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300 Years 'de Motu Aquae Mixto': What Poleni Really Wrote and a New Overflow Theory Based on Momentum Balance

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300 Years 'De motu aquae mixto': What Poleni really wrote and a new Overflow Theory based on Momentum Balance

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Abstract: *In 1717 Johannes Poleni (Poleni, 1717) published a book called De motu aquae mixto which nowadays is cited as the origin of the famous Poleni weir formula. This book contains two separate booklets. The first booklet is on a general discharge theory for a water body consisting of a blocked dead water height at the bottom and a free-flowing vivid water column above that dead water column. Examples for such a situation include the overflow over a weir or over dunes, or the flow from deeper coastal water into a shallow tidal lagoon. Poleni conducted several experiments using a well-designed device and derived by fitting his data a general discharge formulation depending on the dead and vivid water heights. Later the dependency on the dead water (i.e. the weir height) was forgotten, and the Poleni formula is only cited in the form as we know it today.*

Keywords: *Poleni formula, momentum balance, overflow, weirs*.

1. Introduction

In the 17th century, two famous books were regarded as the theoretical basis of open channel and fluvial hydraulics, The first book 'Della misura dell' aque correnti' was written 1639 by Benedetto Castelli. The second book was Torricelli's 'Opera Geometrica' from 1644. In his book Castelli described the continuity equation $Q = v_1A_1 = v_2A_2$ with the following words in the English translation (1661): 'Now applying all that hath been said neerer to our purpose, I consider, that it being most true, that in divers parts of the same River or Current of running water, there doth always passe equal quantity of water in equal time (which thing is also demonstrated in out first Proposition) and it being also true, that in divers parts the same River may have various and different velocity; it follows of necessary consequence, that where the River hath lesse velocity, it shall be of greater measure, and in those parts, in which it hath greater velocity, it shall be of lesse measure; and in sum, the velocity of several parts of the said River, shall have eternally reciprocall and like proportion with their measures. This principle and fundamental well established, that the same Current of Water changeth measure, according to its varying of velocity; that is, lessening the measure, when the velocity encreaseth, and encreasing the measure, when the velocity decreaseth;'

The continuity equation relates the cross section ('measure') of a river and, therefore the water depth to the velocity, but it does not answer the question why the velocity is smaller or larger in a certain cross section. Castelli here assumed, that the velocity is proportional to the water depth itself, leading to $Q= vA \sim h\, Bh \sim h^2$. As a matter of fact, this kind of relation can be observed in every river: With increasing discharge, the elevation of the free surface becomes higher and higher.

In 1644 Evangelista Torricelli published the 'Opera geometrica' with his outflow theory stating that the outflow velocity from a vessel is equal to the filling height within the vessel. Although Torricelli nowhere wrote anything on the application of this formula to open channels or rivers, it seems to be applied to that subject in the following way: When the square of the velocity is proportional to the water depth, then the square of the discharge should also be proportional to the water depth, $v^2 \sim h$, i.e. $h \sim Q^2$. An increase in water depth of a river with the square of the discharge has never been observed in any river; otherwise, we would have many inundations.

From Poleni's introduction, a dispute is documented on the question of whether Castelli or Torricelli was correct, although the latter never mentioned an application of his theory in open channel hydraulics. Poleni cited some scientists who followed Torricelli's hypothesis like Magiotti, Baliani, and Mariotte, and some who followed Castelli, like Baraterius. At the time of Poleni, it was believed that there is a general physical law behind outflow from a vessel and open channel discharge. The role of pressure on the one hand and of slope on the other was not really understood.

Poleni also mentioned Eschinardi who distinguished between the outflow and the open channel discharge problem: 'And that nothing is forgotten here, I have to mention the famous P. Eschinardi and his book on the impetus from 1684, who took a position between the two positions by arguing that the outflow velocity from orifices in vessels is proportional to the square root of the filling height and that the velocity of water coming out of a rectangular channel is proportional to the height of the water in the channel" (Poleni, p. 14) .

But it was Poleni's historical achievement to mediate between the wrong application of Torricelli's outflow theory and Castelli's open channel hydraulics. He found out that we have to integrate the flow velocity over a cross section, which leads to totally different results. But as a matter of fact, Poleni included some correct and some incorrect assumptions to the road of scientific knowledge in hydraulics.

2. Poleni's Theory on simple and mixed motions

In a lot of situations, a body of water can be separated into moving and resting parts. The most important case of such a situation is a weir as it is shown in Figure 1. In front of the weir's canvas the water is at rest. Poleni called such bodies of water 'dead water.' The upper part of the water column above the weir's crest is flowing, and the moving water body is called 'vivid water.' Poleni's hydraulic theory is based on the question of how the discharge in such a situation is related to the vivid and the dead water heights. He further called the situation where the whole water column is moving a simple motion, while he named the case of vivid and dead water bodies a mixed motion.

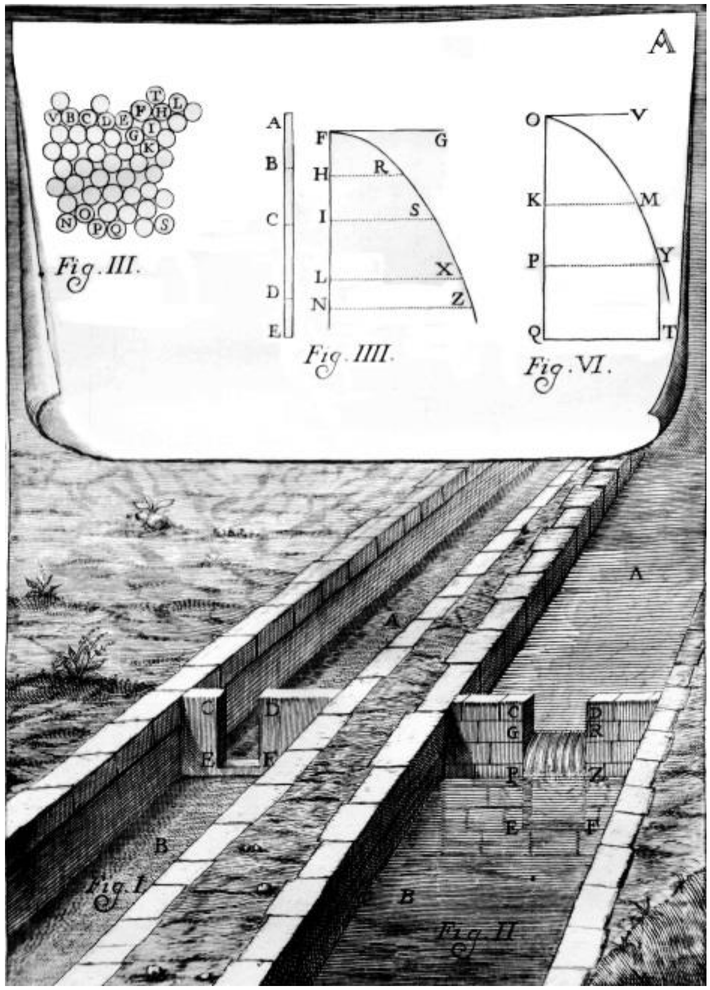

Figure 1. Poleni's first plate for the definition of vivid and dead water and the velocity profile.

This very famous figure from Poleni's original work leads to the impression that Poleni was working on a relation between the discharge and the hydraulic head for the weir overflow situation. Actually the figure is just an example where vivid and dead water bodies can occur. Poleni's experimental results were related to another hydraulic configuration which is shown in figure 2 and discussed later.

But the weir overflow just acts as an example where the velocities directly over the crest are larger than the velocities at the free surface. This is not the case in an undisturbed open channel flow, where the highest velocities can be found at the free surface. But Poleni generalized the situation over the crest of a weir and assumed a parabolic velocity in every free surface flow starting with the smallest velocities at the free surface. Therefore, he assumed that the velocity in a certain depth z can be calculated according to Torricelli's formula, which was written at that time as:

$$v(z)=\sqrt{pz} \qquad [1]$$

The value for earth gravity acceleration g was not known at that time. Today we would identify the coefficient p to be 2g and call the formula the Torricelli theorem.
To obtain the specific discharge Poleni integrated the velocity over the vivid water depth and got:

$$q=\frac{3}{2}\sqrt{ph_L}\,h_L \qquad [2]$$

This formula is referred as the Poleni weir formula or Poleni formula although it was originally a discharge formula for open channels flows. As a matter of fact this formulation is quite correct when we compare its behavior i.e. with the Chezy formula leading also to $q \sim \sqrt{h_L}\,h_L$ for a wide channel. The difference between the Chezy and the Poleni formula is the fact that Chezy also takes the channels slope into account.

For the mixed motion he now stated that the discharge is proportional to the fall velocity out of the vivid water depth and proportional to the total water height:

$$q=\frac{3}{2}\sqrt{Ph_L}\,(h_L+h_T) \qquad [3]$$

Here a new coefficient P was introduced and has to be fitted by experimental data.

3. Poleni's Experiment

In order to determine the coefficient P depending on the living and the dead water depth, Poleni constructed an experimental device as shown in Fig. 2.

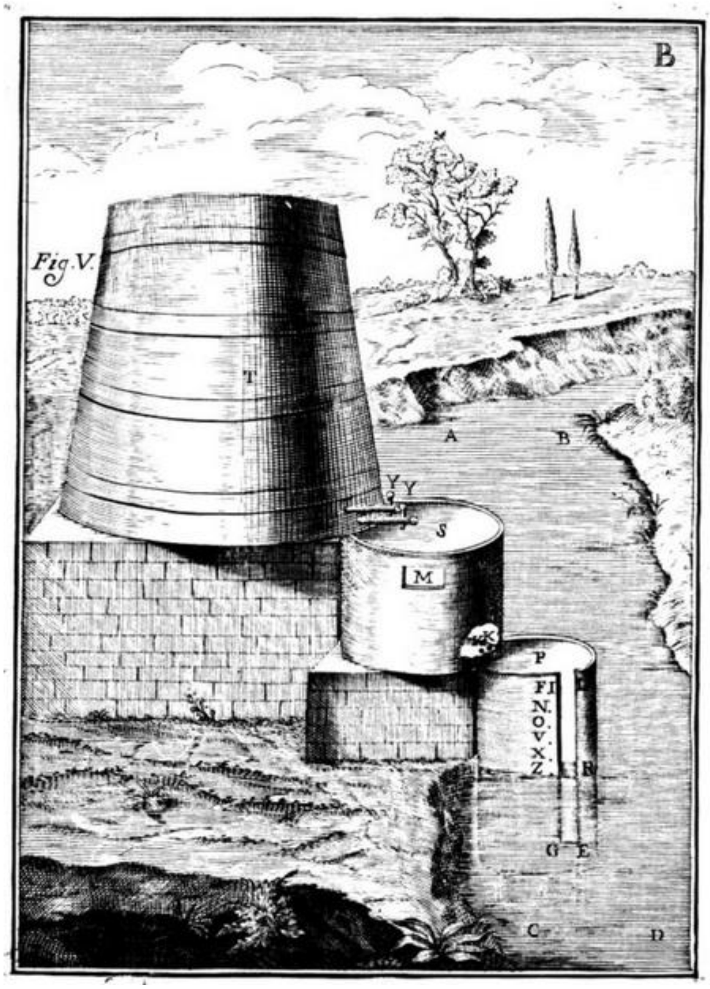

Figure 2. Poleni's experiment to derive a discharge formula for the mixed motion.

It consists first of all of a tank T for the storage of the water used in that experiment. The tank T supplies water to a vessel S (diameter 1.137m), in which the water level remains at a constant value of 56.65cm because of a large rectangular slot M. At the bottom of vessel S are 15 circular orifices having a diameter of 1.805cm each which can be opened or closed. According to Torricelli's law it is guaranteed that the outflow of the vessel S to the vessel P is constant and can be controlled within 15 steps. Finally vessel P has a slit opening and is drowned to a certain height in a creek. In that way the outflow through a vivid and a dead water body is realized. When the water level in vessel P does not change any more, the vivid and the dead water depth can be measured from a scale at the vessel's wall. The results are shown in Table 1 for different widths of the opening slit b.

Table 1. Results from Poleni's experiments in SI-units.

Open holes	b [m]	h_T [m]	h_L [m]
3	0,0349649	0,124069	0,01973825
6	0,0349649	0,124069	0,056395
9	0,0349649	0,124069	0,0947436
12	0,0349649	0,124069	0,1308364
15	0,0349649	0,124069	0,1658013
3	0,0349649	0,2436264	0,00620345
6	0,0349649	0,2436264	0,02312195
9	0,0349649	0,2436264	0,04680785
12	0,0349649	0,2436264	0,0710577
15	0,0349649	0,2436264	0,0969994
3	0,0857204	0,03665675	0,022558
6	0,0857204	0,03665675	0,04793575
9	0,0857204	0,03665675	0,0744414
12	0,0857204	0,03665675	0,0947436
15	0,0857204	0,03665675	0,11448185
8	0,0857204	0,1082784	0,0270696
15	0,0857204	0,1082784	0,06711005
5	0,1782082	0,078953	0,0056395
15	0,1782082	0,078953	0,03552885

In total, Poleni performed 19 experiments varying the dead water height by moving up and down the vessel P and changing the discharge by opening more or less holes. Three widths of the slot in P are investigated. Unfortunately Poleni did not measure the discharge directly. He only specified the number of holes N opened in the vessel P. The results of the experiments clearly show that the vivid water depth decreasing when the dead water depth is increased. This is to be expected because the water also flows out of the vessel in the dead body of water making part of it vivid. On the other hand, the vivid water depth increases when the discharge is increased.

In order to represent his measurements by a formula, Poleni postulated a formulation for the coefficient P having the following shape:

$$q=\frac{3}{2}\sqrt{Ph_L}\,(h_L+h_T) \qquad \text{with } P=p\,\frac{(h_L+8h_T)}{(h_L+8h_T)+\frac{4(10^5 h_T^7)^{1/6}}{c}} \tag{4}$$

There is no derivation for this formula, no explanation for the factor 8 before the dead water depth. For a vanishing dead water height, the coefficient P of the mixed motion becomes the coefficient p of the simple motion. Actually, it is possible to fit the parameters c and p to some of the curves of Poleni's results quite well, but it is not possible to get a good agreement with all of his results using the same values for p and c. Figure 3 shows the results of an inverse calculation of the number of open holes from eq. (4) compared to Poleni's measurements. When c and p are adjusted adequately, better results can be obtained for other curves in the figure.

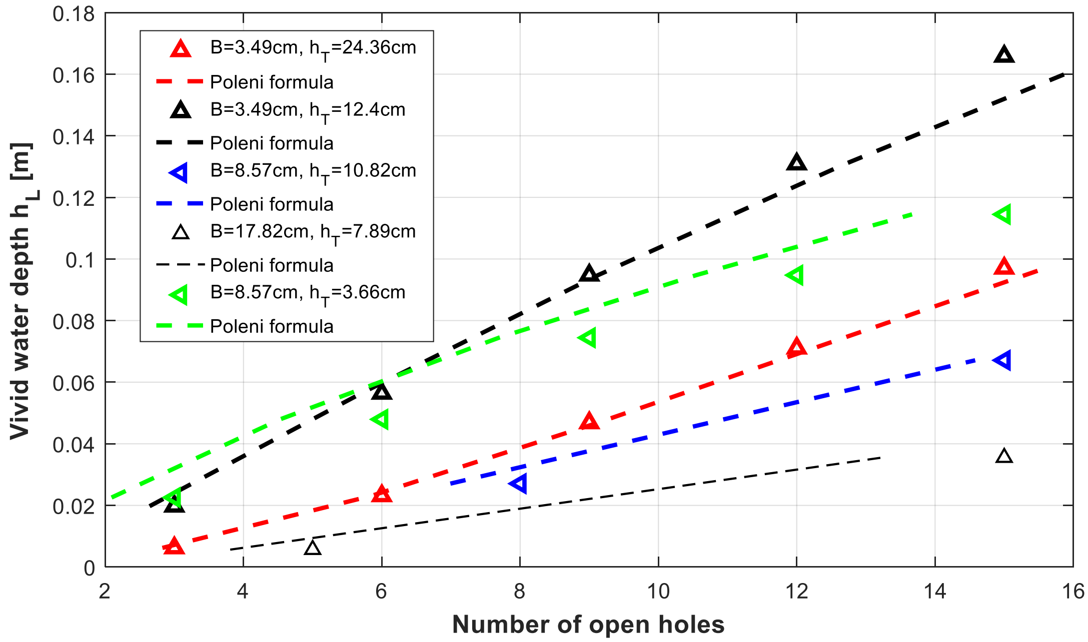

Figure 3. Experimental results of Poleni's experiments (triangles) compared to his formula (dotted lines) The coefficients are c=6 and p=74556000 m/s².

4. The Momentum Balance for Poleni's Experiment

Let us look for a theory describing Poleni's outflow experiment at the present state of the art applying basic physical principles. First of all we have to estimate the vertical discharge into the sloted vessel P. This is actually an outflow problem which is usually solved applying Torricelli's formula with an outflow correction coefficient. Applying the momentum balance to the outflow problem through sharp edged orifices, the formula (Malcherek, 2016a, b)

$$Q = NA_A\sqrt{\frac{gh}{\beta - \frac{A_A}{A}}} \quad \text{with} \quad \beta = 1.25 \qquad [5]$$

agrees very well with experimental results when N is the number of open holes, A is the cross-section of vessel S, and A_A are the holes cross sections.

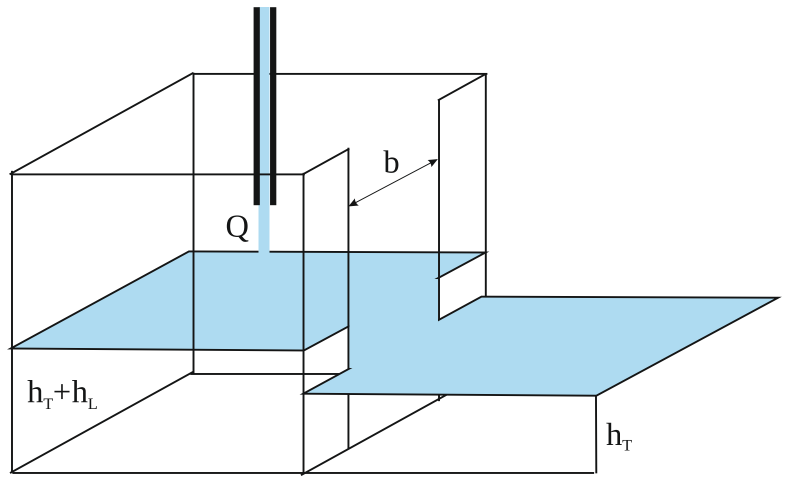

Figure 4. Schematic sketch of Poleni's experiment for the momentum balance.

The experiment is constructed in a way that the mass in vessel P does not change which means that mass balance must not be taken into account. Momentum is a vector with three independent components. The water flow into the vessel only changes the vertical momentum balance. The flow out of the vessel through the slot reduces the horizontal momentum balance. The experiment is also performed in a stationary mode with respect to the horizontal momentum stored in the vessel. For simplicity, let us assume a vessel with a rectangular cross section of width b which is fully opened on one side by the slot. On the wall opposite to the slot, hydrostatic forces ½ ρ g $(h_T+h_L)^2$ can be assumed to act on the water in the vessel. In the cross section of the slot hydrostatic forces ½ ρ g $h_T{}^2$ can be assumed, with respect to the water depth h_T. Finally, the horizontal momentum flux leaving the vessel with the volume flux Q has to be taken into account. It is $\beta\rho Qv$, where β takes into account the effect of the non-homogeneous velocity distribution and v represents the average velocity over the cross section A. Then the full momentum balance reads:

$$\frac{dI}{dt}=0=\frac{1}{2}\rho gb(h_T+h_L)^2-\frac{1}{2}\rho gbh_T^2-\beta\rho Qv \quad [6]$$

When assuming the cross-section A, where the momentum flux leaves the vessel, to be $A=b(h_T+h_L)$, the specific discharge through the slot is:

$$q^2=\frac{g}{2\beta}\left((h_T+h_L)^2-h_T^2\right)\,(h_T+h_L)=\frac{g}{2\beta}\left(h_L^2+2h_Th_L\right)\,(h_T+h_L) \quad [7]$$

Therefore the expression

$$Q=b\sqrt{\frac{g}{2\beta}h_L^3\left(1+2\frac{h_T}{h_L}\right)\,\left(1+\frac{h_T}{h_L}\right)} \quad [8]$$

should give the outflow through a slot when the vessel is drowned to the height h_T.

It should be noted that the velocity coefficient β is not an artificial coefficient to achieve an agreement between this theory and empirical results. The velocity coefficient comes from the integration of the product of the velocity distribution times the mass flux distribution over the outflow cross section. Unfortunately, this coefficient cannot be determined for Poleni's experiments. By setting β=1.77, a perfect agreement between Poleni's measurements and the momentum balance can be achieved, which is shown in figure 5.

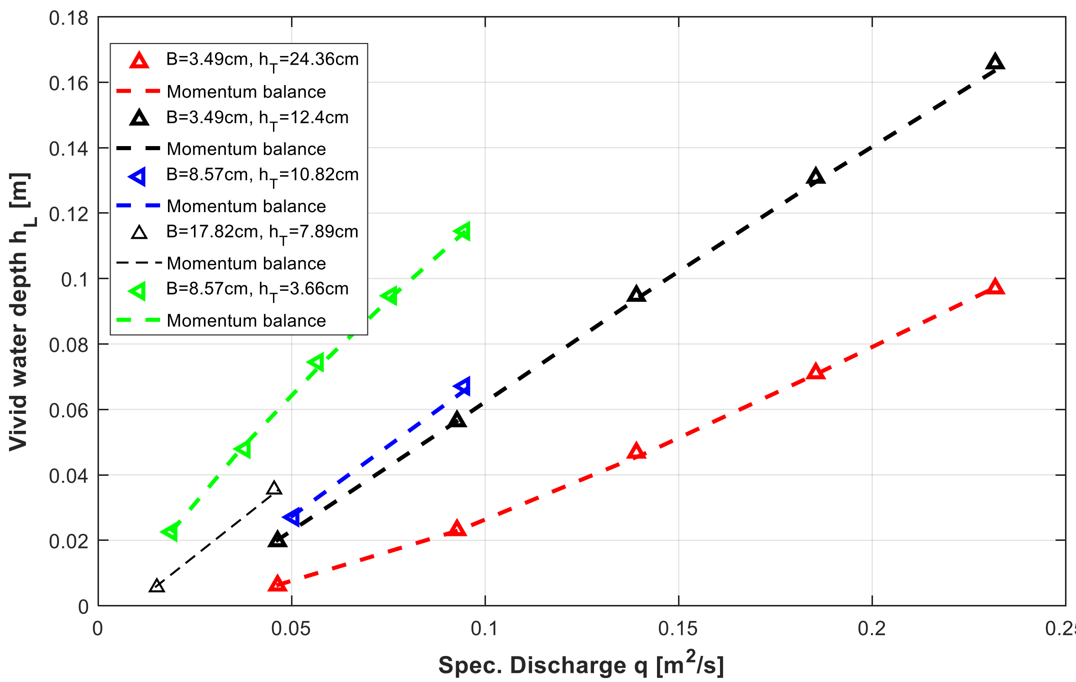

Figure 5. Experimental results of Poleni's experiments (triangles) compared to the momentum balance theory (dotted lines).

5. A New Overflow Theory Based on Momentum Balance

When the momentum balance is able to describe Poleni's experimental data correctly, we should ty to apply it to the overflow over a weir too. Comparing the weir overflow to Poleni's experiment there are two differences to be taken into account. Water does not flow through the weir's plate having the height w, while in Poleni's experiment it con flow through the dead water depth h_T. Second, a momentum flux $\rho Q v_0$ enters the control volume from the upstream flow direction. The momentum balance for the control volume shown in figure 6 reads:

$$0=\frac{1}{2}\rho g b h^2 - \beta\rho Q v+\rho Q v_0 \qquad [9]$$

Applying the continuity equation v=Q/b/h and v_0=Q/B/(w+h) the new overflow formula

$$Q=b\sqrt{\frac{gh^3}{2\left(\beta-\frac{bh}{B(w+h)}\right)}} \qquad [10]$$

is obtained. It was shown recently by Ferro and Aydin that a similar formula is able to reproduce slit weir experiments excellently when the momentum coefficient β is adapted adequately.

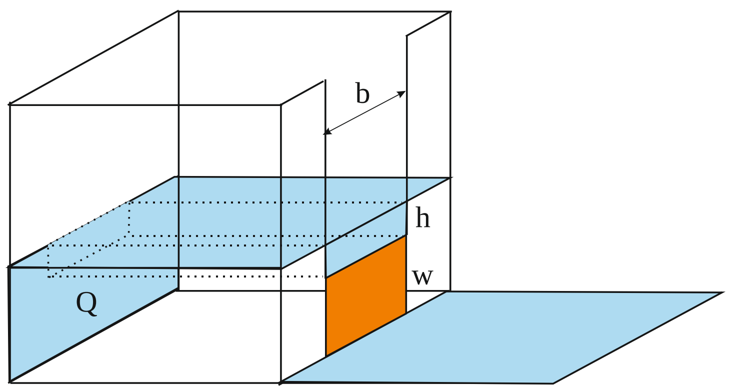

Figure 6. The control volume for the momentum balance for the weir overflow is shown in dashed lines.

6. Summary and Conclusions

Poleni's 'De motu aquae mixto' is a milestone in the history of hydraulics for several reasons. First of all, Poleni was the first hydraulician who documented a carefully-designed experiment and the data to verify his theory. Secondly, he showed how hydraulic engineering works; the starting point was a certain theory or a paradigm that is approximately correct and applicable to the engineer's objective. In the second step, an empirical coefficient is introduced to close the gap between theory and reality. In the third step, experiments are performed where the behaviour of the empirical coefficient is studied under different scenarios. Finally, the coefficient is parameterized to the different scenarios and fitted to the data.

Poleni's theoretical starting point is obviously wrong; the velocity profile in an open channel does not have the shape of a Torricelli square root function with the smallest velocities at the free surface. There is no doubt that Poleni noticed that fact. But after integration over the water depth, a much better discharge formula is obtained showing the correct relation between water depth and discharge. In that way, his approach starting with a wrong assumption was also justified.

In the aftermath, only the integration of the Torricelli formula over the water depth survived the historical path to our modern textbooks. Poleni derived it as a general discharge formula for simple motions in open channel flows. But nowadays, most textbooks on hydraulic engineering cite it as a weir formula because here a situation can be found, where the velocity increases from the free surface to the weirs crest. Poleni's theory on mixed motions, on the other hand, even in books on the history of hydraulics, (Rouse and Ince, 1957) is totally forgotten.

Poleni really worked on the outflow through a drowned slot. The question is how we should describe such a basic hydraulic situation with contemporary fluid mechanics. The author believes that the momentum balance is the driving principle in fluid mechanics. The Navier-Stokes-equations are the momentum balance when applied to an infinitesimal control volume. They do a very good job of reproducing the currents in and around hydraulic structures. In two previous papers, the author showed that also the integrated form of the momentum balance can be applied to basic hydraulics problems. The results obtained by this approach are surprising. The outflow velocity through a sharp-edged orifice comes to the eq. (5) applied in this paper and is in much better agreement with experimental results than the classical Torricelli formula. Applying the integrated momentum balance to a sharp crested sluice gate with an opening height a results in (Malcherek, 2017):

$$Q=ba\sqrt{\frac{3}{4}g\frac{h-a}{\beta-\frac{a}{h}}} \cong 0.6124\sqrt{2gh} \quad \text{for } \beta=1 \qquad [11]$$

This formulation would explain the sluice gate discharge coefficient of 0.61 to be the square root of the fraction 3/8. Therefore, for three fundamental hydraulic problems, new formulations were derived using the momentum principle, i.e. the outflow problem, the sluice gate underflow, and the weir overflow. As mentioned above, following works will concentrate on the application of the new theory to the overflow over different weir types.

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