let

$$
\begin{aligned}
\boldsymbol{\chi}^a &= \mathbf{e}^a + \alpha \eta^{ab} \mathbf{f}_b \\
\boldsymbol{\psi}_a &= \beta \eta_{ab} \mathbf{e}^b + \gamma \mathbf{f}_a
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\langle \boldsymbol{\chi}^a, \boldsymbol{\chi}^b \right\rangle &= \left\langle \mathbf{e}^a + \alpha \eta^{ac} \mathbf{f}_c, \mathbf{e}^b + \alpha \eta^{bd} \mathbf{f}_d \right\rangle \\
&= \left\langle \alpha \eta^{ac} \mathbf{f}_c, \mathbf{e}^b \right\rangle + \left\langle \mathbf{e}^a, \alpha \eta^{bd} \mathbf{f}_d \right\rangle \\
&= 2\alpha \eta^{ab} \\
\left\langle \boldsymbol{\chi}^a, \boldsymbol{\psi}_b \right\rangle &= \left\langle \mathbf{e}^a + \alpha \eta^{ac} \mathbf{f}_c, \beta \eta_{bd} \mathbf{e}^d + \gamma \mathbf{f}_b \right\rangle \\
&= \left\langle \mathbf{e}^a, \gamma \mathbf{f}_b \right\rangle + \left\langle \alpha \eta^{ac} \mathbf{f}_c, \beta \eta_{bd} \mathbf{e}^d \right\rangle \\
&= \gamma \delta^a_b + \alpha \beta \delta^a_b \\
\left\langle \boldsymbol{\psi}_a, \boldsymbol{\psi}_b \right\rangle &= \left\langle \beta \eta_{ac} \mathbf{e}^c + \gamma \mathbf{f}_a, \beta \eta_{bd} \mathbf{e}^d + \gamma \mathbf{f}_b \right\rangle \\
&= \left\langle \beta \eta_{ac} \mathbf{e}^c, \gamma \mathbf{f}_b \right\rangle + \left\langle \gamma \mathbf{f}_a, \beta \eta_{bd} \mathbf{e}^d \right\rangle \\
&= 2\beta\gamma \eta_{ab}
\end{aligned}
$$

To make the basis orthonormal and semi-Lorentzian, we need

$$\begin{aligned} 2\alpha &= 1 \\ \gamma + \alpha\beta &= 0 \\ 2\beta\gamma &= -1 \end{aligned}$$

and therefore

$$\begin{aligned} \alpha &= \frac{1}{2} \\ \beta &= -\frac{1}{2\gamma} \\ \gamma - \frac{1}{4\gamma} &= 0 \\ \gamma^2 - \frac{1}{4} &= 0 \\ \gamma &= \pm\frac{1}{2} \end{aligned}$$

Therefore, choosing $\gamma = \frac{1}{2}$,

$$\begin{aligned} \boldsymbol{\chi}^a &= \mathbf{e}^a + \frac{1}{2}\eta^{ab}\mathbf{f}_b \\ \boldsymbol{\psi}_a &= -\eta_{ab}\mathbf{e}^b + \frac{1}{2}\mathbf{f}_a \end{aligned}$$

Now, what about $\mathbf{f}_a = \mathbf{h}_a + c_{ab}\mathbf{e}^b$? We have seen that this gives

$$\langle \mathbf{h}_a, \mathbf{h}_b \rangle = -\left(b_{ab} + b_{ba}\right) = -2b_{(ab)}$$

Write this the other way around, so that

$$\mathbf{h}_a = \mathbf{f}_a - b_{ab}\mathbf{e}^b$$

Then we may identify

$$\begin{aligned} \mathbf{h}_a &= 2\boldsymbol{\psi}_a \\ &= 2\left(-\eta_{ab}\mathbf{e}^b + \frac{1}{2}\mathbf{f}_a\right) \\ &= \mathbf{f}_a - 2\eta_{ab}\mathbf{e}^b \end{aligned}$$

provides

$$b_{ab} = 2\eta_{ab}$$

This makes $b_{ab}$ into a cosmological constant term.

Then

$$\begin{aligned} \boldsymbol{\chi}^a + \eta^{ab}\psi_b &= \mathbf{e}^a + \frac{1}{2}\eta^{ab}\mathbf{f}_b + \eta^{ab}\left(-\eta_{bc}\mathbf{e}^c + \frac{1}{2}\mathbf{f}_b\right) \\ &= \eta^{ab}\mathbf{f}_b \\ \mathbf{f}_a &= \eta_{ab}\boldsymbol{\chi}^b + \psi_a \\ \boldsymbol{\psi}_a &= -\eta_{ab}\mathbf{e}^b + \frac{1}{2}\mathbf{f}_a \\ &= -\eta_{ab}\mathbf{e}^b + \frac{1}{2}\left(\eta_{ab}\boldsymbol{\chi}^b + \psi_a\right) \\ \frac{1}{2}\boldsymbol{\psi}_a - \frac{1}{2}\eta_{ab}\boldsymbol{\chi}^b &= -\eta_{ab}\mathbf{e}^b \\ \mathbf{e}^b &= \frac{1}{2}\boldsymbol{\chi}^b - \frac{1}{2}\eta^{ba}\boldsymbol{\psi}_a \end{aligned}$$

so,

$$
\begin{aligned}
\mathbf{e}^a &= \frac{1}{2}\boldsymbol{\chi}^a - \frac{1}{2}\eta^{ab}\boldsymbol{\psi}_b \\
&= \frac{1}{2}\boldsymbol{\chi}^a - \frac{1}{4}\eta^{ab}\mathbf{h}_b \\
\mathbf{f}_a &= \eta_{ab}\boldsymbol{\chi}^b + \psi_a \\
&= \eta_{ab}\boldsymbol{\chi}^b + \frac{1}{2}\mathbf{h}_a
\end{aligned}
$$

With this, the curvature becomes, in terms of $\chi$ and $\psi$:

$$
\begin{aligned}
\boldsymbol{\Omega}^a{}_b &= \frac{1}{2}\Omega^a{}_{bcd}\mathbf{e}^c\mathbf{e}^d + 2\chi\Delta^{ac}_{db}\mathbf{f}_c\mathbf{e}^d \\
&= \frac{1}{8}\Omega^a{}_{bcd}\left(\boldsymbol{\chi}^c - \eta^{ce}\boldsymbol{\psi}_e\right)\left(\boldsymbol{\chi}^d - \eta^{df}\boldsymbol{\psi}_f\right) + \frac{1}{2}\chi\Delta^{ac}_{db}\left(\boldsymbol{\psi}_c + \eta_{ce}\boldsymbol{\chi}^e\right)\left(\boldsymbol{\chi}^d - \eta^{df}\boldsymbol{\psi}_f\right) \\
&= \frac{1}{8}\Omega^a{}_{bcd}\left(\boldsymbol{\chi}^c\boldsymbol{\chi}^d + \left(\eta^{ce}\boldsymbol{\chi}^d - \eta^{de}\boldsymbol{\chi}^c\right)\boldsymbol{\psi}_e + \eta^{df}\eta^{ce}\boldsymbol{\psi}_e\boldsymbol{\psi}_f\right) + \frac{1}{2}\chi\Delta^{ac}_{db}\left(\boldsymbol{\psi}_c\boldsymbol{\chi}^d + \eta_{ce}\boldsymbol{\chi}^e\boldsymbol{\chi}^d - \eta^{df}\boldsymbol{\psi}_c\boldsymbol{\psi}_f - \eta^{df}\eta_{ce}\boldsymbol{\chi}^e\boldsymbol{\psi}_f\right)
\end{aligned}
$$

## Appendix E: Octothorpe Dual of a $p$-form

We hinted at the existence of alternative duals earlier in this work. One such possible duals, is what we called the "Octothorpe dual" denoted by the symbol $\#$ . We here present a systematic way of computing the Octothorpe dual. A general a $p$-form $\omega$ is written as

$$
\boldsymbol{\omega} = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\mathbf{d}x^{\mu_1}\wedge\ldots\wedge\mathbf{d}x^{\mu_p}
$$

Where $p\in\{0,1,...,n\}$. The dual of the above form is

$$
\begin{aligned}
{}^*\boldsymbol{\omega} &= {}^*\left(\frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\mathbf{d}x^{\mu_1}\wedge\ldots\wedge\mathbf{d}x^{\mu_p}\right) \\
&= \frac{1}{(n-p)!}\frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\varepsilon^{\mu_1\ldots\mu_p}{}_{\mu_{p+1}\ldots\mu_n}\mathbf{d}x^{\mu_{p+1}}\wedge\ldots\wedge\mathbf{d}x^{\mu_n}
\end{aligned}
$$

The double dual will be :

$$
\begin{aligned}
{}^{**}\boldsymbol{\omega} &= \frac{1}{p!}\left(\frac{1}{(n-p)!}\frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\varepsilon^{\mu_1\ldots\mu_p}{}_{\mu_{p+1}\ldots\mu_n}\right)\varepsilon^{\mu_{p+1}\ldots\mu_n}{}_{\nu_1\ldots\nu_p}\mathbf{d}x^{\nu_1}\wedge\ldots\wedge\mathbf{d}x^{\nu_p} \\
&= (-1)^q\,\boldsymbol{\omega}
\end{aligned}
$$

Where

$$
\begin{aligned}
\varepsilon^{\mu_1\ldots\mu_p}{}_{\mu_{p+1}\ldots\mu_n}\varepsilon^{\mu_{p+1}\ldots\mu_n}{}_{\nu_1\ldots\nu_p} &= \lambda\delta^{[\mu_1}_{\nu_1}\ldots\delta^{\mu_p]}_{\nu_p} \\
(-1)^q\,n! &= \lambda\frac{n!}{p!\,(n-p)!}
\end{aligned}
$$

Hence

$$
\lambda = (-1)^q\,p!\,(n-p)!
$$

A $p$-form on Biconformal space can be written as

$$
\begin{aligned}
\boldsymbol{\omega} &= \frac{1}{p!}\omega^{a_1\ldots a_p}\mathbf{f}_{a_1\ldots a_p} + \frac{1}{(p-1)!}\omega^{a_1\ldots a_{p-1}}{}_{a_p}\mathbf{f}_{a_1}\wedge\ldots\wedge\mathbf{f}_{a_{p-1}}\wedge\mathbf{e}^{a_p} + \ldots\ldots + \frac{1}{p!}\omega_{a_1\ldots a_p}\mathbf{e}^{a_1\ldots a_p} \\
&= \sum_{k=0}^{p}\frac{1}{(p-k)!k!}\omega^{a_1\ldots a_k}{}_{a_{k+1}\ldots a_p}\mathbf{f}_{a_1}\wedge\ldots\wedge\mathbf{f}_{a_k}\wedge\mathbf{e}^{a_{k+1}}\wedge\ldots\mathbf{e}^{a_p}
\end{aligned}
$$

and so

$$*\boldsymbol{\omega} = \sum_{k=0}^{n-p} \frac{1}{(n-p+k)!\,(n-k)!} \frac{(-1)^{n(p-k)}}{(p-k)!k!} \omega^{a_1\ldots a_k}{}_{a_{k+1}\ldots a_p} \varepsilon^{a_{k+1}\ldots.a_p b_1 \ldots b_{n-p+k}}{}_{a_1\ldots a_k c_{k+1}\ldots c_n} \mathbf{f}_{b_1} \wedge \ldots \wedge \mathbf{f}_{b_{n-p+k}} \wedge \mathbf{e}^{c_{k+1}\ldots c_n}$$

Thus, for an $m$-form, in an $n = p + q$ dimensional space, the general dual is

$$\#\boldsymbol{\omega} \equiv \sum_{k=0}^{2n-m} \frac{A^{n,q}_{m,k}}{(n-m+k)!\,(n-k)!} \frac{(-1)^{n(m-k)}}{(m-k)!k!} \omega^{a_1\ldots a_k}{}_{a_{k+1}\ldots a_m} \varepsilon^{a_{k+1}\ldots.a_m b_1 \ldots b_{n-m+k}}{}_{a_1\ldots a_k c_{k+1}\ldots c_n} \mathbf{f}_{b_1} \wedge \ldots \wedge \mathbf{f}_{b_{n-m+k}} \wedge \mathbf{e}^{c_{k+1}\cdots}$$

where

$$\left(A^{n,q}_{m,k}\right)^2 = 1$$

Acting again,

$$\begin{aligned}
\#\#\boldsymbol{\omega} &= \#\left(\sum_{k=0}^{m} \frac{A^{n,q}_{m,k}}{(n-m+k)!\,(n-k)!} \frac{(-1)^{n(m-k)}}{(m-k)!k!} \omega^{a_1\ldots a_k}{}_{a_{k+1}\ldots a_m} \varepsilon^{a_{k+1}\ldots.a_m b_1 \ldots b_{n-m+k}}{}_{a_1\ldots a_k c_{k+1}\ldots c_n} \mathbf{f}_{b_1} \wedge \ldots \wedge \mathbf{f}_{b_{n-m+k}} \wedge \mathbf{e}^{c}\right.\\
&= \sum_{k=0}^{2n-m} \left(\frac{A^{n,q}_{m,k} A^{n,q}_{2n-m,k}}{(n-m+k)!\,(n-k)!} \frac{(-1)^{n(m-k)}}{(m-k)!k!} \omega^{a_1\ldots a_k}{}_{a_{k+1}\ldots a_m} \varepsilon^{a_{k+1}\ldots.a_m b_1 \ldots b_{n-m+k}}{}_{a_1\ldots a_k c_{k+1}\ldots c_n}\right) (-1)^{n(n-m+k)} \varepsilon^{c_{k+1}\cdot}\\
&= \sum_{k=0}^{2n-m} \frac{A^{n,q}_{m,k} A^{n,q}_{2n-m,k}}{(n-m+k)!\,(n-k)!} \frac{1}{(m-k)!k!} (-1)^{n(m-k)} (-1)^{n(n-m+k)} \omega^{a_1\ldots a_k}{}_{a_{k+1}\ldots a_m} \varepsilon^{a_{k+1}\ldots.a_m b_1 \ldots b_{n-m+k}}{}_{a_1\ldots a_k c_{k+1}\ldots c}
\end{aligned}$$

Now use

$$\begin{aligned}
\varepsilon^{a_{k+1}\ldots.a_m b_1\ldots b_{n-m+k}}{}_{a_1\ldots a_k c_{k+1}\ldots c_n} \varepsilon^{c_{k+1}\ldots.c_n d_1 \ldots d_k}{}_{b_1\ldots b_{n-m+k} e_1 \ldots e_{m-k}} &= \varepsilon^{a_{k+1}\ldots.a_m b_1\ldots b_{n-m+k}} \varepsilon_{b_1\ldots b_{n-m+k} e_1\ldots e_{m-k}} \varepsilon_{a_1\ldots a_k c_k}\\
&= (-1)^q (-1)^{(n-m+k)(m-k)} (m-k)!\,(n-m+k)!\delta\\
&= (-1)^{nk-k} (-1)^{nm-nk-(m-k)} (m-k)!\,(n-m+\\
&= (-1)^{m(n-1)} (m-k)!\,(n-m+k)!\delta^{a_{k+1}\ldots a_m}_{e_1\ldots e_{m-k}} k!\,(n
\end{aligned}$$

and therefore

$$\begin{aligned}
\#\#\boldsymbol{\omega} &= \sum_{k=0}^{n-m} A^{n,q}_{m,k} A^{n,q}_{2n-m,k} (-1)^{n(m-k)} (-1)^{n(n-m+k)} (-1)^{m(n-1)} \omega^{a_1\ldots a_k}{}_{a_{k+1}\ldots a_m} \mathbf{f}_{a_1} \wedge \ldots \wedge \mathbf{f}_{a_k} \wedge \mathbf{e}^{a_{k+1}\ldots a_m}\\
&= \sum_{k=0}^{n-m} A^{n,q}_{m,k} A^{n,q}_{n-m,k} (-1)^{nm+n-m} \omega^{a_1\ldots a_k}{}_{a_{k+1}\ldots a_m} \mathbf{f}_{a_1} \wedge \ldots \wedge \mathbf{f}_{a_k} \wedge \mathbf{e}^{a_{k+1}\ldots a_m}
\end{aligned}$$

so the distinct duals depend on each possible

$$A^{n,q}_{m,k} A^{n,q}_{2n-m,k} = \pm 1$$

To get positivity we may drop $q, k$ and require

$$\begin{aligned}
A^n_m A^n_{n-m} (-1)^{nm+n-m} &= +1\\
A^n_m A^n_{2n-m} &= (-1)^n (-1)^{m(n-m)}
\end{aligned}$$

Suppose $m < n - m$. Then we can choose $A^n_m = (-1)^n$

$$\begin{aligned}
(-1)^{n(m-k)} (-1)^{n(n-m+k)} (-1)^{m(n-1)} &= (-1)^{nm-nk+n^2-nm+nk+mn-m^2}\\
&= (-1)^{n^2+mn-m^2}\\
&= (-1)^{n+m(n-m)}\\
&= (-1)^n (-1)^{m(n-m)}
\end{aligned}$$

### 5.0.1 Summary

| Forms | $A^{n,q}_{m,k}$ | | | $A^{n,q}_{2n-m,k}$ | | |
|---|---|---|---|---|---|---|
| 0 | $A_{0,0}$ | | | $A_{2n,0}$ | | |
| 1 | $A_{1,0}$ | $A_{1,1}$ | | $A_{2n-1,0}$ | $A_{2n-1,1}$ | |
| 2 | $A_{2,0}$ | $A_{2,1}$ | $A_{2,2}$ | $A_{2n-2,0}$ | $A_{2n-2,1}$ | $A_{2n-2,2}$ |

1. The first position of the Subscript, $A_{.,.}$represents the order of the form
2. The second position of the subscript, represents the position in the series.
3. For $m \leq n \implies k = 0, 1, ..., m$ whilst for $2n \leq m > m \implies k = m - n, m - n + 1, n$

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