

1978

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## Recommended Citation

On the Structure of Divergence-Free Tensors, *J. Math. Phys* 19 (1978), 2570–2575.

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# On the structure of divergence-free tensors

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(Received 13 February 1978)

Contravariant rank two tensors which are divergence-free on one index and which are constructed from the metric tensor, an auxiliary collection of arbitrary tensor fields, and the first and second partial derivatives of these quantities are classified. The results generalize existing mathematical arguments in support of the Einstein field equations.

## 1. INTRODUCTION

In gravitational field theories such as general relativity, contravariant rank two tensors  $A^{rs}$  which are divergence-free play a significant and well-known role. In this paper we shall classify those tensors which are locally of the form

$$A^{rs} = A^{rs}(g_{ij}; g_{ij,h}; g_{ij,hk}; \rho_Q; \rho_{Q,h}; \rho_{Q,hk}), \quad (1.1)$$

and which satisfy

$$A^{rs}|_s = 0. \quad (1.2)$$

Here  $g_{ij}$  represents the components of a nondegenerate symmetric tensor defined on an  $n$ -dimensional orientable manifold  $M$  while  $\rho_Q$  symbolically denotes a collection of tensor fields of arbitrary rank and weight which are independent of  $g_{ij}$ . Covariant differentiation, as denoted in (1.2) by a vertical bar, is defined in terms of the Christoffel symbols  $\Gamma_{ij}^h$ . It should be emphasized that in classifying  $A^{rs}$  we do *not* assume it to be symmetric, divergence-free with respect to the index  $r$  or a polynomial in any of its variables. Nevertheless, as the following theorem indicates, a rather precise characterization of  $A^{rs}$  is obtained.

*Theorem:* If  $A^{rs}$  is a tensorial concomitant of the type (1.1) and satisfies (1.2), then

$$A^{rs} = B^{rs} + C^{rs}, \quad (1.3)$$

where  $B^{rs}$  is a symmetric divergence-free tensor of the type<sup>1</sup>

$$B^{rs} = B^{rs}(g_{ij}; g_{ij,h}; g_{ij,hk}), \quad (1.4)$$

and  $C^{rs}$  is a skew-symmetric divergence-free tensor of the type (1.1). Moreover,  $C^{rs}$  decomposes into the form

$$C^{rs} = D^{rs} + E^{rst}|_t, \quad (1.5)$$

where  $D^{rs}$  is a skew-symmetric divergence-free tensor of the type (1.4) and  $E^{rst}$  is an appropriate totally skew-symmetric tensor.

As immediate consequences of this theorem we have the following three corollaries.

*Corollary 1:* If  $A^{rs}$  is a tensorial concomitant of the type (1.1), then  $A^{rs}|_s = 0$  if, and only if  $A^{rs}|_r = 0$ .

*Corollary 2:* If  $A^{rs}$  is a symmetric divergence-free concomitant of the type (1.1), then  $A^{rs}$  is independent of the fields  $\rho_Q$  and their derivatives.

*Corollary 3:* In a space of four dimensions, the most general divergence-free tensor of the type (1.1) is

$$A^{rs} = aG^{rs} + bg^{rs} + E^{rst}|_t, \quad (1.6)$$

where  $G^{rs}$  is the Einstein tensor,  $g^{rs}$  the inverse of  $g_{rs}$  and  $a$  and  $b$  are constants. If, in addition,  $A^{rs}$  is independent of  $\rho_{Q,hk}$ , then  $E^{rst}$  is explicitly given by

$$E^{rst} = \epsilon^{rstu}(V^Q \rho_{Q|u} + W_u),$$

where  $V^Q$  and  $W_u$  are arbitrary concomitants of  $g_{ij}$  and  $\rho_Q$ .

If the underlying manifold  $M$  is a spacetime, then we may replace the tensor  $g_{ij}$  by the generalized Pauli spinmatrices and assume that the fields defining  $\rho_Q$  are spin-tensorial in nature. Provided  $A^{rs}$  is now treated as a scalar under spinor (i.e., tetrad) transformations, the previous results can be reproduced verbatim and Eqs. (1.3)–(1.6) remain valid. The proof of these results depend upon several technical innovations and involves numerous lengthy calculations. Consequently in the next section we shall briefly describe the proof by outlining the main steps.<sup>2</sup>

These results lead us to a generalization of that derivation of general relativity in which the gravitational field equations are assumed to be of the form

$$V^{rs} = T^{rs}. \quad (1.7)$$

Here  $T^{rs}$  is the energy-momentum tensor for the external, i.e., nongravitational, source fields and  $V^{rs}$  is a tensor which is usually constructed from only those fields which characterize the gravitational field. To ensure that (1.7) implies

$$T^{rs}|_s = 0$$

it is customary to restrict  $V^{rs}$  by the identity

$$V^{rs}|_s = 0.$$

If, for example,  $V^{rs}$  is assumed to be a tensorial concomitant of the type

$$V^{rs} = V^{rs}(g_{ij}; g_{ij,h}; g_{ij,hk}), \quad (1.8)$$

then it is known<sup>3</sup> that  $V^{rs}$  is a linear combination of  $G^{rs}$  and  $g^{rs}$  in which case (1.7) gives rise to the usual Einstein field equations with cosmological term,

$$aG^{rs} + bg^{rs} = T^{rs}. \quad (1.9)$$

Consequently if for some reason one wishes to modify the Einstein field equations then, as emphasized by Ehlers,<sup>4</sup> the assumption (1.8) would have to be changed. One possible alternative to (1.8) would be to assume that  $V^{rs}$  is of the type (1.1), where the fields  $\rho_Q$  represent either the external source fields or auxiliary gravitational variables. However, the above results clearly indicate that even under these very general circumstances, the Einstein field equations must inevitably be retained. Furthermore, our analysis shows that any attempt to modify  $T^{rs}$  by adding to it a symmetric divergence-free tensor of the type (1.1) cannot lead to different gravitational field equations.<sup>5</sup>

Finally, we would like to point out that a considerable effort has been made in recent years<sup>6-14</sup> to classify tensors  $S^{rs}$  of the type (1.1) which satisfy certain identities arising either from properties enjoyed by the energy-momentum tensors of general relativity or from the conservation laws derived from invariant variational principles. All of these identities may be expressed in the form

$$S^{rs}|_s = U^r, \quad (1.10)$$

where  $U^r$  is a vector which is constructed in a prescribed fashion. Since (1.10) is linear in  $S^{rs}$ , it follows that  $U^r$  uniquely determines  $S^{rs}$  up to a divergence-free tensor of the type (1.1). Therefore, in view of our present work, a knowledge of  $U^r$  suffices to characterize  $S^{rs}$ . Accordingly, it is anticipated that the results of this paper will simplify the existing solutions to classification problems of this kind and perhaps lead to new results in this area.

## 2. THE PROOF OF THE THEOREM

We denote the infinitesimal generators for the coordinate transformations of the tensor fields by<sup>15</sup>  $F_i^j Q^R$ . Consequently, the covariant derivative of  $\rho_Q$  assumes the form

$$\rho_{Q|h} = \rho_{Q,h} - \Gamma_{Qh}^R \rho_R,$$

where  $\Gamma_{Qh}^R = F_i^j Q^R \Gamma_{jh}^i$ , in which case

$$\rho_{Q|h} - \rho_{Q|kh} = -R_{Qhk}^R \rho_R,$$

where  $R_{Qhk}^R = F_i^j Q^R R_j^i{}_{hk}$ . If  $\xi_Q$  denotes another collection of tensor fields of the same type as  $\rho_Q$ , then the principle form of the tensorial concomitant

$$T^U = T^U(g_{ij}, \dots; g_{ij,i}, \dots; \rho_Q, \dots; \rho_{Q,i}, \dots)$$

with respect to  $\xi_Q$  is the tensor<sup>16,17</sup>

$$P^U(\xi) = \sum_{i=0}^m (\partial^{Q_i \dots i} T^U) \xi_{Q_i \dots i}, \quad (2.1)$$

where

$$\partial^{Q_i \dots i} T^U = \frac{\partial T^U}{\partial \rho_{Q_i \dots i}}$$

Since  $\xi_{Q_i \dots i}$  can be expressed as a unique linear combination of the symmetrized covariant derivatives

$$\xi_{Q_i \dots i} = \xi_{Q_i}, \dots, \xi_{Q_i(i_1 \dots i_1)}, \quad (2.2)$$

where  $T^{U; Q_i \dots i}$  is the tensorial derivative<sup>18</sup> of  $T^U$  with respect to  $\rho_{Q_i \dots i}$ . It is not difficult to show<sup>19</sup> that for  $m=2$ ,

$$T^{U; Qhk} = \partial^{Qhk} T^U$$

$$T^{U; Qhk} = \partial^{Qh} T^U + 2\Gamma_{Sh}^Q \partial^{Shk} T^U + \Gamma_{lk}^h \partial^{Qlk} T^U$$

and

$$T^{U; Q} = \partial^Q T^U + \Gamma_{Sh}^Q (\partial^{Sh} T^U + \Gamma_{Tk}^S \partial^{Thk} T^U) + \Gamma_{Sh,k}^Q \partial^{Shk} T^U.$$

The transformation properties of  $T^U$  lead to certain differential invariance identities, one of these being (again for  $m=2$ ),

$$2T^{U; l(j, hk)} g_{li} + T^{U; Q(hk} F_i^j) \rho_{QR} = 0, \quad (2.3)$$

where

$$T^{U; lj, hk} = \frac{\partial T^U}{\partial g_{lj, hk}}.$$

The following lemma will be used to show that if  $A^{rs}$  is a divergence-free concomitant of the type (1.1), then it is a polynomial in the second derivatives of both  $g_{ij}$  and  $\rho_Q$ .

*Lemma 2.1:* Let  $S$  denote a set with  $n$  elements and for integers  $p, q \geq 1$  let

$$f = f(s_1, \dots, s_p, t_1, u_1, \dots, t_q, u_q) : S^{p+2q} \rightarrow \mathbb{R}$$

be any function with the following three properties:

(i) it is totally skew-symmetric in the arguments

$$s_1, s_2, \dots, s_p, \text{ e.g.,}$$

$$f(s_1, s_2, \dots, s_p, t_1, \dots, u_q) = -f(s_2, s_1, \dots, s_p, t_1, \dots, u_q);$$

(ii) it is a symmetric in the arguments  $t_l u_l$  for each

$$l = 1, 2, \dots, q, \text{ e.g.,}$$

$$f(s_1, \dots, s_p, t_1, u_1, \dots, u_q) = f(s_1, \dots, s_p, u_1, t_1, \dots, u_q);$$

and

(iii) it satisfies the cyclic identity with respect to the arguments  $s_k t_l u_l$  for each  $k = 1, \dots, p$  and  $l = 1, \dots, q$  e.g.,

$$f(s_1, \dots, s_p, t_1, u_1, \dots, u_q) + f(u_1, \dots, s_p, s_1, t_1, \dots, u_q) + f(t_1, \dots, s_p, u_1, s_1, \dots, u_q) = 0.$$

Then  $f$  vanishes identically whenever  $p+q > n$ .

*Proof:* We introduce subsets  $A$  and  $B_{\sigma, l}$  of  $S^{p+2q}$  with the definitions

$$A = \{(s_1, \dots, s_p, t_1, u_1, \dots, t_q, u_q) |$$

each  $\sigma \in S$  occurs at least twice amongst the components  $s_1, \dots, s_p, t_1, u_1, \dots, t_q, u_q\}$

for each  $\sigma \in S$  and  $l = 1, 2, \dots, q,$

$$B_{\sigma,l} = \{ (s_1, \dots, s_p, t_1, u_1, \dots, t_q, u_q) \mid \sigma \notin \{t_j, u_j\} \text{ for all } j \neq l \}.$$

The union of all the sets  $A$  and  $B_{\sigma,l}$  equals  $S^{p+2q}$  and so it suffices to prove that the restrictions of  $f$  to  $A$  and to  $B_{\sigma,l}$  vanish. The former may be established using arguments devised by Lovelock<sup>20</sup> while the latter follows by induction on  $n$ .

In the next lemma  $n'$  equals  $n$  or  $n+1$  according to whether  $n$  is even or odd.

**Lemma 2.2:** Let  $T^{rs, i_1 j_1 \dots i_p j_p}$  be a tensorial concomitant of  $g_{ab}$  which enjoys the following symmetry properties:

- (i) it is symmetric in the indices  $i_l j_l$  for all  $l=1, 2, \dots, p$ ; and
- (ii) it satisfies the cyclic identity with respect to the indices  $s_i j_l$  for all  $l=1, 2, \dots, p$ .

Then, provided  $n > 2$  and  $1 \leq p \leq n' - 2$ ,  $T^{rs, i_1 j_1 \dots i_p j_p}$  satisfies the cyclic identity with respect to the indices  $r_i j_l$  for all  $l=1, 2, \dots, p$ . If  $p < n' - 2$ , then  $T^{rs, i_1 j_1 \dots i_p j_p}$  is also symmetric in the indices  $rs$ .

*Proof:* We shall establish this lemma by induction on  $p$ . For  $p=1$  and  $n > 2$ , the result follows from the explicit construction of the most general tensorial concomitant which satisfies (i) and (ii).<sup>21</sup> Let us now suppose that the lemma is true for  $p=q-1$ , where  $2 \leq q \leq n' - 2$ , and proceed to establish its validity when  $p=q$ . On multiplying the invariance identity

$$\begin{aligned} & g^{tr} T^{us, i_1 j_1 \dots i_q j_q} + g^{ts} T^{ru, i_1 j_1 \dots i_q j_q} \\ & + \sum_{l=1}^q [g^{ti_l} T^{rs, i_1 j_1 \dots i_q j_q} + g^{tj_l} T^{rs, i_1 j_1 \dots i_q j_q}] \\ & = g^{ur} T^{ts, i_1 j_1 \dots i_q j_q} + g^{us} T^{rt, i_1 j_1 \dots i_q j_q} \\ & + \sum_{l=1}^q [g^{ui_l} T^{rs, i_1 j_1 \dots i_q j_q} + g^{uj_l} T^{rs, i_1 j_1 \dots i_q j_q}] \end{aligned}$$

by  $g_{ts}$ , on replacing  $u$  by  $s$ , and by repeatedly invoking properties (i) and (ii), it is found that

$$\begin{aligned} & T^{sr, i_1 j_1 \dots i_q j_q} + (n-q-1) T^{rs, i_1 j_1 \dots i_q j_q} = g^{rs} U^{i_1 j_1 \dots i_q j_q} \\ & - \frac{1}{2} \sum_{l=1}^q \left[ g^{si_l} V^{rj_l, i_1 j_1 \dots (i_l j_l) \dots i_q j_q} \right. \\ & \left. + g^{sj_l} V^{ri_l, i_1 j_1 \dots (i_l j_l) \dots i_q j_q} \right] \end{aligned} \quad (2.4)$$

where

$$U^{i_1 j_1 \dots i_q j_q} = T^{ts, i_1 j_1 \dots i_q j_q} g_{ts},$$

$$V^{rs, i_1 j_1 \dots (i_l j_l) \dots i_q j_q} = T^{rs, i_1 j_1 \dots i_{l-1} j_{l-1} a b i_l j_l i_{l+1} j_{l+1} \dots i_q j_q} g_{ab}, \quad (l)$$

and the circumflex  $\hat{\phantom{x}}$  enclosing a pair of indices indicates that those indices are to be deleted.

Since the induction hypothesis is applicable to

$V^{rs, i_1 j_1 \dots i_q j_q}$ , each  $V^{rs, i_1 j_1 \dots i_q j_q}$  is symmetric in the indices  $rs$ . It now follows that each of these tensors is also symmetric under interchange of the pairs of indices  $i_h j_h$  for all  $h=1, 2, \dots, q-1$ . By multiplying (2.4) by  $g_{rs}$  we therefore obtain the relation

$$U^{i_1 j_1 \dots i_q j_q} = \frac{1}{q} \sum_{l=1}^q V^{i_1 j_1 \dots i_q j_q},$$

which implies that  $U^{i_1 j_1 \dots i_q j_q}$  enjoys all the symmetry properties of  $V^{i_1 j_1 \dots i_q j_q}$ . Consequently if we successively cycle on the indices  $r_i j_{2l-1} j_{2l-1}$  and  $s_i j_{2l} j_{2l}$  in (2.4) we find, after a simple analysis of the resulting equation, that

$$U^{i_1 j_1 \dots i_q j_q} = V^{i_1 j_1 \dots i_q j_q} \quad (l)$$

for each  $l=1, 2, \dots, q$ . In view of this result and the various symmetry properties of  $U^{i_1 j_1 \dots i_q j_q}$ , (2.4) simplifies to

$$\begin{aligned} & T^{sr, i_1 j_1 \dots i_q j_q} + (n-q-1) T^{rs, i_1 j_1 \dots i_q j_q} \\ & = g^{rs} U^{i_1 j_1 \dots i_q j_q} - \frac{1}{2} \sum_{l=1}^q [g^{si_l} U^{i_1 j_1 \dots i_q j_q} + g^{sj_l} U^{i_1 j_1 \dots i_q j_q}]. \end{aligned}$$

On account of the invariance identity for  $U^{i_1 j_1 \dots i_q j_q}$ , the left-hand side of this equation is symmetric in  $rs$  and satisfies the cyclic identity on  $s_i j_l$ . Since the same must be true of the right-hand side, this proves the lemma for the case  $p=q$ , thereby completing our induction proof.

**Theorem 2.3:** Let  $A^{rs}$  be a tensorial concomitant of the type

$$A^{rs} = A^{rs} (g_{ij}; g_{ij,h}; g_{ij,hk}).$$

Then  $A^{rs}|_s = 0$  if, and only if  $A^{rs}|_r = 0$ .

*Proof:* On account of the formula<sup>22</sup>

$$A^{rs}|_t = \frac{2}{3} A^{rs; ij,hk} R_{hijk|t},$$

it readily follows that the equations  $A^{rs}|_s = 0$  and  $A^{rs}|_r = 0$  are equivalent to

$$A^{rs; ij,hk} + A^{rk; ij,sh} + A^{rh; ij,ks} = 0, \quad (2.5)$$

and

$$A^{rs; ij,hk} + A^{ks; ij,rh} + A^{hs; ij,kr} = 0, \quad (2.6)$$

respectively. Consequently it suffices to show that (2.5) implies (2.6). From (2.3) it follows that

$$A^{rs; ij, hk} = A^{rs; hk, ij},$$

and so  $A^{rs; ij, hk}$  also satisfies the cyclic identity with respect to the indices  $sij$ . We can apply Lemma 2.1 to the  $p$ th-order derivative of  $A^{rs}$  with respect to  $g_{i_{2l-1}j_{2l-1}i_{2l}j_{2l}}$  ( $l=1, 2, \dots, p$ ), viz.,

$$A^{rs; i_{2l-1}j_{2l-1}i_{2l}j_{2l} \dots i_{2p-1}j_{2p-1}i_{2p}j_{2p}},$$

to deduce that  $A^{rs; i_{2l-1}j_{2l-1}i_{2l}j_{2l}}$  is a polynomial in  $g_{ab, cd}$  of degree no greater than  $m-2$ , where  $m=n'/2$ . This polynomial can be expressed in the form

$$A^{rs; i_{2l-1}j_{2l-1}i_{2l}j_{2l}} = \sum_{l=0}^{m-2} T^{rsi_{2l-1}j_{2l-1}i_{2l}j_{2l} \dots i_{2l}j_{2l}i_{2l-1}j_{2l-1}i_{2l}j_{2l}} \times R_{i_{2l-1}j_{2l-1}i_{2l}j_{2l} \dots i_{2l}j_{2l}i_{2l-1}j_{2l-1}i_{2l}j_{2l}},$$

where the coefficients  $T^{rsi_{2l-1}j_{2l-1}i_{2l}j_{2l}}$  are tensorial concomitants of  $g_{ab}$  alone and enjoy the symmetry properties enumerated in Lemma 2.2. By virtue of this lemma each coefficient satisfies the cyclic identity with respect to the indices  $ri_{2l}$ . Therefore, (2.6) holds and the theorem is proved.

**Lemma 2.4:** If  $A^{rs}$  satisfies (1.1) and (1.2), then

$$A^{rs; ij, hk} + A^{rk; ij, sh} + A^{rh; ij, ks} = 0, \quad (2.7)$$

$$A^{rl; Qhk} + A^{rk; Qlh} + A^{rh; Qkl} = 0, \quad (2.8)$$

$$\frac{1}{2}(A^{rh; Qk} + A^{rk; Qh}) + A^{rs; Qhk} = 0, \quad (2.9)$$

$$A^{rh; Q} + A^{rs; Qh} + A^{rl; Rkh} R_R^Q = 0, \quad (2.10)$$

and

$$\frac{1}{2}R_l^r{}_{hk}(A^{lh; Qk} - 2A^{lh; Qkm}{}_{|m}) - \frac{2}{3}R_l^r{}_{hk} A^{lh; Qkm} = 0. \quad (2.11)$$

*Proof:* Equation (2.7) is the consequence of differentiating (1.2) with respect to  $g_{ij, hks}$ . To derive (2.8)–(2.10) we first remark that the principal forms  $P^r(\xi)$  and  $P^{rs}(\xi)$  of  $A^{rs}{}_{|s}$  and  $A^{rs}$  with respect to  $\xi$  are related by

$$P^r(\xi) = [P^{rs}(\xi)]_{|s}.$$

By substituting into this identity from (2.2) and by equating the coefficients of  $\xi_{Q|(hkl)}$ ,  $\xi_{Q|(hk)}$ , and  $\xi_{Q|h}$  and  $\xi_Q$  in the resulting equation, we can conclude that the tensorial derivatives  $[A^{rs}{}_{|s}]^{Qhkl}$ ,  $[A^{rs}{}_{|s}]^{Qhk}$  and  $[A^{rs}{}_{|s}]^{Qh}$  are given by the left-hand sides of (2.8)–(2.10) respectively while  $[A^{rs}{}_{|s}]^Q = A^{rs; Q} + \frac{1}{2}A^{rh; Rk} R_R^Q + \frac{1}{3}A^{rh; Rkl} R_R^Q{}_{|l}$ .

Equations (2.8)–(2.10) now follow on account of (1.2). Finally, (2.11) arises via the simplification of the identity

$$-[A^{rs}{}_{|s}]^Q + [A^{rs}{}_{|s}]^{Qh} - [A^{rs}{}_{|s}]^{Qhk} + [A^{rs}{}_{|s}]^{Qhkl} = 0.$$

**Lemma 2.5:** If  $A^{rs}$  satisfies (1.1) and (1.2), then

$$D^{rs; Qhk} = 0, \quad (2.12a)$$

$$D^{rs; Qh} = 0, \quad (2.12b)$$

$$D^{rs; Q} = 0, \quad (2.12c)$$

where  $D^{rs} = \frac{1}{2}(A^{rs} + A^{sr})$ .

*Proof:* To derive the first of (2.12), we begin by repeatedly differentiating (2.11), first with respect to  $g_{ab, cde}$  and then with respect to  $\rho_{R_1, t_1 u_1}, \rho_{R_2, t_2 u_2}, \dots, \rho_{R_p, t_p u_p}$  and  $g_{i_{2l-1}j_{2l-1}i_{2l}j_{2l}}$ . On multiplying the resulting equation by a totally symmetric but otherwise arbitrary tensor  $\psi_{cde}$ , we find that

$$[g^{ra} E^{Qb} + g^{rb} E^{Qa} + \psi_{cde} g^{re} D^{ab; Qcd}]^{(\alpha(p); \beta(0, q))} + \sum_{l=1}^q [g^{ri} F^{h, k, Ql, ab} + g^{rj} F^{h, k, Ql, ab} + g^{rh} F^{i, j, Qk, ab} + g^{rk} F^{i, j, Qh, ab}]^{(\alpha(p); \beta(l, q))} - \frac{g}{3} R_l^r{}_{hk} F^{lhQkab, \alpha(p); \beta(0, q)} = 0, \quad (2.13)$$

where

$$E^{Qb} = \psi_{cde} D^{cd; Qeb}, \quad F^{lhQkab} = \psi_{cde} D^{lh; Qke; ab, cd},$$

$$\alpha(p) = R_{1t_1 u_1}; R_{2t_2 u_2}; \dots; R_{pt_p u_p},$$

and

$$\beta(l, q) = i_{j_1, h_1 k_1}; i_{j_2, h_2 k_2}; \dots; (i_{j_l, h_l k_l})^{\wedge}; \dots; i_{qj_q, h_q k_q}.$$

Moreover, on account of (2.8), it follows from Lemma 2.1 that

$$D^{rs; Qhk; \alpha(n-1)} = 0.$$

Hence, to establish the lemma by mathematical induction, it suffices to show that if

$$D^{rs; Qhk; \alpha(p+1)} = 0, \quad (2.14)$$

where  $0 \leq p \leq n-2$ , then

$$D^{rs; Qhk; \alpha(p)} = 0. \quad (2.15)$$

In order to derive (2.15) from (2.14) we shall use a second inductive argument. To start, we note that on account of (2.14)  $D^{rs; Qhk; \alpha(p)}$  is independent of  $\rho_{Q, ij}$  in which case (2.3) leads to

$$D^{rs; Qhk; \alpha(p); ab, cd} = D^{rs; Qhk; \alpha(p); cd, ab}$$

By appealing once more to Lemma 2.1, we find that

$$E^{Qb; \alpha(p); \beta(0, q)} = 0 \quad \text{and} \quad F^{i j_1 Q b h_1 k_1; \alpha(p); \beta(l, q)} = 0$$

whenever  $p + 2q \geq n$ . Therefore, let us suppose that

$$E^{Qb; \alpha(p); \beta(0, q+1)} = 0 \quad \text{and} \quad F^{i j_1 Q b h_1 k_1; \alpha(p); \beta(l, q+1)} = 0 \quad (2.16)$$

for  $p + 2q < n$  and proceed to establish the validity of (2.16) with  $q + 1$  replaced by  $q$ . In view of the second of (2.16), multiplication of (2.13) by  $g_{ra}$  gives rise to

$$n E^{Qb; \alpha(p); \beta(0, q)} - \sum_{l=1}^q [F^{h_1 k_1 Q b i j_1} + F^{i j_1 Q b h_1 k_1}; \alpha(p); \beta(l, q)] = 0. \quad (2.17)$$

Similarly multiplication of (2.13) by  $g_{rh}$  yields (on replacing the indices  $k_1 ab$  by  $bh_1 k_1$ )

$$\begin{aligned} & -E^{Qb; \alpha(p); \beta(0, q)} + n F^{i j_1 Q b h_1 k_1; \alpha(p); \beta(1, q)} \\ & - F^{h_1 k_1 Q b i j_1; \alpha(p); \beta(1, q)} - \sum_{l=2}^q [F^{h_1 k_1 Q b h_1 k_1; i j_1, i j_1} \\ & + F^{i j_1 Q b h_1 k_1; i j_1, h_1 k_1}; \alpha(p); i j_1, h_1 k_1, \dots; (i j_1, h_1 k_1)^{\sim}; \dots; i j_1, h_1 k_1] \\ & = 0. \end{aligned} \quad (2.18)$$

By repeatedly permuting all pairs of indices  $i j_1$  and  $h_1 k_1$  for  $l = 1, \dots, q$  in (2.17) and (2.18) we obtain a homogeneous system of linear equations whose coefficient matrix is non-singular.<sup>23</sup> This implies that (2.16) remains valid with  $q + 1$  replaced by  $q$  and thus, by induction on  $q$ ,

$$E^{Qb; \alpha(p)} = 0 \quad \text{and} \quad F^{ij Q b h k; \alpha(p)} = 0.$$

It is now an elementary matter to obtain (2.15) from (2.13) (with  $q = 0$ ). This completes our original induction argument and establishes (2.12a).

Due to (2.12a), (2.8), (2.9), and Bianchi identities, (2.11) reduces to

$$R_l{}^r{}_{hk} D^{lh; Qk} = 0. \quad (2.19)$$

By cycling on the indices  $rhk$  in (2.9) and by covariantly differentiating (2.9) with respect to  $x^r$  it is found that

$$D^{rh; Qk} + D^{kr; Qh} + D^{hk; Qr} = 0 \quad (2.20)$$

and

$$\frac{1}{2}(A^{rh; Qk} + A^{rk; Qh})_r - \frac{1}{2}A^{rs; Rhk} R_R{}^Q{}_{rs} - \frac{1}{3}A^{kr; Qms} R_m{}^h{}_{rs} = 0. \quad (2.21)$$

Differentiation of (2.21) with respect to  $g_{ab, cds}$  leads to

$$D^{rh; Qs; ab, cd} + D^{rh; Qd; ab, sc} + D^{rh; Qc; ab, ds} = 0. \quad (2.22)$$

On account of the symmetry properties (2.20) and (2.22), it is possible to derive (2.12b) from (2.19) by an induction argument similar to that used to obtain (2.12a).

Finally, on noting (2.21), (2.12c) follows directly from the result of symmetrizing (2.10) on the indices  $r$  and  $h$ .

**Lemma 2.6:** If  $C^{rs}$  is a skew-symmetric divergence-free concomitant of the type (1.1), then

$$P^{rs}(\xi) = V^{rst}{}_{|t}, \quad (2.23)$$

where  $P^{rs}(\xi)$  is the principal form of  $C^{rs}$  and<sup>24</sup>

$$\begin{aligned} V^{rst}(\rho; \xi) &= \frac{3}{2}C^{[rs; Qt]u} \xi_{Q|u} + [C^{[rs; Qt]} - \frac{1}{2}C^{[rs; Qt]u}{}_{|u}] \xi_Q. \end{aligned}$$

**Proof:** Let  $C^r = C^{rs}{}_{|s}$ . Then (2.23) follows immediately from the formula

$$\begin{aligned} P^{rs}(\xi) &= V^{rst}{}_{|t} + \frac{3}{2}C^{[r; Qs]tu} \xi_{Q|tu} \\ &+ [\frac{4}{3}C^{[r; Qs]t} - C^{[r; Qs]tu}{}_{|u}] \xi_{Q|t} \\ &+ [C^{[r; Qs]} - \frac{2}{3}C^{[r; Qs]t}{}_{|t} + \frac{1}{2}C^{[r; Qs]tu}{}_{|tu}] \xi_Q \end{aligned}$$

which may be verified by direct calculation.

It is now a simple matter to prove the theorem stated in the introduction. Indeed, in view of Lemma 2.5, the tensor<sup>25</sup>

$$C_1^{rs} = \int_0^1 [\tilde{A}^{rs; Q} \rho_Q + \tilde{A}^{rs; Qh} \rho_{Q|h}] + \tilde{A}^{rs; Qhk} \rho_{Q|h k} dt,$$

where

$$\tilde{A}^{rs} = A^{rs}(g_{ij}; g_{ij, h}; g_{ij, hk}; t\rho_Q; t\rho_{Q, h}; t\rho_{Q, hk})$$

is skew-symmetric in the indices  $rs$ . Moreover, on recalling (2.1) and (2.2) it is easily seen that

$$A^{rs} = A_0^{rs} + C_1^{rs}, \quad (2.24)$$

where  $A_0^{rs} = A^{rs}(g_{ij}; g_{ij, h}; g_{ij, hk}; 0; 0; 0)$ . Since (1.2) implies that  $A_0^{rs}$  is divergence-free with respect to the index  $s$  we may refer to Theorem 2.3 to conclude that  $A_0^{rs}$  is divergence-free with respect to the index  $r$  and can therefore be expressed in the form

$$A_0^{rs} = B^{rs} + C_2^{rs},$$

where  $B^{rs}$  and  $C_2^{rs}$  are respectively symmetric and skew-symmetric divergence-free concomitants of the type (1.4). Because  $A^{rs}$  and  $A_0^{rs}$  are divergence-free on the index  $s$ , it follows from (2.24) that the same must be true of  $C_1^{rs}$ . How-

ever,  $C_1^{rs}$  is skew-symmetric in its indices and therefore by setting

$$C^{rs} = C_1^{rs} + C_2^{rs}$$

we arrive at (1.3). Finally, (1.5) follows from Lemma 2.6 upon setting

$$E^{rst} = \int_0^1 V^{rst}(\rho; t\rho) dt.$$

## ACKNOWLEDGMENTS

I would like to thank Dr. T. Duchamp, Dr. D. Lovelock, and Dr. K. Kuchař for many fruitful discussions with regard to the preparation of this article.

<sup>1</sup>An explicit characterization of  $B^{rs}$  is available, see D. Lovelock, "The Einstein tensor and its generalizations," *J. Math. Phys.* **12**, 498–501 (1971).

<sup>2</sup>For details, see I.M. Anderson, "A detailed study of divergence-free tensors," preprint (available from the Department of Mathematics, University of Utah).

<sup>3</sup>See Ref. 1 and I.M. Anderson and D. Lovelock, "A characterization of the Einstein tensor in terms of spinors," *J. Math. Phys.* **17**, 1001–3 (1976).

<sup>4</sup>J. Ehlers, "Survey of general relativity theory," in W. Israel (Ed.) *Relativity, Astrophysics and Cosmology* (Reidel, Holland, 1973).

<sup>5</sup>For a discussion of this idea with regard to the electromagnetic energy-momentum tensor, see D. Lovelock, "The Huggins term in curved space," *Nuovo Cimento* **10**, 581–4 (1974).

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<sup>9</sup>I.M. Anderson, "Mathematical foundations of the Einstein field equations," Ph.D. thesis, University of Arizona (1976).

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<sup>17</sup>G.W. Horndeski, "Tensorial concomitants of relative tensors and linear connections," *Utilitas Math.* **9**, 3–31 (1976).

<sup>18</sup>H. Rund, "Variational problems involving combined tensor fields," *Abb. Math. Sem. Univ. Hamburg.* **29**, 243–62 (1966).

<sup>19</sup>See Ref. 9, or Ref. 17.

<sup>20</sup>D. Lovelock, "The four-dimensionality of space and the Einstein tensor," *J. Math. Phys.* **13**, 874–6 (1972).

<sup>21</sup>D. Lovelock, "The uniqueness of the Einstein field equations in a four-dimensional space," *Arch. Ration. Mech. Anal.* **33**, 54–70 (1969).

<sup>22</sup>See Ref. 16.

<sup>23</sup>See Ref. 2, Lemmas 2.2 and 4.2.

<sup>24</sup>Square brackets indicate skew-symmetrization over the enclosed lower case indices only.

<sup>25</sup>It is assumed that  $A^{\alpha}$  is well defined at  $\rho_{\alpha} = 0$  and that the field variables  $\rho_{\alpha}$  form a star-shaped set with respect to  $\rho_{\alpha} = 0$ .