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Understanding Noether’s Theorem by Visualizing the Lagrangian

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By approaching Lagrangian mechanics from a graphical perspective the implications of Noether's Theorem can be made easier to understand. Plotting the Lagrangian for classical single particle systems for one coordinate onto a position-velocity phase space along with the corresponding equations of motion can demonstrate how a system is invariant under continuous transforms in that coordinate. This invariance can be shown to be associated with a quantity in the system that's conserved via Noether's Theorem. The relationship between the symmetry of the system and conserved quantities can then be extended to fields. Invariance in this case is extended to include invariance under any continuous transform in the time, space, and field variables.

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Introduction

In 1918 mathematician Emmy Noether published a mathematical theorem relating continuous symmetry in Lagrangian systems with conservation laws, solving a problem with energy conservation Felix Klein and David Hilbert had run into with Einstein's new General Theory of Relativity.[1]

Noether's theorem for discrete (non-field) systems states that that if the action integral is invariant and stationary under a transform in the form

$$t \rightarrow t' = t + \varepsilon\tau \quad q^{\mu} \rightarrow q'^{\mu} = q^{\mu} + \varepsilon\varphi^{\mu} \qquad (1)$$

then

$$\frac{\partial L}{\partial \dot{q}^{\mu}}\varphi^{\mu} - H\tau \qquad (2)$$

is a conserved quantity. Here H is the Hamiltonian $H = L - \frac{\partial L}{\partial \dot{q}^{\mu}}\dot{q}^{\mu}$.

Hamilton's Principle[2] states that the path of a system is the path that admits a stationary action.

$$\delta S = \delta \int_{t_0}^{t_1} L(q^{\mu}, \dot{q}^{\mu}, t)\, dt = 0 \qquad (3)$$

Using the Calculus of Variations it can be shown that the integral is

$$\int_{t_0}^{t_1} \sum_{\mu} \left(\frac{\partial L}{\partial q^{\mu}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^{\mu}}\right)\delta q^{\mu}\, dt.$$

The fundamental lemma of the Calculus of Variations states that for the integral the term in the parenthesis must =0 as long as the coordinates q_i are independent. This results in the Lagrange Equations of Motion[2],

$$\frac{\partial L}{\partial q^{\mu}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^{\mu}} = 0 \; for \; \mu = 1, 2, \ldots, n. \qquad (4)$$

The action is invariant under the transform in equation (1) if the Rund-Trautman identity[1] is fulfilled

$$\frac{\partial L}{\partial q^{\mu}}\varphi^{\mu}+\frac{\partial L}{\partial \acute{q}^{\mu}}\acute{\varphi}^{\mu}+\frac{\partial L}{\partial t}\tau-H\acute{\tau}=0.$$

(5)

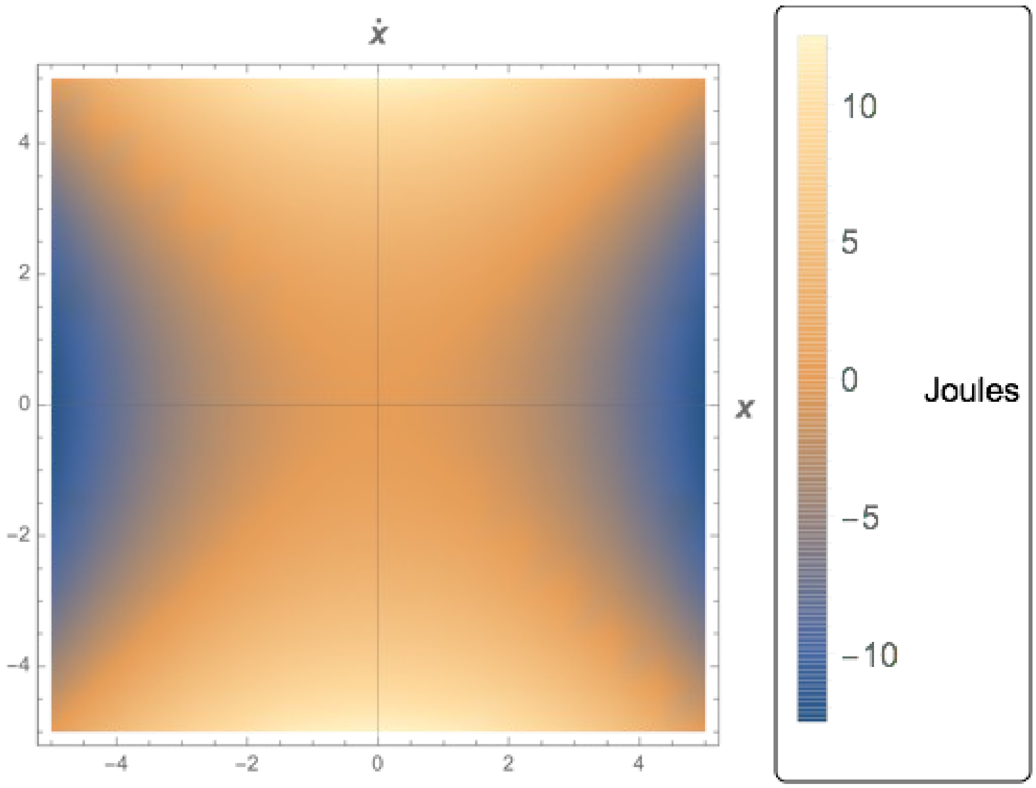
Motion of a Simple Harmonic Oscillator

The Lagrangian for a simple harmonic oscillator

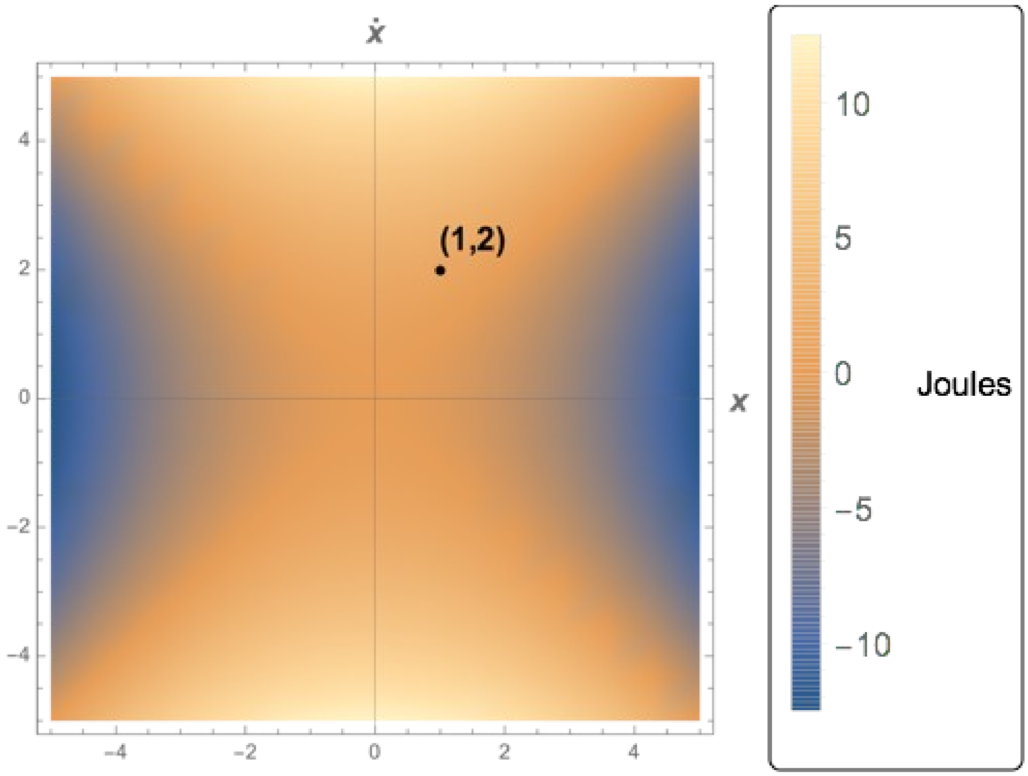
$$L=\frac{1}{2}m\acute{x}^{2}-\frac{1}{2}kx^{2}$$

(5)

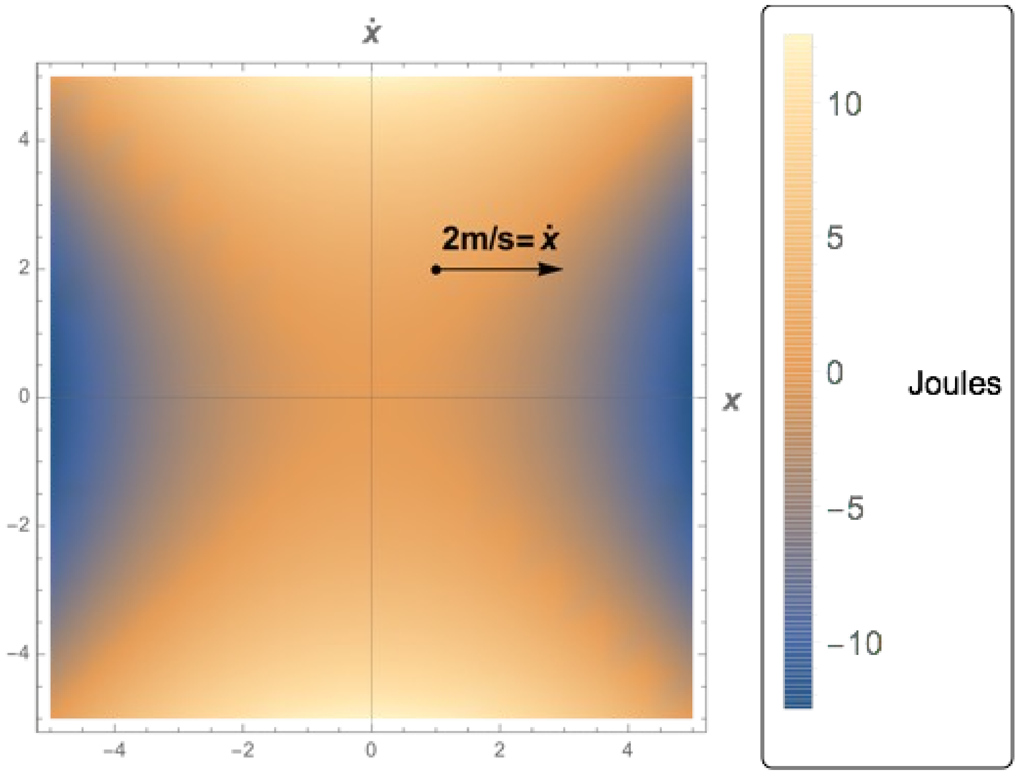
where m is the mass of the oscillator and k is the spring constant. Plotting the Lagrangian in a x-ẋ phase space can assist in understanding the behaviors of the oscillator. Setting the mass to 1kg and the spring constant to 1N/m for simplicity and results in the following plot.



Let's consider the behavior of the system at x=1 m and ẋ=2 m/s.



To figure out what the state will be after some infinitesimal time dt the values $\frac{dx}{dt}$ and $\frac{d\dot{x}}{dt}$. The value for $\frac{dx}{dt} = \dot{x} = 2\,m/s$. Plotting this as vector shows how the position component of the state changes over time.



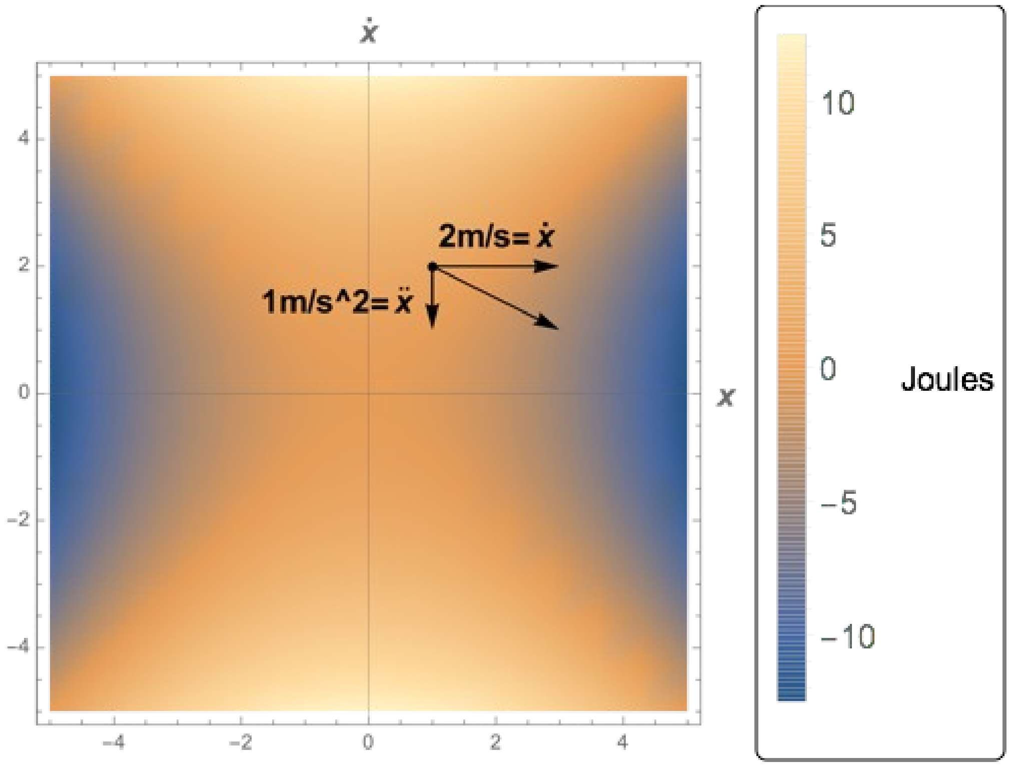
Hamilton's Principle (Equation 3) states that the value for $\dot{x}$ must admit a stationary action. From the Euler-Langrange equations

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=-kx-\frac{d}{dt}(m\dot{x})=0$$

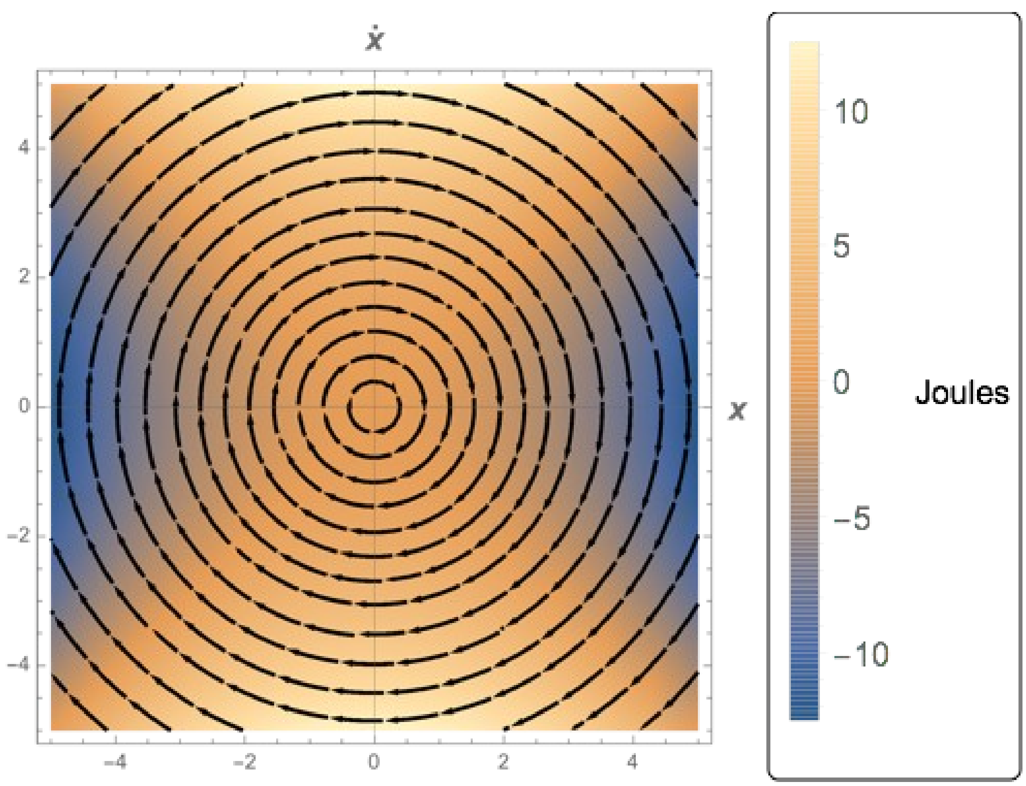
from this equation

$$\dot{x}=\frac{-kx}{m}. \quad (5)$$

When m=1kg, k=1N/m and x=1m $\dot{x}=\frac{-1m}{s^2}$. Plotting vectors representing the change in x and ẋ and adding them together gives a vector representing the change in state of the harmonic oscillator after an infinitesimal change in time at this one point.



Plotting this vector for all points as a stream plot shows how the entire system changes with time.



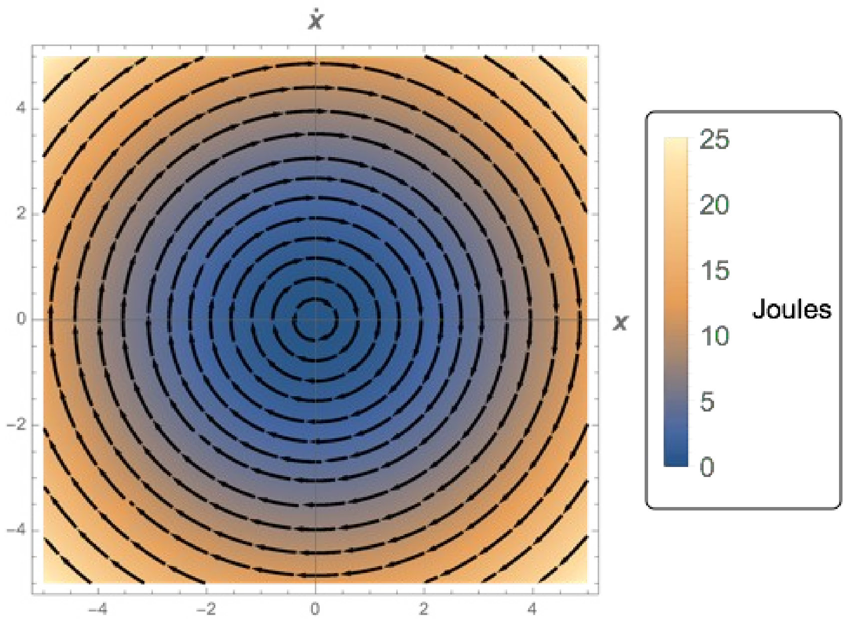
Considering the transform

$t \rightarrow t' = t + dt x \rightarrow x' = x.$

For this transform $\tau = 1$ and $\varphi = 0$, plugging these into the Rund-Trautman identity gives

$$\frac{\partial L}{\partial t} = 0 \rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2\right) = 0$$

which is true so the action is invariant under this transform. Then Noether's Theorem states that (eq. 2) the Hamiltonian is a conserved value. Plotting the path on the Hamiltonian instead of the Lagrangian shows this it does indeed fallow along constant values for the Hamiltonian.



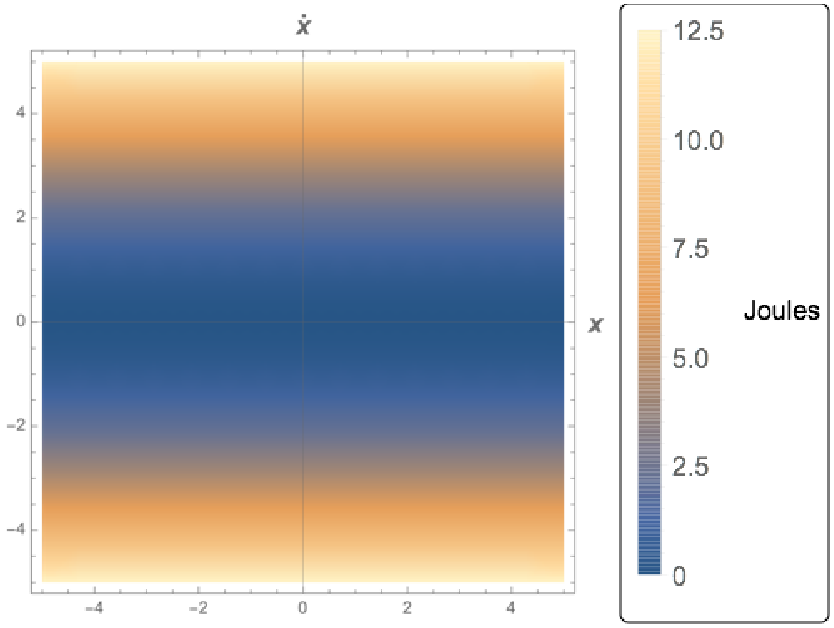
In this case the Hamiltonian is the mechanical energy so the energy of the system is conserved.

Motion of a Free Particle

The Lagrangian for a free particle is

$$L = \frac{1}{2} m \dot{x}^2. \quad (6)$$

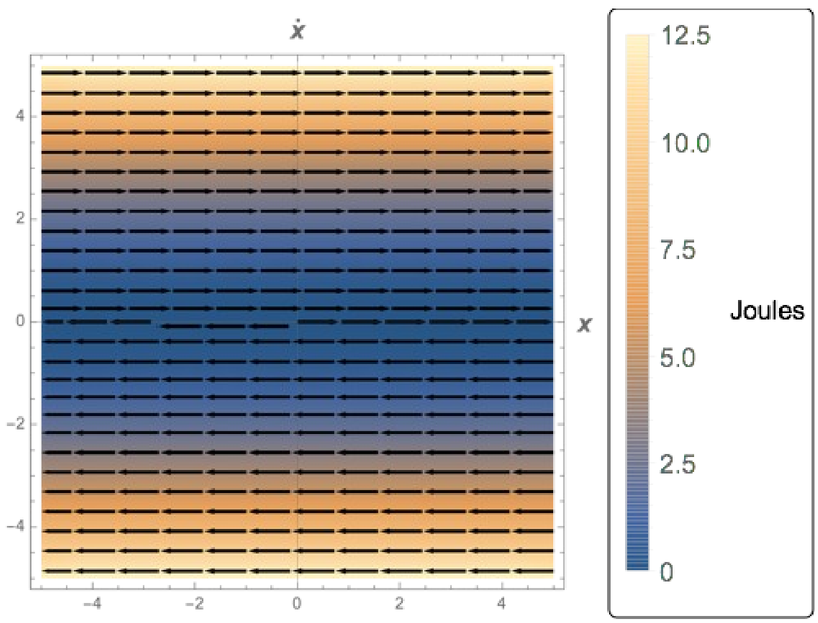
Setting the mass to 1kg and plotting results in the following plot.



Following the same process from the simple harmonic oscillator example to find the acceleration gives

$$\frac{-d}{dt}(m\dot{x})=0 \rightarrow \dot{x}=0.$$

Plotting the motion a stream plot as before.



Here it seems the motion of the particle follows a constant value for the Lagrangian. This time using the transform

$t \rightarrow t^{'} = t\, x \rightarrow x^{'} = x + dx$

$\tau = ¿$ 0 and $\varphi = 1$. The Rund-Trautman identity say that for the system to be invariant under the transform that

$$\frac{\partial L}{\partial x} = 0 \rightarrow \frac{\partial}{\partial x}\left(\frac{1}{2} m \acute{x}^2\right) = 0$$

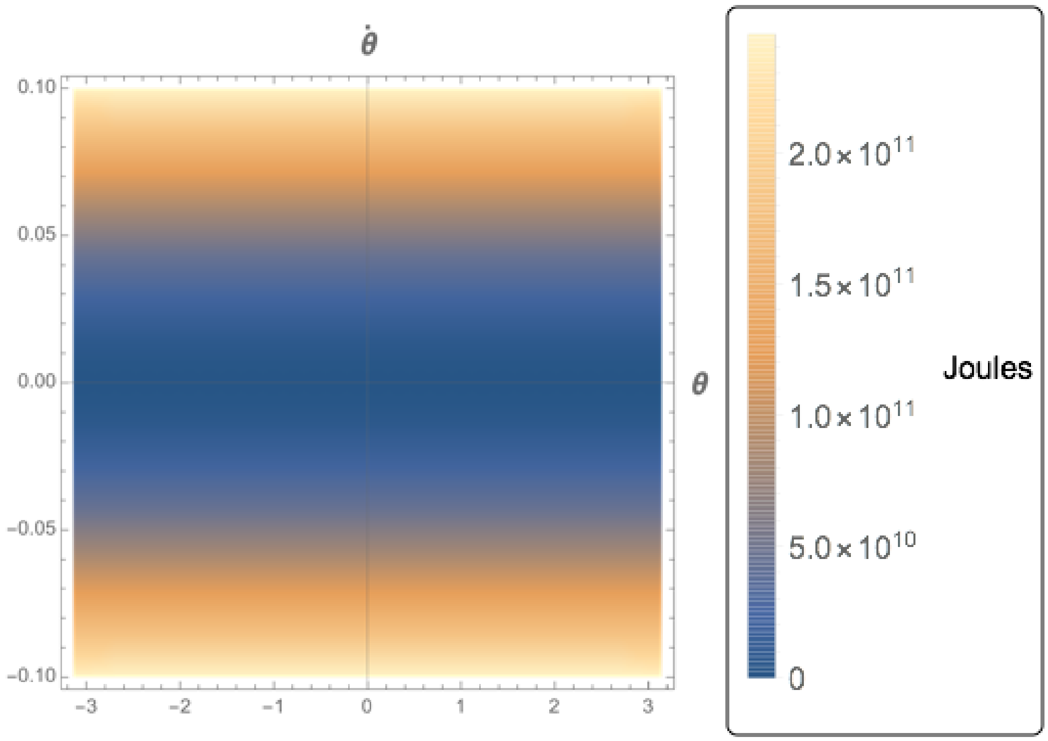
which is true so the action is invariant under the transform and by Noether's Theorem $\frac{\partial L}{\partial \acute{x}} = m \acute{x}$ is a conserved quantity but this is just the classical momentum so the momentum is conserved.

Motion of a Particle in a Spherical Potential

The Lagrangian for a particle in a gravitational field is

$$L = \frac{1}{2} m \left(\acute{r}^2 + \left(r \acute{\theta}\right)^2\right) + \frac{G M m}{r} \quad (7)$$

where G is the universal gravitational constant, M is the mass of the body being orbited, m is the mass of the orbiting particle and r and θ are the position coordinates in polar coordinates. $G = 6.67408 * 10^{11}$ $m^3 kg^{-1} s^{-2}$, $M = 10^{20}$ kg, m=1kg, $r = 7 * 10^6$ m, and $\dot{r} = 0$ m/s and plotting the Lagrangian in θ-θ' phase space gives:



This is similar to the plot from the example for a free particle. Taking a transformation along the θ-axis

$t \rightarrow t' = tr \rightarrow r' = r\theta \rightarrow \theta' = \theta + d\theta$

gives $\tau = ¿\, 0$, $\varphi^r = 0$ and $\varphi^\theta = 1$. Under this transform the Rund-Trautman identity becomes

$$\frac{\partial L}{\partial \theta}\varphi^\theta = 0 \rightarrow \frac{\partial}{\partial \theta}\left(\frac{1}{2}m\left(\acute{r}^2 + \left(r\acute{\theta}\right)^2\right) + \frac{GMm}{r}\right) = 0$$

which is true so the action is invariant under the transform and Noether's Theorem says that

$$\frac{\partial L}{\partial \acute{\theta}}\varphi^\theta = mr^2\acute{\theta}$$

is a conserved value and this is the formula for the angular momentum so the angular momentum is conserved.

Noether's Theorem for Fields

The version of Noether's Theorem stated in the introduction is a special case of a more general version of Noether's Theorem. By including systems involving fields Noether's Theorem becomes broader and more useful. Generally, for a system involving fields the Lagrangian isn't known or easy to find and the Lagrangian Density (L) is used instead. The relationship between the Lagrangian and the Lagrangian Density is

$$L=\iiint L(x^{\mu},\eta_{\rho},\eta_{\rho,v})(dx^{3})$$

where x^{μ} represents ¿, q^{3} ¿,η_{ρ} represents the field equations and $\eta_{\rho,v}=\frac{\partial\eta_{\rho}}{\partial x^{v}}$.

In this case the action integral takes the form

$$S=\int_{\Omega} L\left(x^{\mu},\eta_{\rho},\eta_{\rho,v}\right)\left(dx^{4}\right). \quad (8)$$

Appling Hamilton's Principle (eq. 3) to this and solving via the calculus of variations yields the Euler-Lagrange Equations

$$\frac{d}{dx^{v}}\left(\frac{\partial L}{\partial\eta_{\rho,v}}\right)=\frac{\partial L}{\partial\eta_{\rho}} \quad (9)$$

For such a system Noether's Theorem states that if the action integral is invariant and stationary under a transform in the form

$$x^{v}\rightarrow x^{'v}=x^{v}+\varepsilon_{r}X_{r}^{v}\eta_{\rho}\rightarrow\eta_{\rho}^{'}=\eta_{\rho}+\varepsilon_{r}\psi_{r\rho}$$

where the generators X_{r}^{v} and $\psi_{r\rho}$ can now depend on and of the coordinate or field variables, then there are r conserved currents with conservation equations[2]

$$\frac{d}{d x^{\nu}}\left[H_{\sigma}^{\nu} X_r^{\sigma} - \frac{\partial L}{\partial \eta_{\rho,\nu}} \psi_{r\rho}\right] = 0 \quad (10)$$

where the Hamiltonian Density, H_{σ}^{ν}, is

$$H_{\sigma}^{\nu} = \frac{\partial L}{\partial \eta_{\rho,\nu}} \eta_{\rho,\sigma} - L \delta_{\sigma}^{\nu}$$

Supplementary Materials

TABLE 1. Supplementary materials in "SMoser_Noether's_Theorem.zip".

Filename	Content
Noether's_Theorem_Report.docx	This report
Noether_Plots.nb	Mathematica Notebook used to generate plots

Acknowledgments

S.M. would like to thank Dr O. Varela for his time mentoring on this project.

References

1. **Neuenschwander DE.** *Emmy Noethers wonderful theorem*. Baltimore, MD: Johns Hopkins University Press, 2017.

2. **Goldstein H, Poole C, Safko J.** *Classical mechanics*. San Francisco: Addison Wesley, 2002.