Spring 2020

Lecture Notes: Multi-Objective Optimization

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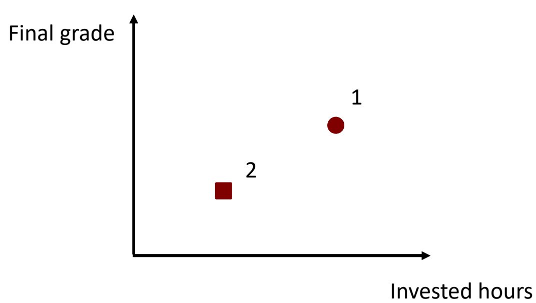
WHAT STARTS HERE CHANGES THE WORLD

Multi-objective optimization

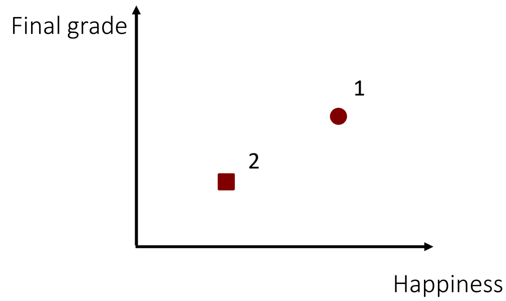
Lecture 19

Tuesday, April 14



Multi-Objective Optimization (MOO)

- Involve **more than one objective** function that are to be minimized or maximized
- Objectives are **competing**
- Multi-objective analysis is used to reveal the **tradeoff** among different objectives
- Find the set of solutions that define the **best tradeoff** between competing objectives



Single-Objective Optimization

$$
\begin{aligned}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \le 0 \quad i = 1, \ldots, n \\
& h_j(x) = 0 \quad j = 1, \ldots, k
\end{aligned}
$$

- $f(x)$ **scalar** objective **function**
- x_n vector of decision variables
- $g_i(x)\ i = 1, \ldots, k$ inequality constraint functions
- $h_j(x)\ i = 1, \ldots, m$ equality constraint functions

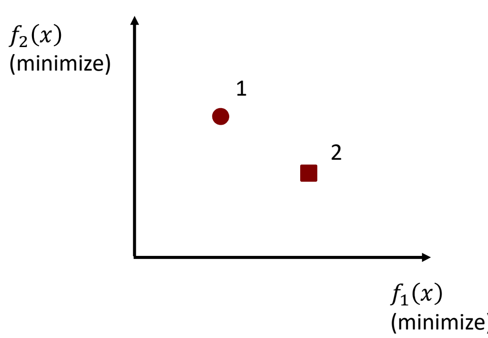
Multi-Objective Optimization

$$\begin{array}{ll} \text{minimize} & f_1(x) \\ \text{maximize} & \vdots \\ \text{minimize} & f_m(x) \\ \text{subject to} & \\ & g_i(x) \le 0 \\ & h_j(x) = 0 \end{array}$$

- $f(x)$: $\boldsymbol{M}$ scalar objective **functions**
- x_n vector of decision variables
- $g_i(x)$ $i = 1, \ldots, k$ inequality constraint functions
- $h_j(x)$ $i = 1, \ldots, m$ equality constraint functions

Dominance

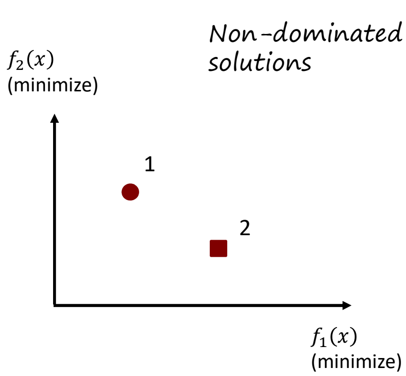
- In **single-objective** optimization problem, the **superiority** of a solution over other solutions is easily determined by comparing their **objective function** values
- In **multi-objective** optimization, the **goodness** of a solution is determined by the **dominance**

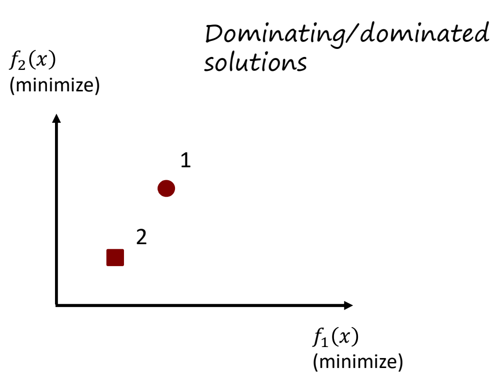


Dominance test

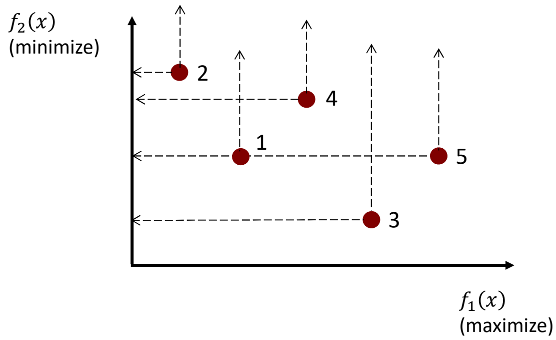
- x_1 dominates x_2, if:
  - ✓ Solution x_1 is **no worse** than x_2 in **all** objectives
  - ✓ Solution x_1 is **strictly better** than x_2 in **at least one** objective
- If x_1 **dominates** x_2, then x_2 is **dominated** by x_1
- If x_1 does not dominate x_2 and is not dominated by x_2, then x_1 and x_2 are **non-dominated** solutions

Dominance test





Dominance test

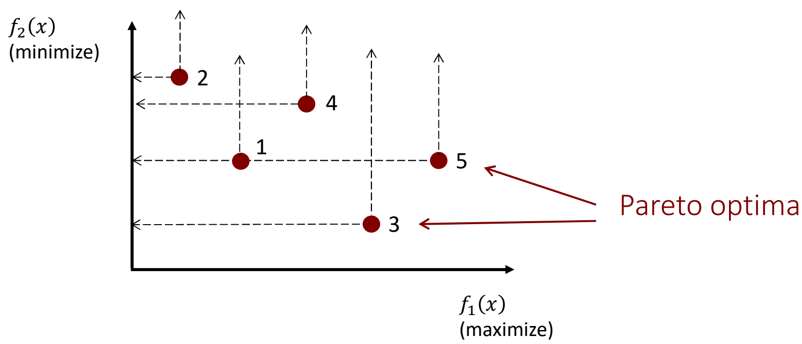


- 1 vs. 2: 1 dominates 2
- 1 vs. 5: 1 is dominated by 5
- 1 vs. 4: 1 & 4 are non-dominated solutions

Pareto optimal solutions

- Given a set of solutions, the **non-dominated solution set** is a set of all the solutions that are **not dominated** by any member of the solution set.
- The non-dominated set of the entire feasible decision space is called the **Pareto-optimal set**.
- The boundary defined by the set of all points mapped from the Pareto optimal set is called the **Pareto-optimal front**.

Dominance test



- 1 vs. 2: 1 dominates 2
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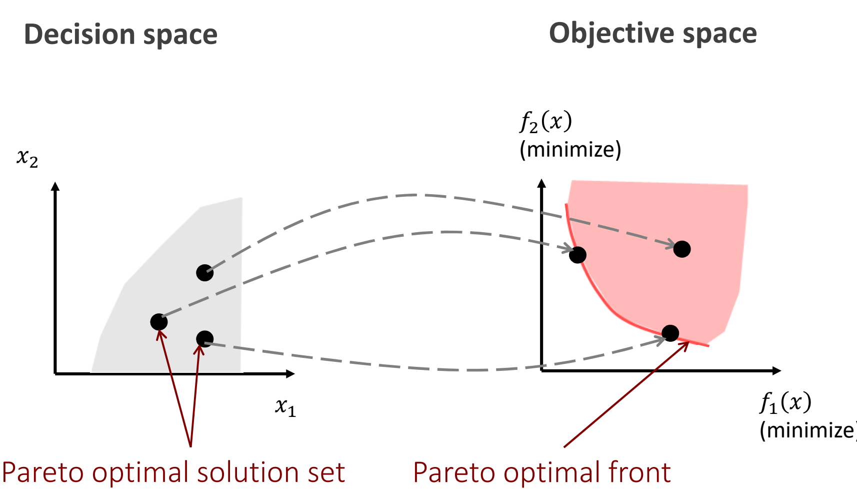
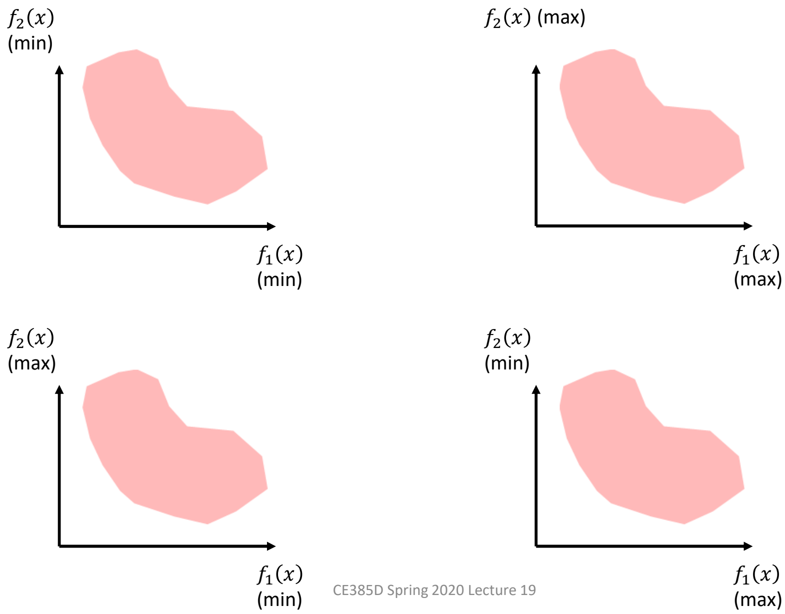
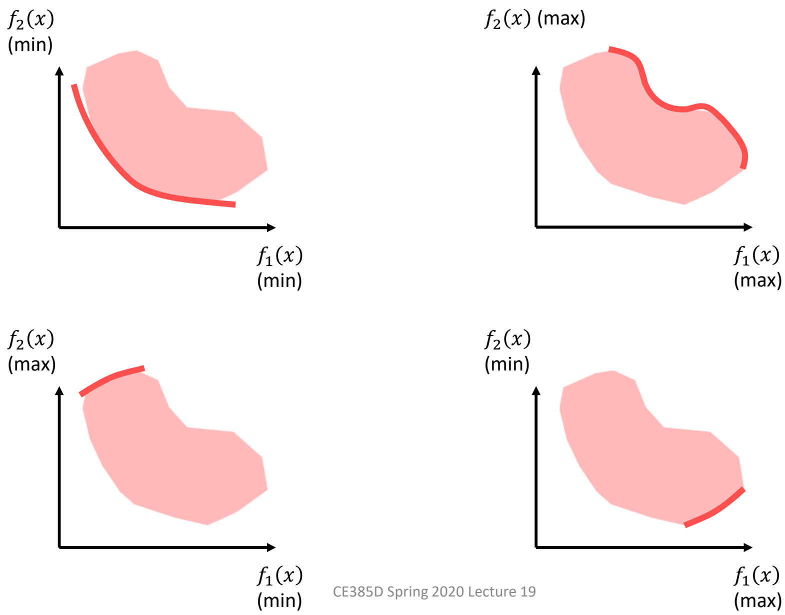
Pareto optimal



Examples

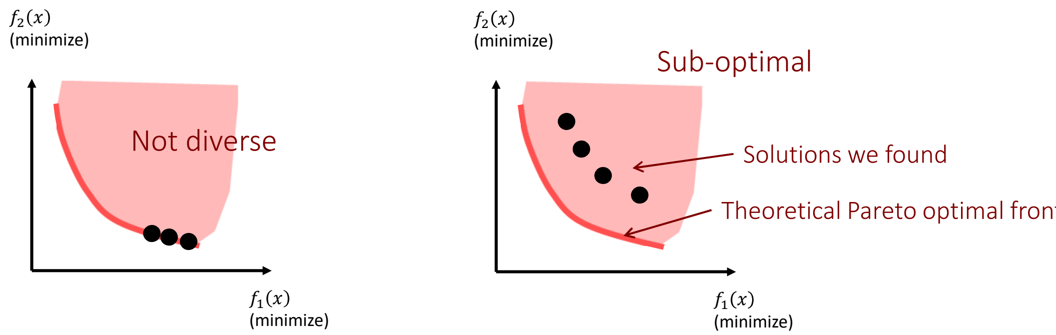


Examples



Goals of MOO

- Find a set of solutions as **close** as possible to Pareto-optimal front
- Find a set of solutions as **diverse** as possible



Classic MOO methods

Classic MOO methods

- **Weighted sum method**
- **ε-Constraint method**
- Weighted metric method
- Rotated weighted metric method
- Dynamically changing the ideal solution
- Benson's method
- Value function method

For further reading:

- *Multi-objective optimization using evolutionary algorithms*. Kalyanmoy Deb.
- *Nonlinear multiobjective optimization*. Kaisa Miettinen.

Weighted sum method

- Scalarize a set of objectives into a **single** objective by adding each objective multiplied by a user-specified **weight**
- Weight of an objective is chosen in proportion to the relative importance of the objective

$$\left.\begin{array}{ll} \text{minimize} & f_1(x) \\ \text{maximize} & \vdots \\ \text{minimize} & f_m(x) \end{array}\right\} \quad \text{minimize} \quad w_1 f_1(x) + w_2 f_2(x) + \ldots + w_m f_m(x)$$

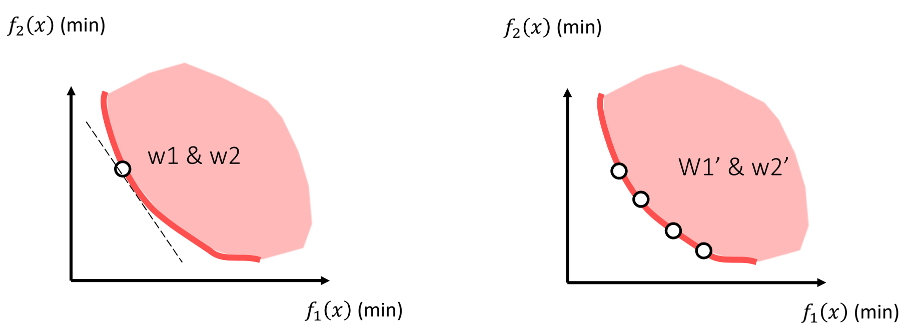
Weighted sum method

$$\begin{aligned} \text{minimize} \quad & F(x) = \sum_{m=1}^{M} w_m f_m(x) & \\ \text{subject to} \quad & g_i(x) \le 0 & i = 1,\dots,n \\ & h_j(x) = 0 & j = 1,\dots,k \end{aligned}$$

- convert all the objectives into one type
- the objectives are normalized
- $w_m \in [0,1]$ is the weight of the m-th objective function
- it is usual practice to choose weights such that: $\sum w_m = 1$
- **Nonlinear** problems are **sensitive** to scale

Weighted sum method

$$\begin{aligned} \text{minimize} \quad & w_1 f_1(x) + w_2 f_2(x) \\ \text{subject to} \quad & g_i(x) \le 0 \\ & h_j(x) = 0 \end{aligned}$$



Weighted sum method

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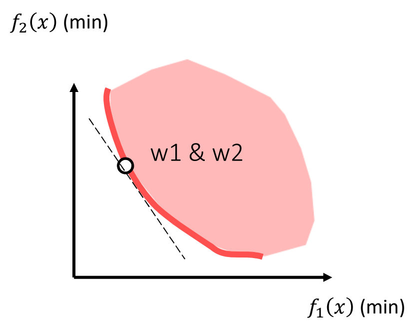
Example: $f_1(x) = 5x_1 + \exp(x_2) \quad f_2(x) = -3x_1$

$$\Downarrow$$

$$F(x) = w_1\left(5x_1 + \exp(x_2)\right) + w_2 \cdot -3x_1$$



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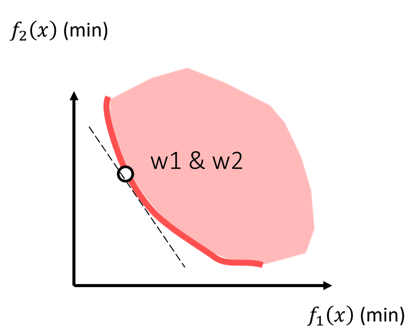
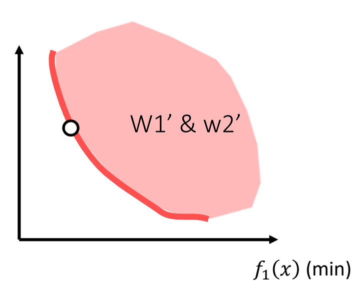
$$F(x) = w_1\left(5x_1 + \exp(x_2)\right) + w_2 \cdot -3x_1$$

Choose weights:

$w_1 = 0.3 \quad w_2 = 0.7 \quad F(x) = -0.6x_1 + 0.3\exp(x_2)$





Weighted sum method

minimize $w_1 f_1(x) + w_2 f_2(x)$
subject to $g_i(x) \le 0$
$h_j(x) = 0$

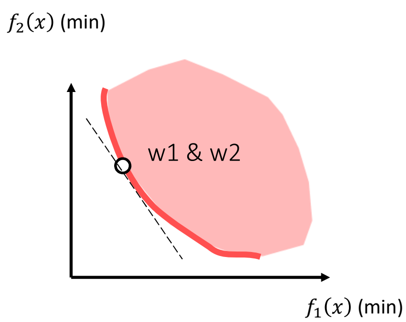
Example: $f_1(x) = 5x_1 + \exp(x_2) \quad f_2(x) = -3x_1$

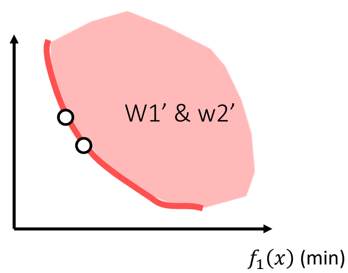
$$\Downarrow$$

$$F(x) = w_1(5x_1 + \exp(x_2)) + w_2 \cdot -3x_1$$

Choose weights:

$w_1 = 0.7 \quad w_2 = 0.3 \quad F(x) = 2.6x_1 + 0.7\exp(x_2)$





Weighted sum method

minimize $w_1 f_1(x) + w_2 f_2(x)$
subject to $g_i(x) \le 0$
$h_j(x) = 0$

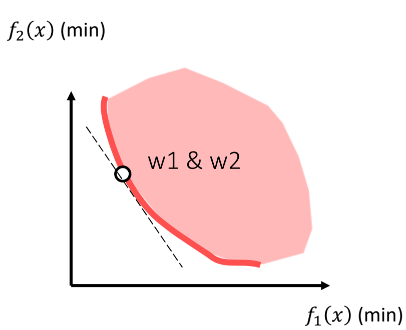
Example: $f_1(x) = 5x_1 + \exp(x_2) \quad f_2(x) = -3x_1$

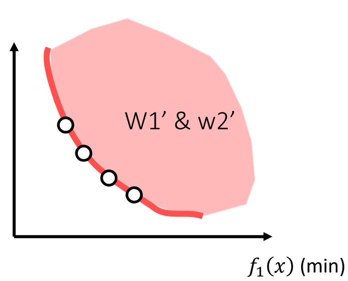
$$\Downarrow$$

$$F(x) = w_1(5x_1 + \exp(x_2)) + w_2 \cdot -3x_1$$

Choose weights:

$w_1 = \ldots \quad w_2 = \ldots \quad F(x) = \ldots$

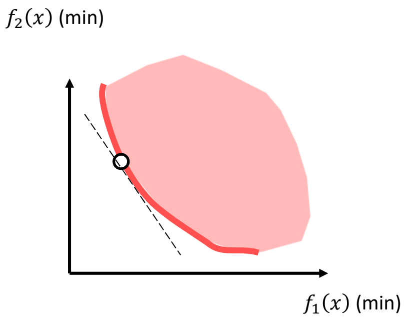


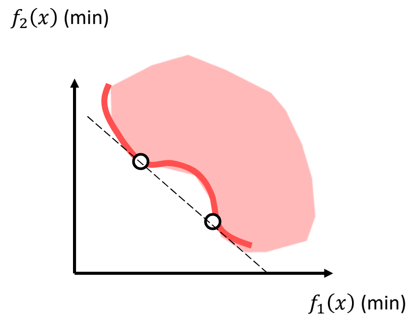


Weighted sum method

$$\begin{aligned} \text{minimize} \quad & w_1 f_1(x) + w_2 f_2(x) \\ \text{subject to} \quad & g_i(x) \le 0 \\ & h_j(x) = 0 \end{aligned}$$

Two different set of weights not necessarily lead to two different Pareto-optimal solutions





Weighted sum method

- **Advantages**
  - ✓ Simple and easy to use
  - ✓ For convex problems it guarantees to find solutions on the entire Pareto-optimal set

Weighted sum method

- **Disadvantages**
  - ✓ For mixed problems (min-max), we need to convert all the objectives into one type
  - ✓ Uniformly distributed set of weights does not guarantee a uniformly distributed set of Pareto-optimal solutions
  - ✓ Two different set of weights not necessarily lead to two different Pareto-optimal solutions
  - ✓ Cannot find certain Pareto-optimal solutions in the case of a nonconvex objective space

ε-Constraint method

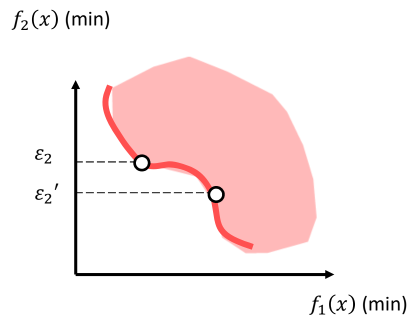
- Keep just **one** of the objectives
- Treat the rest as **constraints** and modify the constraint value.

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{maximize} && \vdots \\ &\text{minimize} && f_m(x) \\ &\text{subject to} && \\ & && g_i(x) \le 0 \\ & && h_j(x) = 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} &\text{minimize} && f_l(x) && \\ &\text{subject to} && f_m(x) \le \varepsilon_m && m = 1,\ldots,M;\, m \ne l \\ & && g_i(x) \le 0 && i = 1,\ldots,n \\ & && h_j(x) = 0 && j = 1,\ldots,k \end{aligned}$$

ε-Constraint method

$$
\begin{aligned}
\text{minimize} \quad & f_1(x) \\
\text{minimize} \quad & f_2(x) \\
\text{subject to} \quad & g_i(x) \le 0 \\
& h_j(x) = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{minimize} \quad & f_1(x) \\
\text{subject to} \quad & f_2(x) \le \varepsilon_2 \\
& g_i(x) \le 0 \\
& h_j(x) = 0
\end{aligned}
$$



ε-Constraint method

$$
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\text{minimize} \quad & f_1(x) \\
\text{minimize} \quad & f_2(x) \\
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& h_j(x) = 0
\end{aligned}
$$

Example:

$f_1(x) = 5x_1 + \exp(x_2) \quad f_2(x) = -3x_1$

$$
\begin{aligned}
\text{minimize} \quad & f_1(x) \\
\text{subject to} \quad & f_2(x) \le \varepsilon_2 \\
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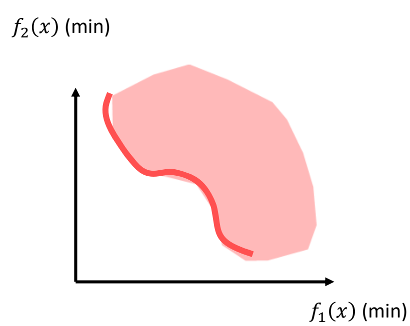
$$



ε-Constraint method

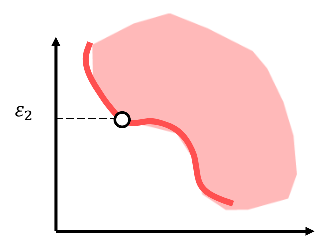
minimize $f_l(x)$
minimize $f_2(x)$
subject to $g_i(x) \le 0$
$h_j(x) = 0$

Example:

$f_1(x) = 5x_1 + \exp(x_2)$ $\quad f_2(x) = -3x_1$

$\varepsilon_2 = 10$ $\quad -3x_1 \le 10$

minimize $f_1(x)$
subject to $f_2(x) \le \varepsilon_2$
$g_i(x) \le 0$
$h_j(x) = 0$



ε-Constraint method

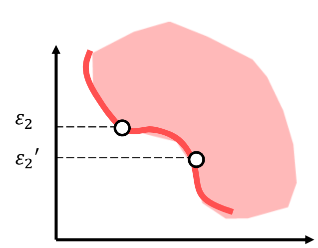
minimize $f_l(x)$
minimize $f_2(x)$
subject to $g_i(x) \le 0$
$h_j(x) = 0$

Example:

$f_1(x) = 5x_1 + \exp(x_2)$ $\quad f_2(x) = -3x_1$

$\varepsilon'_2 = 1$ $\quad -3x_1 \le 1$

minimize $f_1(x)$
subject to $f_2(x) \le \varepsilon_2$
$g_i(x) \le 0$
$h_j(x) = 0$



ε-Constraint method

- **Advantages**
  - ✓ Different Pareto-optimal solutions can be found using different ε values
  - ✓ Applicable to either convex or non-convex problems

Disadvantages

- ✓ The vector has to be chosen carefully so that it is within the minimum and the maximum values of the individual objective function

Making decisions

Making decisions

- You are working on a water resources plan for a large river basin that includes the objectives of municipal and industrial (M&I) water supply, irrigation of cotton, and hydropower.
- The planning committee has proposed 13 planning alternatives for the basin.
- You and your technical staff have assessed the material benefits expected in each objective under each plan.
- Here are the results:

Making decisions

Alternative	Hydropower	M&I Supply	Agriculture
1	0	500	0
2	250	400	0
3	360	50	0
4	0	400	5000
5	150	375	5000
6	150	300	5000
7	290	50	5000
8	250	300	5000
9	0	200	9000
10	50	175	9000
11	95	135	9000
12	50	135	9000
13	150	50	9000

Making decisions

- Are any of these planning alternatives sub-optimal? If so, how would you explain the meaning of sub-optimality to a member of the planning committee?

- You need to determine and explain the trade-off between the alternatives to the planning committee. Suggest different visualization plots you might use to illustrate the trade-off.