5-2024

On the Existence of Periodic Traveling-Wave Solutions to Certain Systems of Nonlinear, Dispersive Wave Equations

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ON THE EXISTENCE OF PERIODIC TRAVELING-WAVE SOLUTIONS TO CERTAIN SYSTEMS OF NONLINEAR, DISPERSIVE WAVE EQUATIONS

by

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A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTERS OF SCIENCE

in

Mathematics

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2024

ABSTRACT

On the existence of periodic traveling-wave solutions to certain systems of nonlinear, dispersive wave equations

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In the field of nonlinear waves, particular interest is given to periodic traveling-wave solutions of nonlinear, dispersive wave equations. This thesis aims to determine the existence of periodic traveling-wave solutions for several systems of water wave equations. These systems are the Schrödinger KdV-KdV, Schrödinger BBM-BBM, Schrödinger KdV-BBM, and Schrödinger BBM-KdV systems, and the $\mathfrak{abcd}$-system. In particular, it is shown that periodic traveling-wave solutions exist and are explicitly given in terms of cnoidal, the Jacobi elliptic function. Certain solitary-wave solutions are also established as a limiting case of the periodic traveling-wave solutions, that is, as the elliptic modulus approaches one.

(79 pages)

PUBLIC ABSTRACT

On the existence of periodic traveling-wave solutions to certain systems of nonlinear, dispersive wave equations

Jacob Daniels

A variety of physical phenomena can be modeled by systems of nonlinear, dispersive wave equations. Such examples include the propagation of a wave through a canal, deep ocean waves with small amplitude and long wavelength, and even the propagation of long-crested waves on the surface of lakes. An important task in the study of water wave equations is to determine whether a solution exists. This thesis aims to determine whether there exists solutions that both travel at a constant speed and are periodic for several systems of water wave equations. The work done in this thesis contributes to the subfields of mathematics known as partial differential equations and nonlinear waves, and has potential applications in the study of fluid dynamics.

ACKNOWLEDGMENTS

I'd first like to thank Dr. Nghiem Nguyen for the constant and unwavering support he has provided me throughout my time at Utah State University. I'd also like to thank Dr. Ian Anderson for the invaluable experiences and knowledge he has shared with me. A special thank you to Professor Claire Watson for the numerous opportunities she has given me to grow as an educator. I'd also like to extend sincere gratitude to Dr. Zhaohu Nie, Dr. Matthew Young, and Dr. James Powell, for the time and effort they have put into my academic studies. Thank you to the Utah State University Mathematics and Statistics department for the continuous support and resources they have provided me. Finally, I'd like to thank my family and my girlfriend Emma Utley for their support and encouragement throughout my academic career.

Jacob Daniels

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CHAPTER 1
INTRODUCTION

The Euler equations [24] are a set of nonlinear partial differential equations derived by Leonhard Euler in 1757. These equations describe the irrotational, inviscid flow of three-dimensional capillary-gravity waves of an incompressible fluid, and are given by

$$\begin{cases} \Delta\phi = 0, \quad \text{in } \Omega_t; \\ \eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0, \quad \text{at } z = \eta(x,y,t); \\ \phi_t + \frac{1}{2}|\nabla\phi|^2 + gz = 0, \quad \text{at } z = \eta(x,y,t); \\ \phi_x h_x + \phi_y h_y + \phi_z = 0, \quad \text{at } z = -h(x,y). \end{cases}$$

Here $\phi(x,y,z,t)$ is the velocity potential, $\eta(x,y,t)$ is the free surface elevation, g denotes the acceleration due to gravity, and $\Omega_t \subset \mathbb{R}^3 \times \mathbb{R}$ is the domain bounded above by $\eta(x,y,t)$ and below by $z = -h(x,y)$. The domain for the two-dimensional Euler equations is drawn in Figure 1.1, where $V(x,z,t)$ denotes the velocity of a water particle at position (x,z) and time t.

For most practical applications, the full Euler equations are more complex than necessary. So in practice, the Euler equations are often approximated with certain physical restraints in mind to achieve a simpler and more manageable equation or system of equations. Some of the most well-studied equations in fluid dynamics, the Korteweg-de Vries (KdV) equation [28]

$$v_t + v_x + vv_x + v_{xxx} = 0,$$

Benjamin-Bona-Mahony (BBM) equation [2]

$$v_t + v_x + vv_x - v_{xxt} = 0,$$

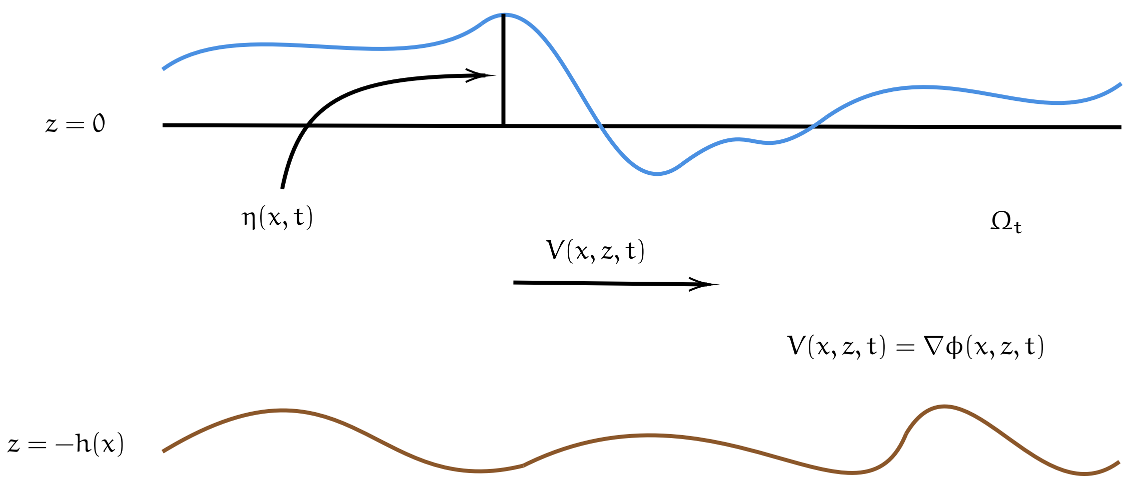


Figure 1.1: Domain for the two-dimensional Euler equations.

and cubic nonlinear Schrödinger (NLS) equation [34]

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0,$$

are examples of such approximations. Relaxing the restraints of an approximation can yield a more general physical setting, but at the cost of a more complex equation or system of equations. Several systems that arise from two distinct approximations of the Euler equations will be the focus of this thesis.

Four systems were recently put forth by Nguyen *et al.* [21, 36, 32] to study the interaction of long and short waves in dispersive media. These systems arise as an approximation of the Euler equations when ϕ and η are assumed to be given by the superposition of two waves, one long and one short. These systems are deemed the Schrödinger KdV-KdV system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} + a_0 \dfrac{\partial^3 u}{\partial x^3} + ib\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} + c\dfrac{\partial^3 v}{\partial x^3} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}; \end{cases}$$

Schrödinger BBM-BBM system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} - a_1 \dfrac{\partial^3 u}{\partial x^2 \partial t} + ib\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} - c\dfrac{\partial^3 v}{\partial x^2 \partial t} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}; \end{cases}$$

Schrödinger KdV-BBM system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} + a_0 \dfrac{\partial^3 u}{\partial x^3} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} - c\dfrac{\partial^3 v}{\partial x^2 \partial t} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}; \end{cases}$$

and Schrödinger BBM-KdV system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} - a_1 \dfrac{\partial^3 u}{\partial x^2 \partial t} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} + c\dfrac{\partial^3 v}{\partial x^3} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}. \end{cases}$$

Here $(x,t) \in \mathbb{R}^2$, $u(x,t)$ is a complex-valued function, $v(x,t)$ is a real-valued function, and $\mu_0, \mu_1, a_0, a_1, b, c$ are real constants with $\mu_0, \mu_1, a_0, a_1, c > 0$. The name "Schrödinger" denotes the presence of the $i\partial_{xx}$ term, whereas "KdV" or "BBM" denotes the presence of the ∂_{xxx} or $-\partial_{xxt}$ term respectively. Restricting our approximation to only long waves, the four systems reduce to the KdV or BBM equation. Which equation is recovered depends on which third-order term is present in the second equation of the system. Imposing no second harmonic resonance, all four systems also reduce to the NLS equation. These relationships are illustrated in Figure 1.2. For a more detailed discussion of these restrictions, and how u and v explicitly relate to ϕ and η from the Euler equations, see [21, 36, 32].

The $abcd$-system is another system of water wave equations, and is given by

$$\begin{cases} \eta_t + w_x + (w\eta)_x + a w_{xxx} - b\eta_{xxt} = 0\,, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0\,, \end{cases}$$

where a, b, c and d are real constants satisfying

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \quad \text{and} \quad a + b + c + d = \frac{1}{3},$$

for $\theta \in [0,1]$. Here $\eta(x,t)$ and $w(x,t)$ are real-valued function for $(x,t) \in \mathbb{R}^2$. This system was derived by Bona, Chen, and Saut [6, 5], and describes small amplitude, long wavelength gravity waves on the surface of water. This system is the approximation of the Euler equations when ϕ and η are both assumed to be strictly long waves, with no restrictions placed on the direction with which these waves travel. Restricting to waves traveling in a single direction, the $abcd$-system is reduced to the KdV and BBM equations. This relationship is also shown in Figure 1.2.

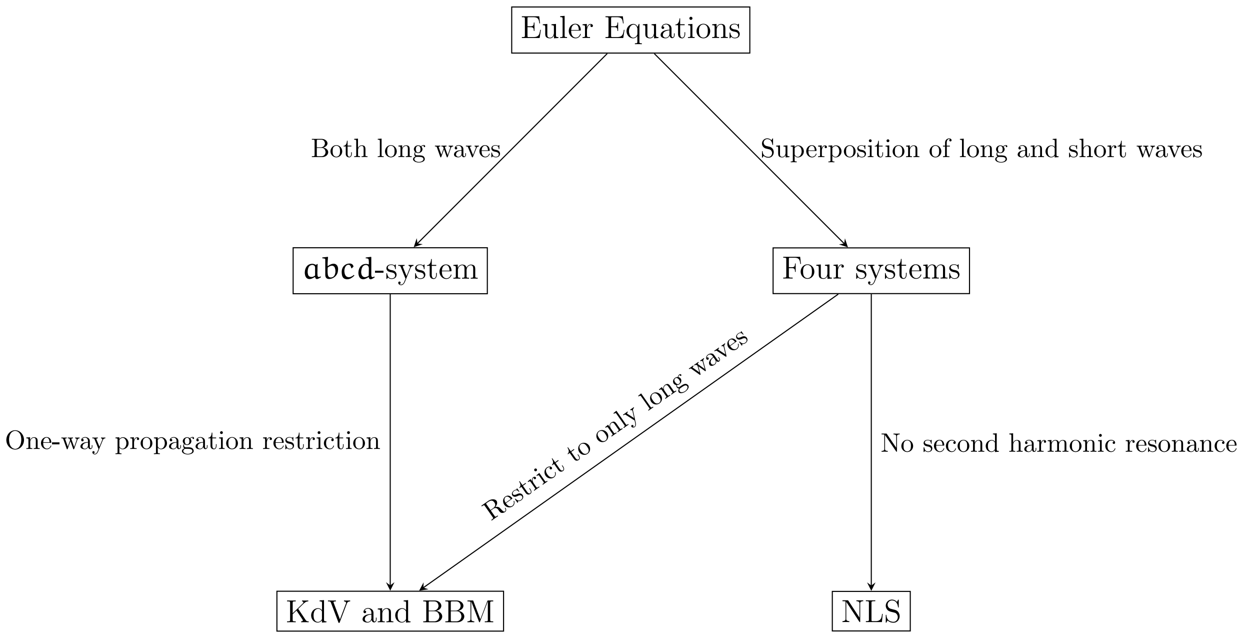


Figure 1.2: Diagram illustrating the relationship of the equations as approximations of the Euler equations.

A natural question to ask is whether there exist traveling-wave solutions to these systems of equations. In the case of the $abcd$-system where $\eta(x,t)$ and $w(x,t)$ are both real, traveling-wave solutions are vector solutions $\big(\eta(x,t), w(x,t)\big)$ of the form

$$\eta(x,t) = f(x-\sigma t) \quad \text{and} \quad w(x,t) = g(x-\sigma t), \tag{1.1}$$

where f and g are smooth, real-valued functions, and $\sigma \neq 0$ denotes the speed of the wave. In the case of the four systems, where $u(x,t)$ is complex-valued and $v(x,t)$ is real-valued, this thesis will study traveling-wave solutions of the form $\big(u(x,t), v(x,t)\big)$, where

$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} f(x-\sigma t) \quad \text{and} \quad v(x,t) = g(x-\sigma t). \tag{1.2}$$

Here f and g are again smooth, real-valued functions and B, σ, $\omega \in \mathbb{R}$ with $\sigma \neq 0$. This is not the only way to write a complex traveling-wave solution, but for the purpose of this thesis we will only consider solutions of this form. If $\sigma > 0$, then the wave is right-propagating, and if $\sigma < 0$, then the wave is left-propagating. Of traveling-wave solutions, periodic traveling-wave solutions and solitary-wave solutions are of particular interest. Periodic traveling-wave solutions are traveling-wave solutions where f and g are periodic functions. Solitary-wave solutions are symmetric solutions around a single maximum that decay to a constant as the moving frame, $x - \sigma t$, approaches $\pm\infty$. This constant is often assumed to be zero, but this is not always the case.

In this thesis, we aim to find explicit periodic traveling-wave solutions to the aforementioned systems. This thesis will study solutions where f and g are given explicitly by cnoidal, the Jacobi elliptic function. To understand why this function is a reasonable candidate for a solution, we consider the following example where we find traveling-wave solutions to the KdV equation. The work done in this example follows that of Drazin and Johnson [22].

Example. The Korteweg-de Vries (KdV) equation is given by

$$v_t + v_x + vv_x + v_{xxx} = 0,$$

where $v(x,t)$ is a real-valued function. Through the transformation $v(x,t) = -1 - 6\tilde{v}(x,t)$, the KdV becomes

$$\tilde{v}_t - 6\tilde{v}\tilde{v}_x + \tilde{v}_{xxx} = 0.$$

To find a traveling-wave solution, we assume that $\tilde{v}(x,t) = f(x - \sigma t)$. Without loss of generality, suppose $\sigma > 0$. Then the KdV equation becomes

$$-\sigma f' - 6ff' + f''' = 0,$$

where the primes denote differentiation with respect to the traveling-wave frame, $\xi = x - \sigma t$. Integrating once yields

$$-\sigma f - 3f^2 + f'' = A,$$

where A is a real constant of integration. Multiplying by f' and integrating again we get

$$-\frac{\sigma}{2}f^2 - f^3 + \frac{1}{2}(f')^2 = Af + B,$$

where B is another real constant of integration. Rearranging this equation, a function $F(f)$ is defined as follows

$$\frac{1}{2}(f')^2 = f^3 + \frac{\sigma}{2}f^2 + Af + B \equiv F(f). \tag{1.3}$$

Thus, the KdV is reduced to a first-order ordinary differential equation. By the fundamental theorem of algebra, $F(f)$ must have three roots with multiplicity. In [22] it is established that if $F(f)$ has two distinct real roots, one with multiplicity two, then the solitary-wave solution is obtained, and if $F(f)$

has three distinct real roots then the periodic traveling-wave solution is found.

To find the periodic traveling-wave solution, label the three distinct real roots as f_1, f_2, and f_3, such that $f_1 > f_2 > f_3$, and define ξ_3 to be $f(\xi_3) = f_3$. Then $F(f)$ can be factored as

$$F(f) = (f - f_1)(f - f_2)(f - f_3) \implies \frac{1}{2}(f')^2 = (f - f_1)(f - f_2)(f - f_3).$$

Solving for f', separating, and integrating yields

$$\pm \int_{f_3}^{f} \frac{dg}{(2(g - f_1)(g - f_2)(g - f_3))^{1/2}} = \int_{\xi_3}^{\xi} d\tau,$$

which can be rearranged to get

$$\xi = \xi_3 \pm \int_{f_3}^{f} \frac{dg}{(2(g - f_1)(g - f_2)(g - f_3))^{1/2}}.$$

Here, the solution f is implicitly defined. By making the following substitution, a closed form solution can be obtained. Let $g = f_3 + (f_2 - f_3)\sin^2(\theta)$, then the above equation reduces to

$$\xi = \xi_3 \pm \sqrt{\frac{2}{f_1 - f_3}} \int_0^{\phi} \frac{d\theta}{(1 - m^2 \sin^2(\theta))^{1/2}}, \tag{1.4}$$

where $m = \sqrt{\frac{f_2 - f_3}{f_1 - f_3}}$ and ϕ is given implicitly by

$$f = f_3 + (f_2 - f_3)\sin^2(\phi) = f_2 - (f_2 - f_3)\cos^2(\phi). \tag{1.5}$$

Note that since $f_1 > f_2 > f_3$, it follows that $0 < m < 1$. Let

$$\nu = \int_0^{\phi} \frac{d\theta}{(1 - m^2 \sin^2(\theta))^{1/2}},$$

then $\nu = G(\phi, m)$, or equivalently, $\phi = G^{-1}(\nu, m)$. Define the Jacobi elliptic function, cnoidal, to be

$$\mathrm{cn}(\nu, m) = \cos(\phi) = \cos\left(G^{-1}(\nu, m)\right).$$

The relation (1.4) then becomes

$$\mathrm{cn}\left((\xi - \xi_3)\sqrt{\frac{f_1 - f_3}{2}}, m\right) = \cos(\phi),$$

where the $\pm$ has been suppressed since cnoidal is an even function. Substituting this into (1.5) finally yields the following traveling-wave solution to the KdV equation

$$f(\xi) = f_2 - (f_2 - f_3)\operatorname{cn}^2\left((\xi - \xi_3)\sqrt{\frac{f_1 - f_3}{2}}, m\right).$$

Moreover, since cnoidal is a periodic function for $m \in [0, 1)$, this solution is a periodic traveling-wave solution.

To find a solitary-wave solution, suppose that f, f', f'' approach zero as ξ approaches $\pm\infty$. This is exactly equivalent to setting the integration constants, A and B, equal to zero in (1.3). The solution is then found by solving the first-order equation

$$\frac{1}{2}(f')^2 = f^3 + \frac{\sigma}{2}f^2,$$

and is given by

$$f(\xi) = -\frac{\sigma}{2}\operatorname{sech}^2\left(\frac{\sqrt{\sigma}}{2}(\xi - \xi_0)\right),$$

where ξ_0 is a constant of integration. A more detailed discussion of these solutions can be found in [22].

From the above example, we see that the KdV equation has periodic traveling-wave solutions given by the Jacobi cnoidal function, and hyperbolic secant solitary-wave solutions. In fact, it has also been shown that the BBM and NLS equations have cnoidal periodic traveling-wave solutions [1, 25] and hyperbolic secant solitary-wave solutions [37, 22]. So, it is natural to assume that the four systems and the $abcd$-system, which reduce to these equations under certain physical restraints, should also have periodic traveling-wave solutions given by the Jacobi cnoidal function. With this in mind, the form of solution studied in this thesis will be exactly of the form (1.1) or (1.2) where

$$f(x - \sigma t) = \sum_{r=0}^{n} j_q \operatorname{cn}^r(\lambda(x - \sigma t), m) \quad \text{and} \quad g(x - \sigma t) = \sum_{r=0}^{n} k_q \operatorname{cn}^r(\lambda(x - \sigma t), m),$$

for some amplitude j_q, $k_q \in \mathbb{R}$, wavelength $\lambda > 0$, speed $\sigma \neq 0$, and elliptic modulus $m \in [0, 1]$.

A fundamental property of the Jacobi cnoidal function is that as the elliptic modulus m approaches one, the cnoidal function limits to the hyperbolic secant function. Just as it is reasonable to assume that there exists Jacobi cnoidal solutions to these systems, it is also reasonable to assume that solitary-wave solutions are given by hyperbolic secant. In fact, hyperbolic secant solitary-wave solutions have been found for the $abcd$-system [17, 18], and synchronized solitary-wave solutions given by hyperbolic secant have been found for the four systems [33]. So, although there is no one-to-one correspondence, finding a periodic traveling-wave solution in terms of the Jacobi cnoidal function could limit to a solitary-wave solution for the same system.

The paper is organized as follows. Chapter 2 will focus on the existence of periodic traveling-wave solutions for the four systems. The existence of periodic traveling-wave solutions for the abcd-system will follow in Chapter 3. Chapter 4 will conclude the thesis with a discussion of future work. Chapters 2 and 3 are stand-alone papers. Chapter 2 has been published by AIMS Mathematics [11], and was coauthored by Nghiem Nguyen and Bruce Brewer. Chapter 3 has been submitted to Water Waves and is awaiting review, but is currently available on arXiv [19] and was also coauthored by Nghiem Nguyen.

CHAPTER 2
EXACT JACOBI ELLIPTIC SOLUTIONS OF SOME MODELS FOR THE INTERACTION OF LONG AND SHORT WAVES

Abstract

Some systems were recently put forth by Nguyen et al. (2020) as models for studying the interaction of long and short waves in dispersive media. These systems were shown to possess synchronized Jacobi elliptic solutions as well as synchronized solitary-wave solutions under certain constraints, i.e., vector solutions, where the two components are proportional to one another. In this paper, periodic traveling-wave solutions given by cnoidal, the Jacobi elliptic function, are studied and explicitly found for these systems. The previously found synchronized solitary-wave solutions are then shown to be subcases of the periodic traveling-wave solutions found here.

2.1 Introduction

The following four systems, termed Schrödinger KdV-KdV, Schrödinger BBM-BBM, Schrödinger KdV-BBM and Schrödinger BBM-KdV, respectively,

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} + a_0 \dfrac{\partial^3 u}{\partial x^3} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} + c\dfrac{\partial^3 v}{\partial x^3} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}; \end{cases} \tag{2.1}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} - a_1 \dfrac{\partial^3 u}{\partial x^2 \partial t} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} - c\dfrac{\partial^3 v}{\partial x^2 \partial t} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}; \end{cases} \tag{2.2}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} + a_0 \dfrac{\partial^3 u}{\partial x^3} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} - c\dfrac{\partial^3 v}{\partial x^2 \partial t} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}; \end{cases} \tag{2.3}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} - a_1 \dfrac{\partial^3 u}{\partial x^2 \partial t} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} + c\dfrac{\partial^3 v}{\partial x^3} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x} \end{cases} \tag{2.4}$$

were recently advocated in [36, 32] (see also [21]) as more suitable models for studying the interaction of long and short waves in dispersive media due to their consistent derivation when compared to the nonlinear Schrödinger-KdV system [27]:

$$\begin{cases} iu_t + u_{xx} + a|u|^2 u = -buv, \\ v_t + cvv_x + v_{xxx} = -\dfrac{b}{2}(|u|^2)_x. \end{cases}$$

Here, the function $u(x,t)$ is a complex-valued function, while $v(x,t)$ is a real-valued function and $x, t \in \mathbb{R}$, where $\mu_0, \mu_1, a_0, a_1, b$ and c are real constants with $\mu_0, \mu_1, a_0, a_1, c > 0$. For a detailed discussion on these systems, we refer our readers to the papers [36, 32, 21].

A traveling-wave solution to the above four systems is a vector solution $\big(u(x,t), v(x,t)\big)$ of the form

$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} f(x - \sigma t), \qquad v(x,t) = g(x - \sigma t), \tag{2.5}$$

where f and g are smooth, real-valued functions with speed $\sigma > 0$ and phase shifts $B, \omega \in \mathbb{R}$. This is not the only way to define traveling-wave solutions with complex-valued components, but for the purpose of this paper we will only consider solutions of this form. Substituting the traveling-wave ansatz (2.5) into the four systems and separating the real and imaginary parts, the following associated systems of ordinary differential equations (ODE) are obtained:

$$\begin{cases} f'g + fg' + a_0 f''' + (\mu_0 - \sigma - 3a_0 B^2 - 2bB)f' = 0, \\ (B + \mu_1)fg + (3a_0 B + b)f'' + (\omega + B\mu_0 - B\sigma - a_0 B^3 - bB^2)f = 0, \\ ff' + gg' + cg''' + (1 - \sigma)g' = 0; \end{cases} \tag{2.6}$$

$$\begin{cases} f'g + fg' + a_1 \sigma f''' + (\mu_0 + 2a_1 B\omega - 3a_1 B^2 \sigma - \sigma - 2bB)f' = 0, \\ (B + \mu_1)fg + (3a_1 B\sigma + b - a_1\omega)f'' + (\omega + B\mu_0 + a_1 B^2 \omega - a_1 B^3 \sigma - B\sigma - bB^2)f = 0, \\ ff' + gg' + c\sigma g''' + (1 - \sigma)g' = 0; \end{cases} \tag{2.7}$$

$$\begin{cases} f'g + fg' + a_0 f''' + (\mu_0 - \sigma - 3a_0 B^2 - 2bB)f' = 0, \\ (B + \mu_1)fg + (3a_0 B + b)f'' + (\omega + B\mu_0 - B\sigma - a_0 B^3 - bB^2)f = 0, \\ ff' + gg' + c\sigma g''' + (1 - \sigma)g' = 0, \end{cases} \tag{2.8}$$

and

$$\begin{cases} f'g + fg' + a_1 \sigma f''' + (\mu_0 + 2a_1 B\omega - 3a_1 B^2\sigma - \sigma - 2bB)f' = 0, \\ (B + \mu_1)fg + (3a_1 B\sigma + b - a_1\omega)f'' + (\omega + B\mu_0 + a_1 B^2\omega - a_1 B^3\sigma - B\sigma - bB^2)f = 0, \\ ff' + gg' + cg''' + (1 - \sigma)g' = 0. \end{cases} \tag{2.9}$$

We refer to semi-trivial solutions as solutions where either f or g is a constant (possibly zero). Of course, the trivial solution $(0, 0)$ is always a solution. In the case when f is a constant multiple of g, the vector solution is termed a synchronized solution. Among the traveling-wave solutions, attention is often given to the solitary-wave and periodic solutions due to the roles they sometimes play in the evolution equations. Solitary-waves are smooth traveling-wave solutions that are symmetric around a single maximum and rapidly decay to zero away from the maximum, while periodic solutions are self-explanatory. Although less common, the term solitary-wave is also sometimes used to describe traveling-wave solutions that are symmetric around a single maximum, but that approach nonzero constants as $\xi \to \pm\infty$.

The topic of existence of synchronized traveling-wave solutions to these four systems has been addressed previously [12]. Notice that when f is a constant multiple of g, i.e., $g = Af$ for some proportional constant A, the three equations in each of the four associated ODEs (2.6)–(2.9) can each be collapsed into one single equations of the form

$$f'^2 = k_3 f^3 + k_2 f^2 + k_1 f + k_0,$$

under certain constraints. The explicit synchronized periodic solutions $\left(e^{i\omega t}e^{iB(x-\sigma t)}f,\ Af\right)$, where f is given by the Jacobi elliptic function cnoidal

$$f(x - \sigma t) :\equiv f(\xi) = C_0 + C_2 \operatorname{cn}^2(\alpha\xi + \beta, m),$$

are then obtained by demanding the coefficients in each of the four cases to satisfy certain constraints. (A brief description of the Jacobi elliptic functions is reviewed below.) In [33], it was shown that the systems possess synchronized solitary-waves with the usual sech^2-profile typical of dispersive equations. In [12], a novel approach was first employed to establish the existence of periodic traveling-wave solutions for these systems, namely, the topological degree theory for positive operators that was introduced by Krasnosel'skii [29, 30] and used in several models [14, 15, 35].

It is worth it to pointing that explicit solitary-wave solutions have been found for another system [17, 18], the $abcd$-system

$$\begin{cases} \eta_t + w_x + (w\eta)_x + a w_{xxx} - b\eta_{xxt} = 0\,, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0\,, \end{cases}$$

where a, b, c and d are real constants satisfying

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \quad a + b + c + d = \frac{1}{3},$$

and $\theta \in [0, 1]$. This system is used to model small-amplitude, long wavelength, gravity waves on the surface of water [6, 5]. Here, $\eta(x, t)$ and $w(x, t)$ are real valued functions and $x, t \in \mathbb{R}$. However, the existence of periodic traveling-wave solutions for this system are still not well understood. The only result that we are aware of is for the special case when $a = c = 0$ and $b = d = 1/6$, where the solutions are given in terms of the Jacobi elliptic cnoidal function [15].

The manuscript is organized as follows. In Section 2.2, some facts about the Jacobi elliptic functions are reviewed, and the results are summarized. In Section 2.3, the explicit cnoidal solutions to the four systems are established, and how these solutions limit to the solitary-wave solutions are analyzed. Section 2.4 is devoted to discussion of the results. Some tedious formulae and expressions are delegated to Appendix A.

2.2 Preliminaries and statement of results

For the readers' convenience, some notions of the Jacobi elliptic functions are briefly recalled here. Let

$$\nu = \int_0^{\phi} \frac{1}{\sqrt{1 - m^2 \sin^2 t}} dt, \qquad \text{for } 0 \leq m \leq 1,$$

then $\nu = F(\phi, m)$ or, equivalently, $\phi = F^{-1}(\nu, m) = \mathrm{am}(\nu, m)$, which is the Jacobi amplitude. The three basic Jacobi elliptic functions: cnoidal $\mathrm{cn}(\nu, m)$, snoidal $\mathrm{sn}(\nu, m)$, and dnoidal $\mathrm{dn}(\nu, m)$, are defined as

$$\mathrm{cn}(\nu, m) = \cos(\phi) = \cos\left(F^{-1}(\nu, m)\right), \quad \mathrm{sn}(\nu, m) = \sin(\phi) = \sin\left(F^{-1}(\nu, m)\right), \quad \text{and}$$

$$\mathrm{dn}(\nu, m) = \frac{d}{d\nu}(\phi) = \frac{d}{d\nu}\left(F^{-1}(\nu, m)\right),$$

where m is referred to as the Jacobi elliptic modulus. These functions are generalizations of the trigonometric and hyperbolic functions, which satisfy

$$\begin{aligned} \mathrm{cn}(\nu, 0) &= \cos(\nu), \qquad & \mathrm{sn}(\nu, 0) &= \sin(\nu), \\ \mathrm{cn}(\nu, 1) &= \mathrm{sech}(\nu), \qquad & \mathrm{sn}(\nu, 1) &= \tanh(\nu). \end{aligned}$$

For $m \in [0, 1)$, the functions $\mathrm{cn}(\nu, m)$ and $\mathrm{sn}(\nu, m)$ are periodic with period $4K(m)$, where $K(m)$ is given by

$$K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - m^2 \sin^2 t}}\, dt.$$

We recall the following relations between these functions:

$$\begin{aligned} &\mathrm{sn}^2(\lambda\xi, m) = 1 - \mathrm{cn}^2(\lambda\xi, m), \\ &\mathrm{dn}^2(\lambda\xi, m) = 1 - m^2 + m^2\,\mathrm{cn}^2(\lambda\xi, m), \\ &\frac{d}{d\xi}\,\mathrm{cn}(\lambda\xi, m) = -\lambda\,\mathrm{sn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m), \\ &\frac{d}{d\xi}\,\mathrm{sn}(\lambda\xi, m) = \lambda\,\mathrm{cn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m), \\ &\frac{d}{d\xi}\,\mathrm{dn}(\lambda\xi, m) = -m^2\lambda\,\mathrm{cn}(\lambda\xi, m)\,\mathrm{sn}(\lambda\xi, m). \end{aligned} \tag{2.10}$$

In this manuscript, the existence of periodic traveling-wave solutions to the above four associated ODE systems (2.6)–(2.9) in general is analyzed. The periodic traveling-wave solutions sought here are of the following form

$$f(\xi) = \sum_{r=0}^{n} d_r\,\mathrm{cn}^r(\lambda\xi, m) \quad \text{and} \quad g(\xi) = \sum_{r=0}^{n} h_r\,\mathrm{cn}^r(\lambda\xi, m)\,, \tag{2.11}$$

where $d_r, h_r \in \mathbb{R}$, $\lambda > 0$, $0 \le m \le 1$, and $n \in \mathbb{N}$ such that $n > 2$. For simplicity, we write the sums in (2.11) as ranging to the same value of n. It will be made clear that the same results will hold even if the sums range to different values of n_1 and n_2, so long as n_1, $n_2 > 2$.

Remark. For the corresponding ODE systems of the Schrödinger KdV-KdV and Schrödinger KdV-BBM systems ((2.6) and (2.8)), the second equation can be rearranged to find

$$fg = \frac{-1}{(B + \mu_1)}\Big((3a_0B + b)f'' + (\omega + B\mu_0 - B\sigma - a_0B^3 - bB^2)f\Big).$$

Substituting this into the first equation of (2.6) and (2.8) yields the third-order ODE

$$\tau_3 f''' + \tau_1 f' = 0,$$

where

$$\begin{aligned} \tau_3 &= a_0 - \frac{3a_0 B + b}{B + \mu_1} \quad \text{and} \\ \tau_1 &= \mu_0 - \sigma - 3a_0 B^2 - 2bB - \frac{\omega + B\mu_0 - B\sigma - a_0 B^3 - bB^2}{B + \mu_1}. \end{aligned} \tag{2.12}$$

The same process for the corresponding ODE systems of the Schrödinger BBM-BBM and Schrödinger BBM-KdV systems ((2.7) and (2.9)) yields the following third-order ODE

$$\tilde{\tau}_3 f''' + \tilde{\tau}_1 f' = 0,$$

where

$$\begin{aligned} \tilde{\tau}_3 &= a_1 \sigma - \frac{3a_1 B\sigma + b - a_1 \omega}{B + \mu_1} \quad \text{and} \\ \tilde{\tau}_1 &= \mu_0 + 2a_1 B\omega - 3a_1 B^2 \sigma - \sigma - 2bB - \frac{\omega + B\mu_0 + a_1 B^2 \omega - a_1 B^3 \sigma - B\sigma - bB^2}{B + \mu_1}. \end{aligned} \tag{2.13}$$

These are simple third-order ODEs that can be solved by standard methods. It is clear that in general, f as defined in (2.11) will not be a solution to these equations. However, this does not mean that our assumption of the form of f in (2.11) is incorrect. For instance, suppose we find a solution of the form (2.5), with f and g as defined in (2.11). Then f and g must simultaneously satisfy the three equations in (2.6)–(2.9). So, it must also be that the corresponding third-order ODE above is satisfied. As stated above, the f in (2.11) will not satisfy this ODE in general, so it must be that B, ω and σ in our solution already force the coefficients in (2.12) and (2.13) to be zero. Therefore, as this paper aims to find solutions that simultaneously satisfy all three equations in (2.6)–(2.9) with no restrictions placed on the system parameters, the form of f and g in (2.11) is completely justified, and any solution found of this form will inherently contain B, ω and σ that force the coefficients of the corresponding third-order ODE to be zero. In fact, straightforward calculation shows that the B, ω, and σ found in all the non-trivial solutions in Section 2.3 satisfy that the coefficients in (2.12) and (2.13) are zero, for their respective system.

Using the above relations (2.10), the following is revealed:

$$\begin{aligned}\frac{d}{d\xi}\operatorname{cn}^r &= -r\lambda\operatorname{cn}^{r-1}\operatorname{sn}\operatorname{dn},\\ \frac{d^2}{d\xi^2}\operatorname{cn}^r &= -r\lambda^2\big[(r+1)m^2\operatorname{cn}^{r+2}+r(1-2m^2)\operatorname{cn}^r+(r-1)(m^2-1)\operatorname{cn}^{r-2}\big],\\ \frac{d^3}{d\xi^3}\operatorname{cn}^r &= r\lambda^3\operatorname{sn}\operatorname{dn}\big[(r+1)(r+2)m^2\operatorname{cn}^{r+1}+r^2(1-2m^2)\operatorname{cn}^{r-1}+(r-1)(r-2)(m^2-1)\operatorname{cn}^{r-3}\big],\end{aligned} \tag{2.14}$$

where the argument $(\lambda\xi, m)$ has been dropped for clarity reasons. Notice that each of the above four associated ODE systems (2.6)–(2.9) involves three equations. Plugging (2.14) into these systems, the following generic forms are obtained:

$$\begin{cases}\operatorname{sn}(\lambda\xi,m)\operatorname{dn}(\lambda\xi,m)\displaystyle\sum_{q=0}^{2n-1}k_{1,q}\operatorname{cn}^q(\lambda\xi,m)=0,\\ \displaystyle\sum_{q=0}^{2n}k_{2,q}\operatorname{cn}^q(\lambda\xi,m)=0,\\ \operatorname{sn}(\lambda\xi,m)\operatorname{dn}(\lambda\xi,m)\displaystyle\sum_{q=0}^{2n-1}k_{3,q}\operatorname{cn}^q(\lambda\xi,m)=0,\end{cases} \tag{2.15}$$

where $k_{j,q} = K_{j,q}(B, \lambda, m, \sigma, \omega, d_i, h_i)$, that is, some function of the parameters defined in (2.5) and (2.11) that depends on the equation j and power on the cnoidal function q. Notice that as (2.15) must hold true for all ξ, the linear independence of cn^q implies that $k_{j,q} = 0$ for each j and q. Since $n > 2$, or equivalently, $2n-1 > n+1$, the sum $(ff' + gg')$ will be the sole contributor to the highest order term $k_{3,2n-1}\operatorname{cn}^{2n-1}$ for all four systems. Requiring $k_{3,2n-1} = 0$ yields the following equation

$$k_{3,2n-1} = -n\lambda(d_n^2 + h_n^2) = 0.$$

By assumption $\lambda, n \neq 0$, which implies that $d_n = h_n = 0$. This argument will hold until g''' also contributes to the highest order term, which will occur when $2n-1 \leq n+1$. So it follows that $d_r = h_r = 0$ for all $r > 2$, which reduces the periodic traveling-wave ansatz (2.11) to

$$f(\xi) = d_0 + d_1\operatorname{cn}(\lambda\xi,m) + d_2\operatorname{cn}^2(\lambda\xi,m) \quad \text{and} \quad g(\xi) = h_0 + h_1\operatorname{cn}(\lambda\xi,m) + h_2\operatorname{cn}^2(\lambda\xi,m). \tag{2.16}$$

The above argument can be generalized if f and g in (2.11) range to different values of n_1 and n_2, as long as $n_1, n_2 > 2$. Suppose $n_1 > n_2$, then the same reasoning above can be used to deduce that $d_r = 0$ for all $r > n_2$. At this point f and g now range to the same value of n_2, and the exact same result found above will hold. The same follows if $n_2 > n_1$.

Next, by demanding all the coefficients $k_{j,q} = 0$, a set of 13 equations is obtained for each of the four systems involving 11 unknowns $d_i, h_i, B, \lambda, \omega, \sigma$ and m with $i = 0, 1, 2$ (Eqs (A.1)-(A.4)). For the Schrödinger KdV-KdV and Schrödinger BBM-BBM systems, the first and last equations in (2.6) and (2.7), respectively, further yield $d_1 = h_1 = 0$. In particular, the only nontrivial periodic solutions for the Schrödinger KdV-KdV and Schrödinger BBM-BBM systems ((2.1) and (2.2)) are of the form

$$f(\xi) = d_0 + d_2 \operatorname{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2 \operatorname{cn}^2(\lambda\xi, m). \tag{2.17}$$

Under these conditions, the sets of 13 equations involving 11 unknowns (Eqs (A.1) and (A.2)) reduce to sets of seven equations with nine unknowns. Similarly, for the Schrödinger KdV-BBM system and the Schrödinger BBM-KdV system, the first and last equations in (2.8) and (2.9), respectively, reveal that $h_1 = 0$. Additionally, when substituting $h_1 = 0$ into (A.3) and (A.4), the coefficients $k_{3,2}$ and $k_{3,0}$ in both systems require that either $d_1 = 0$ or $d_0 = d_2 = 0$. When $d_1 = h_1 = 0$, we have solutions of the form (2.17), where the sets of 13 equations involving 11 unknowns ((A.3) and (A.4)) reduce to seven equations with nine unknowns. When $d_0 = d_2 = h_1 = 0$, we have that the only nontrivial periodic solutions for the Schrödinger KdV-BBM and Schrödinger BBM-KdV systems ((2.3) and (2.4)) are of the form

$$f(\xi) = d_1 \operatorname{cn}(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2 \operatorname{cn}^2(\lambda\xi, m),$$

in which case the sets of 13 equations involving 11 unknowns ((A.3) and (A.4)) reduce to sets of six equations with eight unknowns.

The exact, periodic traveling-wave solutions to the four systems (2.1)–(2.4) could then be established by solving those reduced nonlinear systems with the help of the software Maple. As there are two degrees of freedom, in principle any pair of two unknowns can be chosen as "free parameters" so long as solutions can be found consistently. In most physical situations, though, it is more desirable to think of the wave speed σ and elliptic modulus m as "independent" parameters; that is, the cnoidal solutions are found for fixed elliptic modulus $m \in [0, 1]$ and a certain range of wave speed $\sigma > 0$. For the Schrödinger KdV-KdV system (2.1), these nontrivial periodic traveling-wave solutions are established for each wave speed $\sigma > 0$ with $2c > a_0 > 0$, while for the Schrödinger BBM-BBM system (2.2), $\sigma > 0$ with $2c > a_1 > 0$. For the Schrödinger KdV-BBM (2.3), the range of wave speed is $\sigma > \frac{a_0}{2c} > 0$, while for the Schrödinger BBM-KdV (2.4), $0 < \sigma < \frac{2c}{a_1}$.

2.3 Exact Jacobi elliptic solutions

In this section, we present the explicit periodic traveling-wave solutions for the four systems (2.1)–(2.4) of the form (2.5), where f and g are given by (2.16). The limit of the solutions as the elliptic modulus m approaches one is then taken, and the resulting solitary-wave solutions are compared to the synchronized solitary-wave solutions previously found in [33]. Semi-trivial solutions, i.e. solutions where either f or g in (2.16) is constant, are then presented.

For conciseness, let

$$R = \pm\sqrt{m^4 - m^2 + 1}, \tag{2.18}$$

then $R \in \mathbb{R}$ as $m \in [0, 1]$.

2.3.1 Schrödinger KdV-KdV

Setting all $k_{j,q} = 0$ gives us the following set of parameters, whenever $2c > a_0 > 0$:

$$\begin{cases}
B = \frac{a_0\,\mu_1 - b}{2\,a_0}\,,\\[1em]
d_1 = h_1 = 0\,,\\[1em]
d_0 = \frac{\left(m^4 - 2\,m^2R - m^2 + R + 1\right)\sqrt{2\,c - a_0}\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)}{8\,\sqrt{a_0}R^2\left(a_0 - c\right)}\,,\\[1em]
d_2 = \frac{3\,\sqrt{2\,c - a_0}\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)m^2}{8\,\sqrt{a_0}R\left(a_0 - c\right)}\,,\\[1em]
h_0 = \frac{-1}{8\,a_0R\left(a_0 - c\right)}\Big(6\,a_0{}^3m^2\mu_1{}^2 - 3\,a_0{}^3\mu_1{}^2R + 6\,a_0{}^2c\mu_1{}^2R - 3\,a_0{}^3\mu_1{}^2 - 4\,a_0{}^2bm^2\mu_1\\
\quad +2\,a_0{}^2b\mu_1\,R - 4\,a_0\,bc\mu_1\,R + 2\,a_0{}^2b\mu_1 - 8\,a_0{}^2m^2\mu_0 + 4\,a_0{}^2\mu_0\,R - 8\,a_0{}^2R\sigma - 2\,a_0\,b^2m^2\\
\quad +a_0\,b^2R - 8\,a_0\,c\mu_0\,R + 8\,a_0\,cR\sigma - 2\,b^2cR + 8\,a_0{}^2m^2 + 4\,a_0{}^2\mu_0 + 4\,a_0{}^2R + a_0\,b^2 - 4\,a_0{}^2\Big),\\[1em]
h_2 = \frac{3\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)m^2}{8\,R\left(a_0 - c\right)}\,,\\[1em]
\lambda = \sqrt{\frac{3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0}{16\,a_0R\left(a_0 - c\right)}}\,,\\[1em]
\omega = -\left(a_0\,\mu_1{}^2 - \mu_1\,b - \mu_0 + \sigma\right)\mu_1\,,\\[1em]
\sigma > 0\,,\\[1em]
m \in [0, 1]\,.
\end{cases}$$

Thus, explicit periodic traveling-wave solutions to the Schrödinger KdV-KdV system $\big(u(x,t), v(x,t)\big) = \big(e^{i\omega t}e^{iB(x-\sigma t)}f(x-\sigma t), g(x-\sigma t)\big)$, given in terms of the Jacobi cnoidal function

$$f(\xi) = d_0 + d_2\,\mathrm{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2\,\mathrm{cn}^2(\lambda\xi, m)$$

are established. Notice that $\frac{h_2}{d_2} = \sqrt{\frac{a_0}{2c-a_0}}$ and that as m approaches one, R limits to ± 1. When $m = R = 1$, the above coefficients simplify to $d_0 = 0$ and

$$\begin{cases} \tilde{h}_0 = -\frac{1}{4\,a_0\,(a_0-c)}\left(3a_0^2c\mu_1^2 - 2a_0bc\mu_1 + 4a_0^2 - 4a_0c\mu_0 - b^2c - 4a_0^2\sigma + 4a_0c\sigma\right), \\ \tilde{d}_2 = \frac{3\sqrt{2c-a_0}\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)}{8\sqrt{a_0}\,(a_0-c)}, \\ \tilde{h}_2 = \frac{3\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)}{8\,(a_0-c)}, \\ \tilde{\lambda} = \sqrt{\frac{3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0}{16\,a_0\,(a_0-c)}}, \\ \omega = -\left(a_0\,\mu_1{}^2 - \mu_1\,b - \mu_0 + \sigma\right)\mu_1, \end{cases}$$

from which one obtains the following solitary-wave solution to system (2.1):

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}\tilde{f}(x-\sigma t) \qquad \text{and} \qquad v(x,t) = \tilde{h}_0 \pm \sqrt{\frac{a_0}{2c-a_0}}\tilde{f}(x-\sigma t),$$

where $\tilde{f}(\xi) = \tilde{d}_2\,\mathrm{sech}^2(\tilde{\lambda}\xi)$. Furthermore, when $\sigma = \frac{4a_0^2+3a_0^2c\mu_1^2-2a_0bc\mu_1-4a_0c\mu_0-b^2c}{4a_0(a_0-c)}$, one has $\tilde{h}_0 = 0$, and the synchronized solitary-wave solution established in [33] is recovered.

When $m = -R = 1$, the above coefficients simplify to

$$\begin{cases} \bar{d}_0 = \frac{\sqrt{2c-a_0}\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)}{4\sqrt{a_0}\,(a_0-c)}, \\ \bar{h}_0 = \frac{3a_0^3\mu_1^2 - 2a_0^2b\mu_1 - 4a_0^2\mu_0 - a_0b^2 - 3a_0^2c\mu_1^2 + 2a_0bc\mu_1 + 4a_0c\mu_0 + b^2c + 4a_0\sigma(a_0-c)}{4\,a_0\,(a_0-c)}, \\ \bar{d}_2 = \frac{3\sqrt{2c-a_0}\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)}{8\sqrt{a_0}\,(a_0-c)}, \\ \bar{h}_2 = -\frac{3\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)}{8\,(a_0-c)}, \\ \bar{\lambda} = \sqrt{-\frac{3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0}{16\,a_0\,(a_0-c)}}, \\ \omega = -\left(a_0\,\mu_1{}^2 - \mu_1\,b - \mu_0 + \sigma\right)\mu_1, \end{cases}$$

and one arrives at the solitary-wave solution

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}[\bar{d}_0 + \bar{f}(x-\sigma t)] \qquad \text{and} \qquad v(x,t) = \bar{h}_0 \pm \sqrt{\frac{a_0}{2c-a_0}}\bar{f}(x-\sigma t),$$

where $\bar{f}(\xi) = \bar{d}_2 \operatorname{sech}^2(\bar{\lambda}\xi)$.

Aside from the above nontrivial solutions, system (2.1) also possesses the following trivial and semi-trivial solutions:

(1)

$$u(x,t) = 0 \qquad \text{and} \qquad v(x,t) = h_0,$$

for any $h_0 \in \mathbb{R}$.

(2)

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}d_0 \qquad \text{and} \qquad v(x,t) = h_0,$$

where $\sigma = \frac{\omega - a_0B^3 - bB^2 + Bh_0 + B\mu_0 + h_0\mu_1}{B}$, for any $B, d_0, h_0, \omega \in \mathbb{R}$.

(3)

$$u(x,t) = 0 \qquad \text{and} \qquad v(x,t) = -\frac{2}{3}h_2 + \frac{1}{3}\frac{h_2}{m^2} + \sigma - 1 + h_2 \operatorname{cn}^2\big(\lambda(x-\sigma t), m\big),$$

where $\lambda = \sqrt{\frac{h_2}{12cm^2}}$, for any $h_2, \sigma > 0$ and $m \in [0,1]$.

2.3.2 Schrödinger BBM-BBM

Setting all $k_{j,q} = 0$ gives us the following, whenever $2c > a_1 > 0$ and R is as defined in (2.18):

$$\begin{cases}
B = \frac{a_1\,\mu_0\,\mu_1 - b}{2a_1\sigma\,(a_1\,\mu_1{}^2+1)}\,, \\
d_1 = h_1 = 0\,, \\
d_0 = \frac{\sqrt{a_1\,(2\,c-a_1)}\left(m^4+2\,m^2R-m^2-R+1\right)}{8\,a_1R^2\sigma\,(a_1\,\mu_1{}^2+1)^2(a_1-c)}\Big(4\,a_1{}^3\mu_1{}^4\sigma-4\,a_1{}^2b\mu_1{}^3\sigma-a_1{}^2\mu_0{}^2\mu_1{}^2-4\,a_1{}^2\mu_0\,\mu 1^2\sigma+8\,a_1{}^2\mu_1{}^2\sigma \\
\quad +2\,a_1\,b\mu_0\,\mu 1-4\,a_1\,b\mu_1\,\sigma-4\,a_1\,\mu_0\,\sigma+4\,a_1\,\sigma-b^2\Big)\,, \\
d_2 = \frac{-3\,m^2\sqrt{a_1\,(2\,c-a_1)}}{8\,a_1R\sigma\,(a_1\,\mu_1{}^2+1)^2(a_1-c)}\Big(4\,a_1{}^3\mu_1{}^4\sigma-4\,a_1{}^2b\mu_1{}^3\sigma-a_1{}^2\mu_0{}^2\mu_1{}^2-4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma+8\,a_1{}^2\mu_1{}^2\sigma \\
\quad +2\,a_1\,b\mu_0\,\mu_1-4\,a_1\,b\mu_1\,\sigma-4\,a_1\,\mu_0\,\sigma+4\,a_1\,\sigma-b^2\Big)\,, \\
h_0 = \frac{1}{8\,a_1R\sigma\,(a_1\,\mu_1{}^2+1)^2(a_1-c)}\Big(8\,a_1{}^4\mu_1{}^4r\sigma^2-8\,a_1{}^3c\mu_1{}^4R\sigma^2+8\,a_1{}^4m^2\mu_1{}^4\sigma-4\,a_1{}^4\mu_1{}^4R\sigma-4\,a_1{}^4\mu_1{}^4\sigma \\
\quad -8\,a_1{}^3bm^2\mu_1{}^3\sigma-4\,a_1{}^3b\mu_1{}^3R\sigma+8\,a_1{}^2bc\mu_1{}^3R\sigma+4\,a_1{}^3b\mu_1{}^3\sigma-2\,a_1{}^3m^2\mu_0{}^2\mu_1{}^2 \\
\quad -8\,a_1{}^3m^2\mu_0\,\mu_1{}^2\sigma-a_1{}^3\mu_0{}^2\mu_1{}^2R-4\,a_1{}^3\mu_0\,\mu_1{}^2R\sigma+16\,a_1{}^3\mu_1{}^2R\sigma^2+2\,a_1{}^2c\mu_0{}^2\mu_1{}^2R \\
\quad +8\,a_1{}^2c\mu_0\,\mu 1^2R\sigma-16\,a_1{}^2c\mu_1{}^2R\sigma^2+16\,a_1{}^3m^2\mu_1{}^2\sigma+a_1{}^3\mu_0{}^2\mu_1{}^2+4\,a_1{}^3\mu_0\,\mu_1{}^2\sigma-8\,a_1{}^3\mu_1{}^2R\sigma \\
\quad -8\,a_1{}^3\mu_1{}^2\sigma+4\,a_1{}^2bm^2\mu_0\,\mu_1-8\,a_1{}^2bm^2\mu_1\,\sigma+2\,a_1{}^2b\mu_0\,\mu_1\,R-4\,a_1{}^2b\mu_1\,R\sigma-4\,a_1\,bc\mu_0\,\mu_1\,R \\
\quad +8\,a_1\,bc\mu_1\,R\sigma-2\,a_1{}^2\,b\,\mu_0\,\mu_1+4\,a_1{}^2b\mu_1\,\sigma-8\,a_1{}^2m^2\mu_0\,\sigma-4\,a_1{}^2\mu_0\,R\sigma+8\,a_1{}^2R\sigma^2+8\,a_1\,c\mu_0\,R\sigma \\
\quad -8\,a_1\,cR\sigma^2+8\,a_1{}^2m^2\sigma+4\,a_1{}^2\mu_0\,\sigma-4\,a_1{}^2R\sigma-2\,a_1\,b^2m^2-a_1\,b^2R+2\,b^2cR-4\,a_1{}^2\sigma+a_1\,b^2\Big)\,, \\
h_2 = \frac{-3\left(4\,a_1{}^3\mu_1{}^4\sigma-4\,a_1{}^2b\mu_1{}^3\sigma-a_1{}^2\mu_0{}^2\mu_1{}^2-4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma+8\,a_1{}^2\mu_1{}^2\sigma+2\,a_1\,b\mu_0\,\mu_1-4\,a_1\,b\mu_1\,\sigma-4\,a_1\,\mu_0\,\sigma+4\,a_1\,\sigma-b^2\right)m^2}{8\,R\sigma\,(a_1\,\mu_1{}^2+1)^2(a_1-c)}\,, \\
\lambda = \sqrt{\frac{4\,a_1{}^3\mu_1{}^4\sigma-4\,a_1{}^2b\mu_1{}^3\sigma-a_1{}^2\mu_0{}^2\mu_1{}^2-4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma+8\,a_1{}^2\mu_1{}^2\sigma+2\,a_1\,b\mu_0\,\mu_1-4\,a_1\,b\mu_1\,\sigma-4\,a_1\,\mu_0\,\sigma+4\,a_1\,\sigma-b^2}{-16\,a_1R\sigma^2(a_1-c)(a_1\,\mu_1{}^2+1)^2}}\,, \\
\omega = -\frac{\left(a_1\,\mu_1{}^2\sigma-b\mu_1-\mu_0+\sigma\right)\mu_1}{a_1\,\mu_1{}^2+1}\,, \\
\sigma > 0\,, \\
m \in [0,1]\,.
\end{cases}$$

Thus, explicit periodic traveling-wave solutions to the Schrödinger BBM-BBM system $\big(u(x,t), v(x,t)\big) = \big(e^{i\omega t}e^{iB(x-\sigma t)}f(x-\sigma t), g(x-\sigma t)\big)$, given in terms of the Jacobi cnoidal function

$$f(\xi) = d_0 + d_2\,\mathrm{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2\,\mathrm{cn}^2(\lambda\xi, m)$$

are established. Notice that $\frac{h_2}{d_2} = \sqrt{\frac{a_1}{2c-a_1}}$. When $m = R = 1$, the above coefficients simplify to

$$
\begin{cases}
\tilde{d}_0 = \frac{\sqrt{a_1(2c-a_1)}}{4a_1\sigma(a_1\mu_1^2+1)^2(a_1-c)}\Big(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma \\
\quad +2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\
\tilde{d}_2 = \frac{-3\sqrt{a_1(2c-a_1)}}{8a_1\sigma(a_1\mu_1^2+1)^2(a_1-c)}\Big(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma \\
\quad +2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\
\tilde{h}_0 = \frac{1}{4a_1\sigma(a_1\mu_1^2+1)^2(a_1-c)}\Big(4a_1^4\mu_1^4\sigma^2 - 4a_1^3c\mu_1^4\sigma^2 - 4a_1^3b\mu_1^3\sigma + 4a_1^2bc\mu_1^3\sigma - a_1^3\mu_0^2\mu_1^2 \\
\quad -4a_1^3\mu_0\mu_1^2\sigma + 8a_1^3\mu_1^2\sigma^2 + a_1^2c\mu_0^2\mu_1^2 + 4a_1^2c\mu_0\mu_1^2\sigma - 8a_1^2c\mu_1^2\sigma^2 + 2a_1^2b\mu_0\mu_1 \\
\quad -4a_1^2b\mu_1\sigma - 2a_1bc\mu_0\mu_1 + 4a_1bc\mu_1\sigma - 4a_1^2\mu_0\sigma + 4a_1^2\sigma^2 + 4a_1c\mu_0\sigma - 4a_1c\sigma^2 \\
\quad - a_1b^2 + b^2c\Big), \\
\tilde{h}_2 = \frac{-3\left(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\right)}{8\sigma(a_1\mu_1^2+1)^2(a_1-c)}, \\
\tilde{\lambda} = \sqrt{\frac{4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2}{-16a_1\sigma^2(a_1-c)(a_1\mu_1^2+1)^2}}, \\
\omega = -\frac{(a_1\mu_1^2\sigma - b\mu_1 - \mu_0 + \sigma)\mu_1}{a_1\mu_1^2+1}, \\
\sigma > 0,
\end{cases}
$$

from which one obtains the following solitary-wave solution to system (2.2):

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}[\tilde{d}_0 + \tilde{f}(x-\sigma t)] \qquad \text{and} \qquad v(x,t) = \tilde{h}_0 \pm \sqrt{\frac{a_1}{2c-a_1}}\tilde{f}(x-\sigma t),$$

where $\tilde{f}(\xi) = \tilde{d}_2\,\mathrm{sech}^2(\tilde{\lambda}\xi)$.

When $m = -R = 1$, the above coefficients simplify to $d_0 = 0$ and

$$\begin{cases} \bar{d}_2 = \frac{3\sqrt{a_1\,(2\,c-a_1)}}{8 a_1\sigma\,(a_1\,\mu_1{}^2+1)^2\,(a_1-c)}\Big(4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2 b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma \\ \quad +2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1\,b\mu_1\,\sigma - 4\,a_1\,\mu_0\,\sigma + 4\,a_1\,\sigma - b^2\Big), \\ \bar{h}_0 = \frac{1}{-8a_1\sigma\,(a_1\,\mu_1{}^2+1)^2\,(a_1-c)}\Big(-8\,a_1{}^4\mu_1{}^4\sigma^2 + 8\,a_1{}^3 c\mu_1{}^4\sigma^2 + 8\,a_1{}^4\mu_1{}^4\sigma - 8\,a_1{}^2 bc\mu_1{}^3\sigma - 16\,a_1{}^3\mu_1{}^2\sigma^2 \\ \quad -2\,a_1{}^2 c\mu_0{}^2\mu_1{}^2 - 8\,a_1{}^2 c\mu_0\,\mu_1{}^2\sigma + 16\,a_1{}^2 c\mu_1{}^2\sigma^2 + 16\,a_1{}^3\mu_1{}^2\sigma + 4\,a_1\,bc\mu_0\,\mu_1 - 8\,a_1\,bc\mu_1\sigma \\ \quad -8\,a_1{}^2\sigma^2 - 8\,a_1\,c\mu_0\sigma + 8\,a_1\,c\sigma^2 + 8\,a_1{}^2\sigma - 2\,b^2 c\Big), \\ \bar{h}_2 = \frac{3\left(4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2 b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma + 2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1\,b\mu_1\,\sigma - 4\,a_1\,\mu_0\,\sigma + 4\,a_1\,\sigma - b^2\right)}{8\sigma\,(a_1\,\mu_1{}^2+1)^2\,(a_1-c)}, \\ \bar{\lambda} = \sqrt{\frac{4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2 b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma + 2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1\,b\mu_1\,\sigma - 4\,a_1\,\mu_0\,\sigma + 4\,a_1\,\sigma - b^2}{16 a_1\sigma^2\,(a_1-c)(a_1\,\mu_1{}^2+1)^2}}, \\ \omega = -\frac{\left(a_1\,\mu_1{}^2\sigma - b\mu_1 - \mu_0 + \sigma\right)\mu_1}{a_1\,\mu_1{}^2+1}, \end{cases}$$

and one arrives at the solitary-wave solution

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}\bar{f}(x-\sigma t) \qquad \text{and} \qquad v(x,t) = \bar{h}_0 \pm \sqrt{\frac{a_1}{2c-a_1}}\bar{f}(x-\sigma t),$$

where $\bar{f}(\xi) = \bar{d}_2\,\text{sech}^2(\bar{\lambda}\xi)$. Furthermore, when $B = \frac{a_1\,\mu_0\,\mu_1 - b}{2\,\sigma\,a_1\,(a_1\,\mu_1{}^2+1)}$ satisfies the following equation:

$$(a_1^2 c\mu_0\mu_1 - a_1 bc)B^2 + (2a_1 bc\mu_1 + 2a_1 c\mu_0 - 2a_1^3\mu_1^2 - 2a_1^2)B + (a_1^2\mu_0\mu_1 + bc - a_1 b - a_1 c\mu_0\mu_1) = 0,$$

one has $\bar{h}_0 = 0$, and the synchronized solitary-wave solution established in [33] is recovered.

Aside from the above nontrivial solutions, system (2.2) also possesses the following trivial and semi-trivial solutions:

(1)

$$u(x,t) = 0 \qquad \text{and} \qquad v(x,t) = h_0,$$

for any $h_0 \in \mathbb{R}$.

(2)

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}d_0 \qquad \text{and} \qquad v(x,t) = h_0,$$

where $\sigma = \frac{a_1 B^2\omega - bB^2 + Bh_0 + B\mu_0 + h_0\mu_1 + \omega}{B(a_1 B^2+1)}$, for any $B, d_0, h_0, \omega \in \mathbb{R}$.

(3)

$$u(x,t) = 0 \qquad \text{and} \qquad v(x,t) = -\frac{2}{3}h_2 + \frac{1}{3}\frac{h_2}{m^2} + \sigma - 1 + h_2\,\text{cn}^2\left(\lambda(x-\sigma t), m\right),$$

where $\lambda = \sqrt{\frac{h_2}{12cm^2\sigma}}$, for any $h_2, \sigma > 0$ and $m \in [0, 1]$.

2.3.3 Schrödinger KdV-BBM

Setting all $k_{j,q} = 0$ gives us the following set of parameters, with R as defined in (2.18):

$$\begin{cases} B = \frac{a_0\,\mu_1 - b}{2\,a_0}\,, \\ d_1 = h_1 = 0\,, \\ d_0 = \frac{\left(m^4 - 2\,m^2 R - m^2 + R + 1\right)\sqrt{2\,c\sigma - a_0}\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)}{8\sqrt{a_0}\,R^2(a_0 - c\sigma)}\,, \\ d_2 = \frac{3\sqrt{2\,c\sigma - a_0}\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)m^2}{8\sqrt{a_0}R(a_0 - c\sigma)}\,, \\ h_0 = \frac{1}{8a_0 R\,(c\sigma - a_0)}\Big(6\,a_0{}^2c\mu_1{}^2R\sigma + 6\,a_0{}^3m^2\mu_1{}^2 - 3\,a_0{}^3\mu_1{}^2R - 4\,a_0\,bc\mu_1\,R\sigma - 3\,a_0{}^3\mu_1{}^2 - 4\,a_0{}^2bm^2\mu_1 \\ \quad +2\,a_0{}^2b\mu_1\,R - 8\,a_0\,c\mu_0\,R\sigma + 8\,a_0\,cR\sigma^2 - 2\,b^2cR\sigma + 2\,a_0{}^2b\mu_1 - 8\,a_0{}^2m^2\mu_0 + 4\,a_0{}^2\mu_0\,R \\ \quad -8\,a_0{}^2R\sigma - 2\,a_0\,b^2m^2 + a_0\,b^2R + 8\,a_0{}^2m^2 + 4\,a_0{}^2\mu_0 + 4\,a_0{}^2R + a_0\,b^2 - 4\,a_0{}^2\Big)\,, \\ h_2 = \frac{3\left(3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0\right)m^2}{8R(a_0 - c\sigma)}\,, \\ \lambda = \sqrt{\frac{3\,a_0{}^2\mu_1{}^2 - 2\,a_0\,b\mu_1 - 4\,a_0\,\mu_0 - b^2 + 4\,a_0}{16a_0\,R\,(a_0 - c\sigma)}}\,, \\ \omega = -\left(a_0\,\mu_1{}^2 - \mu_1\,b - \mu_0 + \sigma\right)\mu_1\,, \\ \sigma > \frac{a_0}{2\,c}\,, \\ m \in [0, 1]\,. \end{cases}$$

Thus, explicit periodic traveling-wave solutions to the Schrödinger KdV-BBM system $\big(u(x,t), v(x,t)\big) = \big(e^{i\omega t}e^{iB(x-\sigma t)}f(x - \sigma t), g(x - \sigma t)\big)$, given in terms of the Jacobi cnoidal function

$$f(\xi) = d_0 + d_2\,\mathrm{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2\,\mathrm{cn}^2(\lambda\xi, m)$$

are established. Notice that $\frac{h_2}{d_2} = \sqrt{\frac{a_0}{2c\sigma - a_0}}$. When $m = R = 1$, the above coefficients simplify to $d_0 = 0$ and

$$\begin{cases} \tilde{d}_2 = \frac{3\sqrt{2c\sigma - a_0}\left(3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8\sqrt{a_0}(a_0 - c\sigma)}, \\ \tilde{h}_0 = \frac{1}{8a_0(c\sigma - a_0)}\left(6a_0^2c\mu_1^2\sigma - 4a_0bc\mu_1\sigma - 8a_0c\mu_0\sigma + 8a_0c\sigma^2 - 2b^2c\sigma - 8a_0^2\sigma + 8a_0^2\right), \\ \tilde{h}_2 = \frac{3\left(3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8(a_0 - c\sigma)}, \\ \tilde{\lambda} = \sqrt{\frac{3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0}{16a_0(a_0 - c\sigma)}}, \\ \omega = -\left(a_0\mu_1^2 - \mu_1 b - \mu_0 + \sigma\right)\mu_1, \\ \sigma > \frac{a_0}{2c}, \end{cases}$$

from which one obtains the following solitary-wave solution to system (2.3):

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}\tilde{f}(x - \sigma t) \qquad \text{and} \qquad v(x,t) = \tilde{h}_0 \pm \sqrt{\frac{a_0}{2c\sigma - a_0}}\tilde{f}(x - \sigma t),$$

where $\tilde{f}(\xi) = \tilde{d}_2\,\text{sech}^2(\tilde{\lambda}\xi)$. Furthermore, when σ satisfies the condition $\frac{\sigma + 3a_0B^2 + 2bB - \mu_0}{a_0} = \frac{\sigma - 1}{c\sigma}$, one has $\tilde{h}_0 = 0$, and the synchronized solitary-wave solution established in [33] is recovered.

When $m = -R = 1$, the above coefficients simplify to

$$\begin{cases} \bar{d}_0 = \frac{\sqrt{2c\sigma - a_0}\left(3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{4\sqrt{a_0}(a_0 - c\sigma)}, \\ \bar{d}_2 = \frac{3\sqrt{2c\sigma - a_0}\left(3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8\sqrt{a_0}(c\sigma - a_0)}, \\ \bar{h}_0 = \frac{1}{4a_0(c\sigma - a_0)}\left(3a_0^2c\mu_1^2\sigma - 3a_0^3\mu_1^2 - 2a_0bc\mu_1\sigma + 2a_0^2b\mu_1 - 4a_0c\mu_0\sigma + 4a_0c\sigma^2\right. \\ \qquad \left. - b^2c\sigma + 4a_0^2\mu_0 - 4a_0^2\sigma + a_0b^2\right), \\ \bar{h}_2 = \frac{3\left(3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8(c\sigma - a_0)}, \\ \bar{\lambda} = \sqrt{\frac{3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0}{16a_0(c\sigma - a_0)}}, \\ \omega = -\left(a_0\mu_1^2 - \mu_1 b - \mu_0 + \sigma\right)\mu_1, \\ \sigma > \frac{a_0}{2c}, \end{cases}$$

and one arrives at the solitary-wave solution

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}[\bar{d}_0 + \bar{f}(x - \sigma t)] \qquad \text{and} \qquad v(x,t) = \bar{h}_0 \pm \sqrt{\frac{a_0}{2c\sigma - a_0}}\bar{f}(x - \sigma t),$$

where $\bar{f}(\xi) = \bar{d}_2 \operatorname{sech}^2(\bar{\lambda}\xi)$.

In addition, there exists another nontrivial solution to system (2.3) of the form

$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} d_1 \operatorname{cn}\left(\lambda(x-\sigma t), m\right) \qquad \text{and} \qquad v(x,t) = h_0 + h_2 \operatorname{cn}^2\left(\lambda(x-\sigma t), m\right),$$

where $h_0 = \frac{9\,a_0{}^2 c m^2 \mu_1{}^2 - 6\,a_0\, b c m^2 \mu_1 - 12\,a_0\, c h_2\, m^2 - 12\,a_0\, c m^2 \mu_0 - 3\,b^2 c m^2 + 2\,a_0{}^2 m^2 + 6\,a_0\, c h_2}{12\,a_0\, c m^2}$; $h_2 > 0$ such that $9a_0{}^2 m^2 \mu_1{}^2 - 6a_0 b m^2 \mu_1 - 4a_0 h_2 m^2 - 12a_0 m^2 \mu_0 - 3b^2 m^2 + 2a_0 h_2 + 12a_0 m^2 < 0$; $d_1 = \pm \frac{\sqrt{-6\,a_0 h_2 m^2\,(9\,a_0{}^2 m^2 \mu_1{}^2 - 6\,a_0\, b m^2 \mu_1 - 4\,a_0\, h_2\, m^2 - 12\,a_0\, m^2 \mu_0 - 3\,b^2 m^2 + 2\,a_0\, h_2 + 12\,a_0\, m^2)}}{6\,a_0\, m^2}$; $B = \frac{a_0\,\mu_1 - b}{2\,a_0}$; $\omega = \frac{-\mu_1\left(6\,a_0\, c\mu_1{}^2 - 6\,bc\mu_1 - 6\,c\mu_0 + a_0\right)}{6\,c}$; $\lambda = \sqrt{\frac{h_2}{2\,a_0 m^2}}$; $\sigma = \frac{a_0}{6\,c}$; and any $m \in [0,1]$.

Aside from the above nontrivial solutions, system (2.3) also possesses the following trivial and semi-trivial solutions:

(1)

$$u(x,t) = 0 \qquad \text{and} \qquad v(x,t) = h_0,$$

for any $h_0 \in \mathbb{R}$.

(2)

$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} d_0 \qquad \text{and} \qquad v(x,t) = h_0,$$

where $\sigma = \frac{\omega - a_0 B^3 - bB^2 + Bh_0 + B\mu_0 + h_0\mu_1}{B}$, for any $B, d_0, h_0, \omega \in \mathbb{R}$.

(3)

$$u(x,t) = 0 \qquad \text{and} \qquad v(x,t) = -\frac{2}{3}\,h_2 + \frac{1}{3}\,\frac{h_2}{m^2} + \sigma - 1 + h_2 \operatorname{cn}^2\left(\lambda(x-\sigma t), m\right),$$

where $\lambda = \sqrt{\frac{h_2}{12 c m^2 \sigma}}$, for any $h_2, \sigma > 0$ and $m \in [0,1]$.

2.3.4 Schrödinger BBM-KdV

Setting all $k_{j,q} = 0$ gives us the following set of parameters, with R as defined in (2.18):

$$
\begin{cases}
B = \frac{a_1\,\mu_0\,\mu_1 - b}{2a_1\,\sigma\,(a_1\,\mu_1{}^2+1)}\,, \\[2ex]
d_1 = h_1 = 0\,, \\[2ex]
d_0 = \frac{\sqrt{2\,c - a_1\,\sigma}\left(m^4 - 2\,m^2 R - m^2 + R + 1\right)}{8\,R^2\sqrt{a_1\,\sigma}(a_1\,\mu_1{}^2+1)^2(a_1\,\sigma - c)}\Big(4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2 b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma \\
\qquad +2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1 b\mu_1\,\sigma - 4\,a_1\mu_0\,\sigma + 4\,a_1\,\sigma - b^2\Big)\,, \\[2ex]
d_2 = \frac{3\,m^2\sqrt{2\,c - a_1\,\sigma}}{8R\sqrt{a_1\,\sigma}(a_1\,\mu_1{}^2+1)^2(a_1\,\sigma - c)}\Big(4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2 b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma \\
\qquad +2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1 b\mu_1\,\sigma - 4\,a_1\mu_0\,\sigma + 4\,a_1\,\sigma - b^2\Big)\,, \\[2ex]
h_0 = \frac{-1}{8\,a_1\,R\sigma\,(a_1\,\mu_1{}^2+1)^2(a_1\,\sigma - c)}\Big(-8\,a_1{}^4\mu_1{}^4 R\sigma^3 + 8\,a_1{}^4 m^2\mu_1{}^4\sigma^2 + 4\,a_1{}^4\mu_1{}^4 R\sigma^2 + 8\,a_1{}^3 c\mu_1{}^4 R\sigma^2 \\
\qquad -4\,a_1{}^4\mu_1{}^4\sigma^2 - 8\,a_1{}^3 b m^2\mu_1{}^3\sigma^2 + 4\,a_1{}^3 b\mu_1{}^3 R\sigma^2 + 4\,a_1{}^3 b\mu_1{}^3\sigma^2 - 2\,a_1{}^3 m^2\mu_0{}^2\mu_1{}^2\sigma \\
\qquad -8\,a_1{}^3 m^2\mu_0\,\mu_1{}^2\sigma^2 + a_1{}^3\mu_0{}^2\mu_1{}^2\sigma\,R + 4\,a_1{}^3\mu_0\,\mu_1{}^2 R\sigma^2 - 16\,a_1{}^3\mu_1{}^2 R\sigma^3 - 8\,a_1^2 bc\mu_1{}^3 R\sigma \\
\qquad +16\,a_1{}^3 m^2\mu_1{}^2\sigma^2 + a_1{}^3\mu_0{}^2\mu_1{}^2\sigma + 4\,a_1{}^3\mu_0\,\mu_1{}^2\sigma^2 + 8\,a_1{}^3\mu_1{}^2 R\sigma^2 - 2\,a_1{}^2 c\mu_0{}^2\mu_1{}^2 R \\
\qquad -8\,a_1{}^2 c\mu_0\,\mu_1{}^2 R\sigma + 16\,a_1{}^2 c\mu_1{}^2 R\sigma^2 - 8\,a_1{}^3\mu_1{}^2\sigma^2 + 4\,a_1{}^2 b m^2\mu_0\,\mu 1\,\sigma - 8\,a_1{}^2 b m^2\mu_1\,\sigma^2 \\
\qquad -2\,a_1{}^2 b\mu_0\,\mu_1 R\sigma + 4\,a_1{}^2 b\mu_1\,R\sigma^2 - 8\,a_1{}^2 m^2\mu_0\,\sigma^2 + 4\,a_1{}^2\mu_0\,R\sigma^2 - 8\,a_1{}^2 R\sigma^3 + 4\,a_1\,bc\mu_0\,\mu_1\,R \\
\qquad -8\,a_1 bc\mu_1\,R\sigma + 8\,a_1{}^2 m^2\sigma^2 + 4\,a_1{}^2\mu_0\,\sigma^2 + 4\,a_1{}^2 R\sigma^2 - 2\,a_1 b^2 m^2\sigma + a_1 b^2 R\sigma - 8\,a_1 c\mu_0\,R\sigma \\
\qquad +8\,a_1\,cR\sigma^2 - 4\,a_1{}^2\sigma^2 + a_1 b^2\sigma - 2\,b^2 cR\Big)\,, \\[2ex]
h_2 = \frac{3\left(4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2 b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma + 2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1 b\mu_1\,\sigma - 4\,a_1\mu_0\,\sigma + 4\,a_1\,\sigma - b^2\right)m^2}{8R(a_1\,\mu_1{}^2+1)^2(a_1\,\sigma - c)}\,, \\[2ex]
\lambda = \sqrt{\frac{4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2 b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma + 2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1 b\mu_1\,\sigma - 4\,a_1\mu_0\,\sigma + 4\,a_1\,\sigma - b^2}{16\,a_1\,R\sigma\,(a_1\,\mu_1{}^2+1)^2(a_1\,\sigma - c)}}\,, \\[2ex]
\omega = -\frac{\left(a_1\,\mu_1{}^2\sigma - b\mu_1 - \mu_0 + \sigma\right)\mu_1}{a_1\,\mu_1{}^2+1}\,, \\[2ex]
\sigma < \frac{2\,c}{a_1}\,, \\[2ex]
m \in [0, 1]\,.
\end{cases}
$$

Thus, explicit periodic traveling-wave solutions to the Schrödinger BBM-KdV system $\big(u(x,t), v(x,t)\big) = \big(e^{i\omega t}e^{iB(x-\sigma t)}f(x-\sigma t), g(x-\sigma t)\big)$, given in terms of the Jacobi cnoidal function

$$f(\xi) = d_0 + d_2\,\mathrm{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2\,\mathrm{cn}^2(\lambda\xi, m)$$

are established. Notice that $\frac{h_2}{d_2} = \sqrt{\frac{a_1\sigma}{2c-a_1\sigma}}$. When $m = R = 1$, the above coefficients simplify to $d_0 = 0$ and

$$\begin{cases} \tilde{d}_2 = \frac{3\sqrt{2c-a_1\sigma}}{8\sqrt{a_1\sigma}(a_1\mu_1^2+1)^2(a_1\sigma-c)}\Big(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma \\ \quad +2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\ \tilde{h}_0 = \frac{-1}{8a_1\sigma(a_1\mu_1^2+1)^2(a_1\sigma-c)}\Big(-8a_1^4\mu_1^4\sigma^3 + 8a_1^4\mu_1^4\sigma^2 + 8a_1^3c\mu_1^4\sigma^2 - 16a_1^3\mu_1^2\sigma^3 - 8a_1^2bc\mu_1^3\sigma \\ \quad +16a_1^3\mu_1^2\sigma^2 - 2a_1^2c\mu_0^2\mu_1^2 - 8a_1^2c\mu_0\mu_1^2\sigma + 16a_1^2c\mu_1^2\sigma^2 - 8a_1^2\sigma^3 + 4a_1bc\mu_0\mu_1 \\ \quad -8a_1bc\mu_1\sigma + 8a_1^2\sigma^2 - 8a_1c\mu_0\sigma + 8a_1c\sigma^2 - 2b^2c\Big), \\ \tilde{h}_2 = \frac{3\left(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\right)}{8(a_1\mu_1^2+1)^2(a_1\sigma-c)}, \\ \tilde{\lambda} = \sqrt{\frac{4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2}{16a_1\sigma(a_1\mu_1^2+1)^2(a_1\sigma-c)}}, \\ \omega = -\frac{(a_1\mu_1^2\sigma - b\mu_1 - \mu_0 + \sigma)\mu_1}{a_1\mu_1^2+1}, \\ \sigma < \frac{2c}{a_1}, \end{cases}$$

from which one obtains the following solitary-wave solution to system (2.4):

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}\tilde{f}(x-\sigma t) \qquad \text{and} \qquad v(x,t) = \tilde{h}_0 \pm \sqrt{\frac{a_1\sigma}{2c-a_1\sigma}}\tilde{f}(x-\sigma t),$$

where $\tilde{f}(\xi) = \tilde{d}_2\,\text{sech}^2(\tilde{\lambda}\xi)$. Furthermore, when $B = \frac{a_1\mu_0\mu_1 - b}{2\sigma a_1(a_1\mu_1^2+1)}$ satisfies the condition

$$\begin{aligned}&(2a_1^2bc\mu_1^2 + 2a_1bc - 2a_1^3c\mu_0\mu_1^3 - 2a_1^2c\mu_0\mu_1)B^3 + (-4a_1^2bc\mu_1^3 - 4a_1^2c\mu_0\mu_1^2 - 4a_1bc\mu_1 - 4a_1c\mu_0)B^2 \\ &+ (2a_1^3\mu_0\mu_1^3 + 2a_1^2c\mu_0\mu_1^3 + 2a_1^2\mu_0\mu_1 + 2a_1c\mu_0\mu_1 - 2a_1^2b\mu_1^2 - 2a_1b - 2a_1bc\mu_1^2 - 2bc)B \\ &+ (2a_1b\mu_0\mu_1 - a_1^2\mu_0^2\mu_1^2 - b^2) = 0,\end{aligned}$$

one has $\tilde{h}_0 = 0$, and the synchronized solitary-wave solution established in [33] is recovered.

When $m = -R = 1$, the above coefficients simplify to

$$\begin{cases} \bar{d}_0 = \frac{\sqrt{2c - a_1\sigma}}{4\sqrt{a_1\sigma}(a_1\mu_1^2+1)^2(a_1\sigma - c)}\Big(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma \\ \quad +2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\ \bar{d}_2 = \frac{-3\sqrt{2c - a_1\sigma}}{8\sqrt{a_1\sigma}(a_1\mu_1^2+1)^2(a_1\sigma - c)}\Big(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma \\ \quad +2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\ \bar{h}_0 = \frac{1}{4a_1\sigma(a_1\mu_1^2+1)^2(a_1\sigma - c)}\Big(4a_1^4\mu_1^4\sigma^3 - 4a_1^3c\mu_1^4\sigma^2 - 4a_1^3b\mu_1^3\sigma^2 - a_1^3\mu_0^2\mu_1^2\sigma \\ \quad -4a_1^3\mu_0\mu_1^2\sigma^2 + 8a_1^3\mu_1^2\sigma^3 + 4a_1^2bc\mu_1^3\sigma + a_1^2c\mu_0^2\mu_1^2 + 4a_1^2c\mu_0\mu_1^2\sigma \\ \quad -8a_1^2c\mu_1^2\sigma^2 + 2a_1^2b\mu_0\mu_1\sigma - 4a_1^2b\mu_1\sigma^2 - 4a_1^2\mu_0\sigma^2 + 4a_1^2\sigma^3 - 2a_1bc\mu_0\mu_1 \\ \quad +4a_1bc\mu_1\sigma - a_1b^2\sigma + 4a_1c\mu_0\sigma - 4a_1c\sigma^2 + b^2c\Big), \\ \bar{h}_2 = \frac{-3\left(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\right)}{8(a_1\mu_1^2+1)^2(a_1\sigma - c)}, \\ \bar{\lambda} = \sqrt{\frac{4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2}{-16a_1\sigma(a_1\mu_1^2+1)^2(a_1\sigma - c)}}, \\ \omega = -\frac{(a_1\mu_1^2\sigma - b\mu_1 - \mu_0 + \sigma)\mu_1}{a_1\mu_1^2+1}, \\ \sigma < \frac{2c}{a_1}, \end{cases}$$

and one arrives at the solitary-wave solution

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}[\bar{d}_0 + \bar{f}(x-\sigma t)] \qquad \text{and} \qquad v(x,t) = \bar{h}_0 \pm \sqrt{\frac{a_1\sigma}{2c - a_1\sigma}}\bar{f}(x-\sigma t),$$

where $\bar{f}(\xi) = \bar{d}_2 \operatorname{sech}^2(\bar{\lambda}\xi)$.

In addition, there exists another nontrivial solution to system (2.3) of the form

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}d_1 \operatorname{cn}\big(\lambda(x-\sigma t), m\big) \qquad \text{and} \qquad v(x,t) = h_0 + h_2 \operatorname{cn}^2\big(\lambda(x-\sigma t), m\big),$$

where

$$d_1 = \pm\frac{1}{6cm^2(a_1\mu_1^2+1)}\Big(3ch_2m^2\left(8a_1^2ch_2m^2\mu_1^4 - 4a_1^2ch_2\mu_1^4 - 24a_1^2cm^2\mu_1^4 + a_1^2m^2\mu_0^2\mu_1^2\right. \\ + 24a_1bcm^2\mu_1^3 + 16a_1ch_2m^2\mu_1^2 + 24a_1cm^2\mu_0\mu_1^2 - 2a_1bm^2\mu_0\mu_1 - 8a_1ch_2\mu_1^2 \\ \left. - 48a_1cm^2\mu_1^2 + 24bcm^2\mu_1 + b^2m^2 + 8ch_2m^2 + 24cm^2\mu_0 - 4ch_2 - 24cm^2\right)\Big)^{1/2};$$

$$h_0 = \frac{-1}{24\,(a_1{}^2\mu_1{}^4 + 2\,a_1\mu_1{}^2 + 1)\,a_1\,cm^2}\Big(24\,a_1{}^3ch_2\,m^2\mu_1{}^4 - 12\,a_1{}^3ch_2\,\mu_1{}^4 - 144\,a_1{}^2c^2m^2\mu_1{}^4$$
$$+ a_1{}^3m^2\mu_0{}^2\mu_1{}^2 + 24\,a_1{}^2bcm^2\mu_1{}^3 + 48\,a_1{}^2ch_2\,m^2\mu_1{}^2 + 24\,a_1{}^2cm^2\mu_0\,\mu_1{}^2 - 2\,a_1{}^2bm^2\mu_0\,\mu_1$$
$$- 24\,a_1{}^2ch_2\,\mu_1{}^2 - 288\,a_1\,c^2m^2\mu_1{}^2 + 24\,a_1\,bcm^2\mu_1 + a_1\,b^2m^2 + 24\,a_1\,ch_2\,m^2 + 24\,a_1\,cm^2\mu_0$$
$$- 12\,a_1\,ch_2 - 144\,c^2m^2\Big);$$

any $h_2 > 0$ such that $8\,a_1{}^2ch_2\,m^2\mu_1{}^4 - 4\,a_1{}^2ch_2\mu_1{}^4 - 24\,a_1{}^2cm^2\mu_1{}^4 + a_1{}^2m^2\mu_0{}^2\mu_1{}^2 + 24\,a_1\,bcm^2\mu_1{}^3 + 16\,a_1\,ch_2\,m^2\mu_1{}^2 + 24\,a_1\,cm^2\mu_0\,\mu_1{}^2 - 2\,a_1\,bm^2\mu_0\,\mu_1 - 8\,a_1\,ch_2\mu_1{}^2 - 48\,a_1\,cm^2\mu_1{}^2 + 24\,bcm^2\mu_1 + b^2m^2 + 8\,ch_2\,m^2 + 24\,cm^2\mu_0 - 4\,ch_2 - 24\,cm^2 > 0$; $B = \frac{a_1\,\mu_0\,\mu_1 - b}{12\,c(a_1\mu_1{}^2+1)}$; $\lambda = \sqrt{\frac{h_2}{12\,cm^2}}$; $\omega = \frac{\mu_1\,(-6\,a_1\,c\mu_1{}^2 + a_1\,b\mu_1 + a_1\,\mu_0 - 6\,c)}{(a_1\mu_1{}^2+1)a_1}$; $\sigma = \frac{6\,c}{a_1}$; and any $m \in [0,1]$.

Aside from the above nontrivial solutions, system (2.4) also possesses the following trivial and semi-trivial solutions:

(1)

$$u(x,t) = 0 \qquad \text{and} \qquad v(x,t) = h_0,$$

for any $h_0 \in \mathbb{R}$.

(2)

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}d_0 \qquad \text{and} \qquad v(x,t) = h_0,$$

where $\sigma = \frac{a_1B^2\omega - bB^2 + Bh_0 + B\mu_0 + h_0\mu_1 + \omega}{B(a_1B^2+1)}$, for any $B, d_0, h_0, \omega \in \mathbb{R}$.

(3)

$$u(x,t) = 0 \qquad \text{and} \qquad v(x,t) = -\frac{2}{3}\,h_2 + \frac{1}{3}\,\frac{h_2}{m^2} + \sigma - 1 + h_2\,\mathrm{cn}^2\left(\lambda(x-\sigma t), m\right),$$

where $\lambda = \sqrt{\frac{h_2}{12cm^2}}$, for any $h_2, \sigma > 0$ and $m \in [0,1]$.

2.4 Conclusions

The periodic traveling-wave solutions for the four systems (2.1)–(2.4) given by (2.5) and (2.16) were found. Furthermore, these cnoidal solutions limited to solitary-wave solutions when $m \to 1$. This was expected since it is well known that the ODE equation

$$f'^2 = k_3f^3 + k_2f^2 + k_1f + k_0$$

has a unique solitary-wave solution as well as a periodic cnoidal solution, and that the periodic cnoidal solution limits to the solitary-wave solution when the Jacobi elliptic modulus m approaches one. Consequently, under certain restraints on the system parameters, these solitary-wave solutions were shown to limit to the synchronized solitary-wave solutions established in [33]. It can similarly be shown that the periodic traveling-wave solutions here reduce to the synchronized periodic traveling-wave solutions found in [12]. Note that there is no one-to-one correspondence between cnoidal solutions and solitary-wave solutions. So in general, it is not known if all solitary-wave solutions of the form (2.5) are recovered as the limit of the periodic traveling-wave solutions found here. This is not further pursued here, but would be an interesting question to investigate.

Notice that substituting $u = 0$ into systems (2.1) and (2.3) yields

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + v\frac{\partial v}{\partial x} + c\frac{\partial^3 v}{\partial x^3} = 0,$$

which is the KdV equation. Similarly, substituting $u = 0$ into systems (2.2) and (2.4) yields

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + v\frac{\partial v}{\partial x} - c\frac{\partial^3 v}{\partial x^2 \partial t} = 0,$$

which is the BBM equation. For each system, a solution was found where $u = 0$ and v is given by

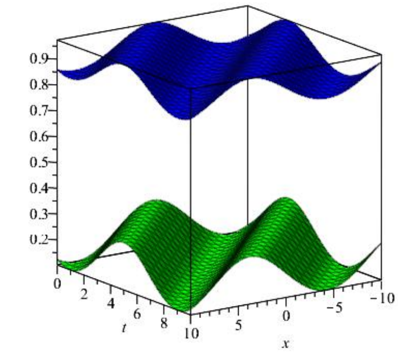
$$v(x,t) = \tilde{h}_0 + h_2 \operatorname{cn}^2\left(\tilde{\lambda}(x - \sigma t), m\right),$$

where $\tilde{h}_0$ and $\tilde{\lambda}$ are constants that depend on which system is considered. This corresponds with the form of cnoidal solutions found for the KdV and BBM equations in [22, 37].

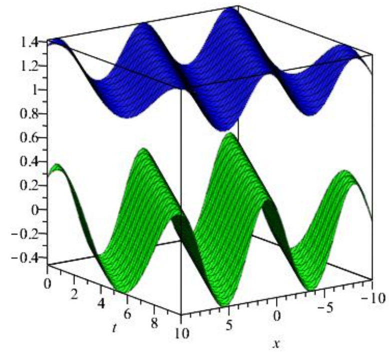
Figure 2.1 below shows some graphs for the cnoidal wave solutions for the four systems (2.1)–(2.4). Recall that the traveling-wave solutions studied here to the above four systems are vector solutions $\big(u(x,t), v(x,t)\big)$ of the form

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}f(x - \sigma t), \qquad v(x,t) = g(x - \sigma t),$$

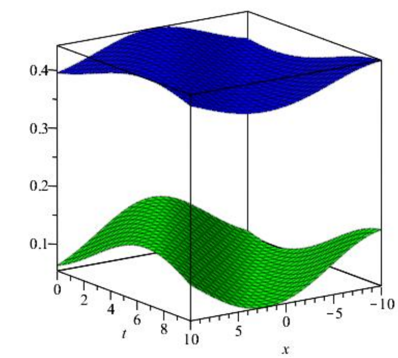
where f and g are smooth, real-valued functions with speed $\sigma > 0$ and phase shifts $B, \omega \in \mathbb{R}$. For ease of graphing, the imaginary terms in $u(x,t)$ were suppressed, as they define a phase shift and, thus, a rotation of the real function f, which is graphed below. For all four vector solutions, $m = \frac{1}{2}$ and $R = \frac{\sqrt{13}}{4}$ were chosen, while the remaining parameters were then fixed to ensure real solutions and are listed here; KdV-KdV: $\sigma = 2,\ a_0 = 1,\ b = -1,\ c = \frac{3}{2},\ \mu_0 = 1,\ \mu_1 = \frac{1}{4}$; BBM-BBM: $\sigma = 1,\ a_1 = 1,\ b = -1,\ c = \frac{5}{2},\ \mu_0 = 1,\ \mu_1 = 1$; KdV-BBM: $\sigma = \frac{3}{2},\ a_0 = 1,\ b = -1,\ c = \frac{3}{2},\ \mu_0 = 1,\ \mu_1 = \frac{1}{4}$; BBM-KdV: $\sigma = \frac{1}{2},\ a_1 = 1,\ b = -1,\ c = \frac{3}{2},\ \mu_0 = 1,\ \mu_1 = \frac{1}{4}$. The graphs are now listed below, with $u(x,t)$ in blue and $v(x,t)$ in green.



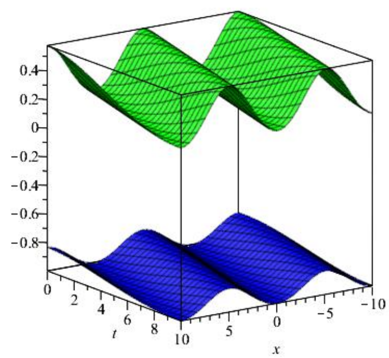
(a) Cnoidal solution for the KdV-KdV system.



(b) Cnoidal solution for the BBM-BBM system.



(c) Cnoidal solution for the KdV-BBM system.



(d) Cnoidal solution for the BBM-KdV system.

Figure 2.1: Graphs of some cnoidal solutions for the four systems.

CHAPTER 3

EXACT JACOBI ELLIPTIC SOLUTIONS OF THE ABCD-SYSTEM

Abstract

In this manuscript, consideration is given to the existence of periodic traveling-wave solutions to the abcd-system. This system was derived by Bona, Saut, and Chen to describe small amplitude, long wavelength gravity waves on the surface of water. These exact solutions are formulated in terms of cnoidal the Jacobi elliptic function. The existence of explicit traveling-wave solutions is very useful in theoretical investigations such as stability of solutions, as well as other numerical analysis of the system.

3.1 Introduction

Let Ω_t be a domain in $\mathbb{R}^3 \times \mathbb{R}$ whose boundary consists of two parts: the fixed surface located at $z = -h(x, y)$, and the free surface $z = \eta(x, y, t)$. It is well-known that the three-dimensional capillary-gravity waves on an inviscid, irrotational, and incompressible fluid layer of uniform depth h that is impermeable on the fixed surface are governed by the Euler equations [24]

$$\begin{cases} \Delta\phi = 0, \quad \text{in } \Omega_t; \\ \eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0, \quad \text{at } z = \eta(x, y, t); \\ \phi_t + \frac{1}{2}|\nabla\phi|^2 + gz = 0, \quad \text{at } z = \eta(x, y, t); \\ \phi_x h_x + \phi_y h_y + \phi_z = 0, \quad \text{at } z = -h(x, y). \end{cases} \tag{3.1}$$

Here, g denotes the acceleration of gravity; $\phi(x, y, z, t)$ is the velocity potential function. The full Euler equations (3.1) are often far more complex than necessary for many applications, so several approximation models have been derived in its place under certain restricted physical regimes. One such model is the abcd-system, introduced by Bona *et al.* [6, 5], to describe small amplitude, long wavelength gravity waves on the surface of water

$$\begin{cases} \eta_t + w_x + (w\eta)_x + a w_{xxx} - b\eta_{xxt} = 0\,, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0\,, \end{cases} \tag{3.2}$$

where a, b, c, and d are real constants and $\theta \in [0, 1]$ that satisfy

$$\begin{aligned} a + b &= \frac{1}{2}(\theta^2 - \frac{1}{3}), \\ c + d &= \frac{1}{2}(1 - \theta^2) \geq 0, \\ a + b + c + d &= \frac{1}{3}. \end{aligned} \tag{3.3}$$

Precisely, the regime for which the above system is derived to approximate the Euler equations is when the maximal deviation α of the free surface is small and a typical wavelength λ is large as compared to the undisturbed water depth h, such that the Stokes number $S = \frac{\alpha\lambda^2}{h^3}$ is of order one. The functions $\eta(x, t)$ and $w(x, t)$ are real valued and $x, t \in \mathbb{R}$. By choosing specific values for the parameters a, b, c, and d, the system (3.2) includes a wide range of other systems that have been derived over the last few decades such as the classical Boussinesq system [10, 8, 9], the Kaup system [26], the coupled Benjamin-Bona-Mahony system (BBM-system) [4], the coupled Korteweg-de Vries system (KdV-system) [6, 5], the Bona-Smith system [3], and the integrable version of Boussinesq system [31]. In particular, these specializations are:

- Classical Boussinesq system

$$\begin{cases} \eta_t + w_x + (w\eta)_x = 0, \\ w_t + \eta_x + w w_x - \frac{1}{3} w_{xxt} = 0; \end{cases}$$

- Kaup system

$$\begin{cases} \eta_t + w_x + (w\eta)_x + \frac{1}{3} w_{xxx} = 0, \\ w_t + \eta_x + w w_x = 0; \end{cases}$$

- Bona-Smith system

$$\begin{cases} \eta_t + w_x + (w\eta)_x - b\eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - b w_{xxt} = 0; \end{cases}$$

- Coupled BBM-system

$$\begin{cases} \eta_t + w_x + (w\eta)_x - \dfrac{1}{6}\eta_{xxt} = 0, \\ w_t + \eta_x + ww_x - \dfrac{1}{6}w_{xxt} = 0; \end{cases}$$

- Coupled KdV-system

$$\begin{cases} \eta_t + w_x + (w\eta)_x + \dfrac{1}{6}w_{xxx} = 0, \\ w_t + \eta_x + ww_x + \dfrac{1}{6}\eta_{xxx} = 0; \end{cases}$$

- Coupled KdV-BBM system

$$\begin{cases} \eta_t + w_x + (w\eta)_x + \dfrac{1}{6}w_{xxx} = 0, \\ w_t + \eta_x + ww_x - \dfrac{1}{6}w_{xxt} = 0; \end{cases}$$

- Coupled BBM-KdV system

$$\begin{cases} \eta_t + w_x + (w\eta)_x - \dfrac{1}{6}\eta_{xxt} = 0, \\ w_t + \eta_x + ww_x + \dfrac{1}{6}\eta_{xxx} = 0. \end{cases}$$

For a more detailed discussion on this, we refer our readers to the papers [6, 5].

In this manuscript, attention is given to the existence of periodic traveling-wave solutions to the $\mathfrak{abcd}$-system. A traveling-wave solution to the system (3.2) is a vector solution $\big(\eta(x,t), w(x,t)\big)$ of the form

$$\eta(x,t) = \eta(x - \sigma t) \quad \text{and} \quad w(x,t) = w(x - \sigma t), \tag{3.4}$$

where σ denotes the speed of the waves. Notice that when $\sigma > 0$, one obtains a right-propagating solution, while for $\sigma < 0$, one has a left-propagating solution. This feature of bidirectional propagation of (η, w) is a hallmark of the $\mathfrak{abcd}$-system in its derivation, and is the main difference between system (3.2) and the one-way approximation models such as

$$v_t + v_x + \frac{1}{p}(v^p)_x \pm Lv_x = 0, \quad p > 0,$$

(which is commonly referred to as the generalized KdV equation when $L = \partial_{xx}$, the generalized BBM equation when $L = -\partial_{xt}$, and simply the KdV and BBM equations, respectively, when $p = 2$), and the cubic, nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} + |u|^2 u = 0.$$

It would also be interesting to analyze the existence of periodic solutions when the two components propagate at different speeds, and/or when the two components travel in opposite directions. Such scenarios are not considered in the scope of this paper.

The most studied types of traveling-wave solutions are periodic traveling-wave solutions and solitary-wave solutions. Periodic traveling-wave solutions are self-explanatory, while solitary-waves are smooth traveling-wave solutions that are symmetric around a *single* maximum and rapidly decay to zero away from the maximum. Though less common, the term solitary-wave is sometimes used to describe traveling-wave solutions that are symmetric around a single maximum but approach nonzero constants as $\xi = x - \sigma t \to \pm\infty$. In this manuscript, we consider non-trivial solutions to be solutions where both components are non-constant, and semi-trivial solutions to be solutions where one component is a constant. The constant solution, (f, g) for any $f,\ g \in \mathbb{R}$, is always a solution and will be referred to as the trivial solution.

The topic of existence of explicit periodic traveling-wave solutions for many single equations are readily available such as for the KdV equation [22], generalized KdV equation [16], BBM equation [1], NLS equation [25], and Whitham equation [23]. In contrast, there are far less of these results for systems due to the complexity of the coupled equations involved. Leveling up the scalar results to systems is no minor task, requiring new insights and new approaches. For the classical Boussinesq system, Krishnan showed that the periodic traveling-wave solutions exist and are given by Jacobi elliptic functions [31]. Chen and Li [13] (see also [39]) have recently established the periodic traveling-wave solutions for the beta derivative of the Kaup system where the first temporal derivative ∂_t is replaced by the fractional derivative ∂_t^β, with $\beta \in (0, 1]$, given in terms of normal trigonometric functions as well as the Jacobi cnoidal functions. Four systems were recently derived by Nguyen *et al.* [21, 36, 32] to describe the interaction of long and short waves in dispersive media, and periodic traveling-wave solutions given by the Jacobi elliptic function cnoidal have been explicitly calculated for all four systems [11].

Another important subject regarding traveling-wave solutions is the stability of these solutions. The topic of stability of periodic traveling-wave solutions such as cnoidal waves has attracted far less consideration than that of solitary-wave solutions. Within this limited attention, explicitly known formulae for periodic traveling-wave solutions often play an instrumental role. Angula, Bona and Scialom [38] proved that the periodic traveling-waves for the KdV equation are nonlinearly orbitally stable with respect to perturbations of the same period. Central to the argument in their proof is the fact that the cnoidal waves lie in the set of minimizers, a conclusion that would be hard to establish without having explicit formula for the cnoidal waves at hand. Deconinck *et al.* established that the KdV cnoidal waves are spectrally stable with respect to perturbations of arbitrary period [7], and orbitally stable with respect to sub-harmonic perturbations [20]. Again, the explicit formulae for cnoidal waves play pivotal roles in their numerical computations.

In this paper, periodic traveling-wave solutions to the system (3.2) of the form

$$\eta(\xi) = \sum_{r=0}^{n} j_r \operatorname{cn}^r(\lambda\xi, m) \quad \text{and} \quad w(\xi) = \sum_{r=0}^{n} k_r \operatorname{cn}^r(\lambda\xi, m)\,, \tag{3.5}$$

are analyzed, where j_r, $k_r \in \mathbb{R}$, $\lambda > 0$, $m \in (0, 1]$, $\xi = x - \sigma t$, $n \in \mathbb{N}$ such that $n > 4$, and cn denotes cnoidal the Jacobi elliptic function. For the readers' convenience, a brief introduction to Jacobi elliptic functions can be found in Section 3.2. If $m = 0$, then (3.5) reduces to a cosine series which is not the aim of this paper and is excluded. For simplicity the sums in (3.5) are both truncated at n terms, but it will be made clear that truncating at different values, n_1 and n_2, does not change the results so long as n_1, $n_2 > 4$.

The analysis of (3.5) requires consideration of the cases $c \neq 0$ and $c = 0$ separately. For the first case, $c \neq 0$, it is shown that non-trivial periodic traveling-wave solutions only exist when $a^2 + b^2 \neq 0$, and in particular, $j_r = k_r = 0$ for $r \geq 3$. When $a^2 + b^2 = 0$, it is established that only semi-trivial periodic traveling-wave solutions exist, and that $j_r = 0$ for all $r \geq 1$ and $k_r = 0$ for $r \geq 3$. For the second case, $c = 0$, it is shown that non-trivial periodic traveling-wave solutions only exist when $b^2 + d^2 \neq 0$, and that $j_r = 0$ for $r = 1, 3$ and $r \geq 5$ and $k_r = 0$ for $r = 1$ and $r \geq 3$.

The paper is organized as follows. In Section 3.2, some facts and identities of Jacobi elliptic functions are reviewed. In Section 3.3, the existence of periodic traveling-wave solutions to the system (3.2) is established. The periodic traveling-wave solutions are then explicitly computed in Section 3.4. Finally, a discussion of results is given in Section 3.5, including remarks about how these cnoidal solutions limit to the solitary-wave solutions previously established in [17, 18].

3.2 Preliminaries

For the readers' convenience, some notions of the Jacobi elliptic functions are briefly recalled here. Let

$$\nu = \int_0^{\phi} \frac{1}{\sqrt{1 - m^2 \sin^2 t}} dt, \qquad \text{for } 0 \le m \le 1.$$

Then $\nu = F(\phi, m)$, or equivalently, $\phi = F^{-1}(\nu, m) = \mathrm{am}(\nu, m)$ which is the Jacobi amplitude and m is referred to as the Jacobi elliptic modulus. The three basic Jacobi elliptic functions: cnoidal $\mathrm{cn}(\nu, m)$, snoidal $\mathrm{sn}(\nu, m)$, and dnoidal $\mathrm{dn}(\nu, m)$, are defined as

$$\mathrm{cn}(\nu, m) = \cos(\phi) = \cos\left(F^{-1}(\nu, m)\right), \quad \mathrm{sn}(\nu, m) = \sin(\phi) = \sin\left(F^{-1}(\nu, m)\right), \quad \text{and}$$
$$\mathrm{dn}(\nu, m) = \frac{d}{d\nu}(\phi) = \frac{d}{d\nu}\left(F^{-1}(\nu, m)\right).$$

These functions are generalizations of the trigonometric and hyperbolic functions which satisfy

$$\mathrm{cn}(\nu, 0) = \cos(\nu), \qquad \mathrm{sn}(\nu, 0) = \sin(\nu),$$
$$\mathrm{cn}(\nu, 1) = \mathrm{sech}(\nu), \qquad \mathrm{sn}(\nu, 1) = \tanh(\nu).$$

For $m \in [0, 1)$, the functions $\mathrm{cn}(\nu, m)$ and $\mathrm{sn}(\nu, m)$ are periodic with period $4K(m)$, where $K(m)$ is given by

$$K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - m^2 \sin^2 t}} dt.$$

We recall the following identities:

$$\begin{aligned}
&\mathrm{sn}^2(\lambda\xi, m) = 1 - \mathrm{cn}^2(\lambda\xi, m), \\
&\mathrm{dn}^2(\lambda\xi, m) = 1 - m^2 + m^2\, \mathrm{cn}^2(\lambda\xi, m), \\
&\frac{d}{d\xi} \mathrm{cn}(\lambda\xi, m) = -\lambda\, \mathrm{sn}(\lambda\xi, m)\, \mathrm{dn}(\lambda\xi, m), \\
&\frac{d}{d\xi} \mathrm{sn}(\lambda\xi, m) = \lambda\, \mathrm{cn}(\lambda\xi, m)\, \mathrm{dn}(\lambda\xi, m), \\
&\frac{d}{d\xi} \mathrm{dn}(\lambda\xi, m) = -m^2\lambda\, \mathrm{cn}(\lambda\xi, m)\, \mathrm{sn}(\lambda\xi, m).
\end{aligned}$$

The following relations will be useful

$$\begin{aligned}
\frac{d}{d\xi} \mathrm{cn}^r &= -r\lambda\, \mathrm{cn}^{r-1}\, \mathrm{sn}\, \mathrm{dn}, \\
\frac{d^2}{d\xi^2} \mathrm{cn}^r &= -r\lambda^2 \left[(r+1)m^2\, \mathrm{cn}^{r+2} + r(1-2m^2)\, \mathrm{cn}^r + (r-1)(m^2-1)\, \mathrm{cn}^{r-2}\right], \\
\frac{d^3}{d\xi^3} \mathrm{cn}^r &= r\lambda^3\, \mathrm{sn}\, \mathrm{dn} \left[(r+1)(r+2)m^2\, \mathrm{cn}^{r+1} + r^2(1-2m^2)\, \mathrm{cn}^{r-1} + (r-1)(r-2)(m^2-1)\, \mathrm{cn}^{r-3}\right],
\end{aligned} \tag{3.6}$$

where the argument $(\lambda\xi, m)$ has been dropped for clarity reason.

3.3 Existence of Periodic Traveling-Wave Solutions

Substituting the traveling-wave ansatz (3.4) into the $abcd$-system (3.2) yields the following system of ordinary differential equations (ODE)

$$\begin{cases} -\sigma\eta' + w' + (\eta w)' + a w''' + b\sigma\eta''' = 0\,, \\ -\sigma w' + \eta' + w w' + c\eta''' + d\sigma w''' = 0\,, \end{cases} \tag{3.7}$$

where the primes denote the derivatives with respect to the moving frame $\xi = x - \sigma t$. Replacing η and w in (3.7) with the periodic traveling-wave ansatz (3.5) and using cnoidal derivative identities (3.6), the following generic forms are obtained

$$\begin{cases} -\lambda\,\mathrm{sn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m) \displaystyle\sum_{q=0}^{2n-1} h_{1,q}\,\mathrm{cn}^q(\lambda\xi, m) = 0, \\ -\lambda\,\mathrm{sn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m) \displaystyle\sum_{q=0}^{2n-1} h_{2,q}\,\mathrm{cn}^q(\lambda\xi, m) = 0, \end{cases} \tag{3.8}$$

where the subscript q in $h_{1,q}$ and $h_{2,q}$ corresponds with the power of cn in their respective equations. Here the $h_{p,q} = H_{p,q}(\lambda, m, \sigma, j_i, k_i)$, that is, some function of the parameters defined in (3.5) that depends on the equation p and the power q of cn in the equation.

As (3.8) must hold true for all ξ, the linear independence of the cnoidal function implies that $h_{1,q} = 0$ and $h_{2,q} = 0$ for all q. Consequently, finding a periodic traveling-wave solution for the $abcd$-system (3.2) of the form (3.5) is now equivalent to finding a solution to the system of algebraic equations, $h_{1,q} = 0$ and $h_{2,q} = 0$ for all q. Using these equations, we show that the j_r and k_r from (3.5) must be zero for all $r \geq r_1$ and $r \geq r_2$, respectively. The exact values of r_1 and r_2 are found to be dependent on first, whether $c \neq 0$ or $c = 0$, and second, the values of the other parameters a, b, and d. The analysis is now split to independently study when $c \neq 0$ and $c = 0$.

3.3.1 The case when $c \neq 0$

As $h_{1,q} = 0$ and $h_{2,q} = 0$ for all q, the following theorem is established. (Recall that j_r and k_r are the coefficients of cn^r as given in (3.5) for $r \in \mathbb{Z}_{\geq 0}$.)

Theorem 1. *Suppose* $c \neq 0$*, then the periodic traveling-wave ansatz* (3.5) *will take on one of the following forms depending on the values of* a *and* b*:*

1. *If* $a^2 + b^2 \neq 0$*, then* $j_r = k_r = 0$ *for all integers* $r \geq 3$ *and* (3.5) *reduces to*

$$\eta(\xi) = j_0 + j_1 \operatorname{cn}(\lambda\xi, m) + j_2 \operatorname{cn}^2(\lambda\xi, m) \quad \textit{and} \quad w(\xi) = k_0 + k_1 \operatorname{cn}(\lambda\xi, m) + k_2 \operatorname{cn}^2(\lambda\xi, m)\,. \tag{3.9}$$

2. *If* $a^2 + b^2 = 0$*, then* $j_r = 0$ *for all integers* $r \geq 1$ *and* $k_r = 0$ *for all integers* $r \geq 3$*. Thus,* (3.5) *reduces to*

$$\eta(\xi) = j_0 \quad \textit{and} \quad w(\xi) = k_0 + k_1 \operatorname{cn}(\lambda\xi, m) + k_2 \operatorname{cn}^2(\lambda\xi, m)\,. \tag{3.10}$$

Consequently, only semi-trivial periodic traveling-wave solutions exist in this case.

Proof. (1.) Let $\rho(\eta')$ denote the largest power of cn in η'. Then,

$$\rho(\eta') = \rho(w') = n - 1, \quad \rho(\eta''') = \rho(w''') = n + 1, \quad \text{and} \quad \rho((\eta w)') = \rho(ww') = 2n - 1.$$

From this, since $n > 2$ it follows that $2n - 1 > n + 1$, which implies that the coefficient of cn^{2n-1} in the second equation of (3.8) will solely come from the ww'-term. Thus, $h_{2,2n-1} = nk_n^2$. Setting this coefficient equal to zero gives $k_n = 0$. Recalculating the above orders yields

$$\rho(\eta') = n - 1, \quad \rho(w') = n - 2, \quad \rho(\eta''') = n + 1, \quad \rho(w''') = n, \quad \rho((\eta w)') = 2n - 2, \quad \text{and} \quad \rho(ww') = 2n - 3.$$

After i iterations, the orders become

$$\rho(\eta') = n - 1, \quad \rho(w') = n - 2 - i, \quad \rho(\eta''') = n + 1, \quad \rho(w''') = n - i, \quad \rho((\eta w)') = 2n - 2 - i, \quad \text{and} \quad \rho(ww') = 2n - 3 - 2i.$$

This argument can be repeated until $\rho(ww') \leq \rho(\eta''')$. Now, consider the two possibilities for n.

- If n is even: let $z \in \mathbb{Z}_{\geq 1}$ such that $n = 2z$. Then $\rho(ww') = 2n - 3 - 2i = 4z - 3 - 2i$ and $\rho(\eta''') = n + 1 = 2z + 1$. Since $\rho(ww')$ and $\rho(\eta''')$ are both odd, there exists an integer i such that $\rho(ww') = \rho(\eta''')$. Solving for i gives

$$4z - 3 - 2i = 2z + 1.$$

  Thus, $\rho(ww') = \rho(\eta''')$ when $i = z - 2$. Recalculating the orders reveals

$$\rho(\eta') = 2z - 1, \quad \rho(w') = z, \quad \rho(\eta''') = 2z + 1, \quad \rho(w''') = z + 2, \quad \rho((\eta w)') = 3z, \quad \text{and} \quad \rho(ww') = 2z + 1.$$

- If n is odd: let $z \in \mathbb{Z}_{\geq 1}$ such that $n = 2z + 1$. Then $\rho(ww') = 2n - 3 - 2i = 4z - 1 - 2i$ and $\rho(\eta''') = n + 1 = 2z + 2$. Since $\rho(ww')$ is odd and $\rho(\eta''')$ is even, there will not be an integer i such that $\rho(ww') = \rho(\eta''')$. So this argument will break when $\rho(ww') < \rho(\eta''')$, which will happen when

$$4z - 1 - 2i < 2z + 2,$$

which gives $i > z - \frac{3}{2}$. Thus, $i = z - 1$ will be the first iteration such that the above inequality is achieved. From here, the orders become

$$\rho(\eta') = 2z, \quad \rho(w') = z, \quad \rho(\eta''') = 2z + 2, \quad \rho(w''') = z + 2, \quad \rho((\eta w)') = 3z + 1, \quad \text{and} \quad \rho(ww') = 2z + 1.$$

However, $\rho(\eta''') > \rho(ww')$, so from the second equation, the coefficient of cn^{2z+2} will solely come from the η'''-term, which implies $h_{2,2z+2} = -c\lambda^2 m^2 (2z+1)(2z+2)(2z+3) j_{2z+1}$. Since c, λ, m are not zero, it must be the case that $j_{2z+1} = 0$. Recalculating the orders reveals

$$\rho(\eta') = 2z - 1, \quad \rho(w') = z, \quad \rho(\eta''') = 2z + 1, \quad \rho(w''') = z + 2, \quad \rho((\eta w)') = 3z, \quad \text{and} \quad \rho(ww') = 2z + 1.$$

So, regardless of whether n is even or odd, the same reduction of orders is achieved. The orders of the second equation are now balanced, but the orders in the first equation are not. Specifically, $\rho((\eta w)')$ is greater than any other term in the first equation. This means that the coefficient of cn^{3z} in the first equation of (3.8) will solely come from the $(\eta w)'$-term, which implies $h_{1,3z} = (3z + 1) j_{2z} k_{z+1}$. Demanding $h_{1,3z} = 0$ yields that either $j_{2z} = 0$ or $k_{z+1} = 0$.

- If $k_{z+1} = 0$: recalculating the orders

$$\rho(\eta') = 2z - 1, \quad \rho(w') = z - 1, \quad \rho(\eta''') = 2z + 1, \quad \rho(w''') = z + 1, \quad \rho((\eta w)') = 3z - 1, \quad \text{and} \quad \rho(ww') = 2z - 1.$$

Like in the odd case above, $\rho(\eta''') > \rho(ww')$, so the coefficients of cn^{2z+1} and cn^{2z} are solely given by the η'''-term. This implies that $h_{2,2z+1} = -c\lambda^2 m^2 (2z)(2z + 1)(2z + 2) j_{2z}$. Since c, λ, m are not zero, it must be the case that $j_{2z} = 0$. Similarly, since $h_{2,2z} = -c\lambda^2 m^2 (2z - 1)(2z)(2z + 1) j_{2z-1}$, the same logic yields $j_{2z-1} = 0$. Thus,

$$\rho(\eta') = 2z - 3, \quad \rho(w') = z - 1, \quad \rho(\eta''') = 2z - 1, \quad \rho(w''') = z + 1, \quad \rho((\eta w)') = 3z - 3, \quad \text{and} \quad \rho(ww') = 2z - 1.$$

- If $j_{2z} = 0$: recalculating the orders

$$\rho(\eta') = 2z-2, \quad \rho(w') = z, \quad \rho(\eta''') = 2z, \quad \rho(w''') = z+2, \quad \rho((\eta w)') = 3z-1, \quad \text{and} \quad \rho(ww') = 2z+1.$$

Now, $\rho(ww') > \rho(\eta''')$, so it is similarly deduced that $h_{2,2z+1} = (z+1)k_{z+1}^2$ which implies $k_{z+1} = 0$. Therefore,

$$\rho(\eta') = 2z-2, \quad \rho(w') = z-1, \quad \rho(\eta''') = 2z, \quad \rho(w''') = z+1, \quad \rho((\eta w)') = 3z-2, \quad \text{and} \quad \rho(ww') = 2z-1.$$

Again $\rho(\eta''') > \rho(ww')$, so $j_{2z-1} = 0$. Consequently,

$$\rho(\eta') = 2z-3, \quad \rho(w') = z-1, \quad \rho(\eta''') = 2z-1, \quad \rho(w''') = z+1, \quad \rho((\eta w)') = 3z-3, \quad \text{and} \quad \rho(ww') = 2z-1.$$

So, regardless of whether $j_{2z} = 0$ or $k_{z+1} = 0$, the same order reduction is achieved. After i iterations of this argument, the orders become

$$\rho(\eta') = 2z-3-2i, \quad \rho(w') = z-1-i, \quad \rho(\eta''') = 2z-1-2i, \quad \rho(w''') = z+1-i, \quad \rho((\eta w)') = 3z-3-3i, \quad \text{and} \quad \rho(ww') = 2z-1-2i.$$

By balancing the orders, there are three instances in which this argument breaks. From the first equation, the argument breaks when $\rho((\eta w)') \leq \rho(w''')$ or $\rho((\eta w)') \leq \rho(\eta''')$. From the second equation, the argument will break when $\rho(ww') = \rho(\eta''') \leq \rho(w''')$.

$$\begin{aligned} \rho((\eta w)') \leq \rho(w''') : &\quad 3z-3-3i \leq z+1-i \quad \iff \quad z-2 \leq i; \\ \rho((\eta w)') \leq \rho(\eta''') : &\quad 3z-3-3i \leq 2z-1-2i \quad \iff \quad z-2 \leq i; \\ \rho(ww') = \rho(\eta''') \leq \rho(w''') : &\quad 2z-1-2i \leq z+1-i \quad \iff \quad z-2 \leq i. \end{aligned}$$

So this argument will break after $(z-2)$ iterations for all three cases. Plugging $i = z-2$ and recalculating the orders a final time yields

$$\rho(\eta') = 1, \quad \rho(w') = 1, \quad \rho(\eta''') = 3, \quad \rho(w''') = 3, \quad \rho((\eta w)') = 3, \quad \text{and} \quad \rho(ww') = 3. \tag{3.11}$$

Since $c \neq 0$ and $a^2 + b^2 \neq 0$, the highest order terms in both equations are balanced. Furthermore, since $\rho(\eta') = \rho(w') = 1$, the identities in (3.6) imply that $\rho(\eta) = \rho(w) = 2$. Therefore, if $c \neq 0$ and $a^2 + b^2 \neq 0$, then $j_r = k_r = 0$, for all $r \geq 3$.

(2.) If $a^2 + b^2 = 0$, then $a = b = 0$. With this, the first equation in (3.7) is reduced to

$$-\sigma\eta' + w' + (\eta w)' = 0.$$

The orders in (3.11) are now unbalanced since $\rho((\eta w)') > \rho(\eta') = \rho(w')$. So, $h_{1,3} = 4j_2k_2$ which implies either $j_2 = 0$ or $k_2 = 0$.

- If $j_2 = 0$: then it follows that $h_{1,2} = 3j_1k_2$, so either $j_1 = 0$ or $k_2 = 0$.
  - If $j_1 = 0$, then the desired result is obtained.
  - If $k_2 = 0$, then the orders in the second equation of (3.7) are unbalanced, and $h_{2,1} = k_1^2$ which yields $k_1 = 0$. Now $\rho(ww') < \rho(\eta''')$ so from the same argument used in the proof of *(1.)*, it follows that $j_1 = 0$.
- If $k_2 = 0$: then $h_{1,2} = 3j_2k_1$, so either $j_2 = 0$ or $k_1 = 0$.
  - If $j_2 = 0$, then we arrive at the same case as above, and it follows that $j_1 = k_1 = 0$.
  - If $k_1 = 0$, then $w = k_0$ which implies $ww' = 0$. Now, the second equation in (3.7) becomes

    $$\eta' + c\eta''' = 0,$$

    but $\rho(\eta''') > \rho(\eta')$, so it quickly follows that $j_2 = j_1 = 0$.

Therefore, when $c \neq 0$ and $a^2 + b^2 = 0$, $j_r = 0$ for all integers $r \geq 1$. Moreover, the same result proved in *(1.)* holds for w, that is, $k_r = 0$ for all integers $r \geq 3$. Thus, η is constant, so there are only semi-trivial periodic traveling-wave solutions in this case. □

Remark. The above theorem will still hold if η and w in (3.5) are truncated at different values of n_1 and n_2, so long as $n_1, n_2 > 2$. However, the proof would need case splits for when $n_1 > n_2$ and $n_1 < n_2$, and inside these additional case splits there would be further splitting for when the difference of n_1 and n_2 is even or odd. For this reason, we have only considered the case when $n_1 = n_2 = n$.

From the above analysis, we see that non-trivial periodic traveling-wave solutions to (3.7) may only occur when $c \neq 0$ and $a^2 + b^2 \neq 0$, so we will focus on this case. Under these conditions,

$j_r = k_r = 0$ for all integers $r \geq 3$, so (3.8) reduces to

$$\begin{cases} -\lambda \operatorname{sn}(\lambda\xi, m) \operatorname{dn}(\lambda\xi, m) \displaystyle\sum_{q=0}^{3} h_{1,q} \operatorname{cn}^q(\lambda\xi, m) = 0, \\ -\lambda \operatorname{sn}(\lambda\xi, m) \operatorname{dn}(\lambda\xi, m) \displaystyle\sum_{q=0}^{3} h_{2,q} \operatorname{cn}^q(\lambda\xi, m) = 0. \end{cases}$$

Next, by demanding $h_{1,q} = 0$ and $h_{2,q} = 0$, we obtain the following system of 8 equations with 9 unknowns, λ, m, σ, j_i, and k_i for $i = 0, 1, 2$:

$$\begin{cases} h_{1,3} = -24\, b\lambda^2 m^2 \sigma\, j_2 - 24\, a\lambda^2 m^2 k_2 + 4\, j_2 k_2, \\ h_{1,2} = -6\, b\lambda^2 m^2 \sigma\, j_1 - 6\, a\lambda^2 m^2 k_1 + 3\, j_1 k_2 + 3\, j_2 k_1, \\ h_{1,1} = 16\, b\lambda^2 m^2 \sigma\, j_2 + 16\, a\lambda^2 m^2 k_2 - 8\, b\lambda^2 \sigma\, j_2 - 8\, a\lambda^2 k_2 - 2\,\sigma\, j_2 + 2\, j_0 k_2 \\ \qquad +2\, j_1 k_1 + 2\, j_2 k_0 + 2\, k_2, \\ h_{1,0} = 2\, b\lambda^2 m^2 \sigma\, j_1 + 2\, a\lambda^2 m^2 k_1 - b\lambda^2 \sigma\, j_1 - a\lambda^2 k_1 - \sigma\, j_1 + j_0 k_1 + j_1 k_0 + k_1, \\ h_{2,3} = -24\, d\lambda^2 m^2 \sigma\, k_2 - 24\, c\lambda^2 m^2 j_2 + 2\, k_2^2, \\ h_{2,2} = -6\, d\lambda^2 m^2 \sigma\, k_1 - 6\, c\lambda^2 m^2 j_1 + 3\, k_1 k_2, \\ h_{2,1} = 16\, d\lambda^2 m^2 \sigma\, k_2 + 16\, c\lambda^2 m^2 j_2 - 8\, d\lambda^2 \sigma\, k_2 - 8\, c\lambda^2 j_2 - 2\,\sigma\, k_2 + 2\, k_0 k_2 \\ \qquad +k_1^2 + 2\, j_2, \\ h_{2,0} = 2\, d\lambda^2 m^2 \sigma\, k_1 + 2\, c\lambda^2 m^2 j_1 - d\lambda^2 \sigma\, k_1 - c\lambda^2 j_1 - \sigma\, k_1 + k_0 k_1 + j_1\,. \end{cases} \tag{3.12}$$

The exact periodic traveling-wave solutions to (3.7) could then be established by solving the system of nonlinear equations (3.12) with the help of the computer software Maple. As there is one degree of freedom, any of the 9 unknowns can be chosen as a free parameter. In most physical settings, it is desirable to have the elliptic modulus m as the free parameter. With this in mind, the solutions found in the next section will be given in terms of m, along with the systems' parameters a, b, c, and d. In Section 3.4.1, a solution is found with $j_1 = k_1 = 0$. For this special case, (3.12) is reduced to a system of 4 equations with 7 unknowns which creates two new degrees of freedom. In this case λ and σ, the wavelength and the speed of the wave respectively, will be chosen along with m as the free parameters in our solutions. Furthermore, semi-trivial solutions for the case when $c \neq 0$ and $a^2 + b^2 = 0$ will be presented in Section 3.4.3.

3.3.2 The case when $c = 0$

When $c = 0$, system (3.7) reduces to

$$\begin{cases} -\sigma\eta' + w' + (\eta w)' + a w''' + b\sigma\eta''' = 0\,, \\ -\sigma w' + \eta' + w w' + d\sigma w''' = 0\,. \end{cases} \tag{3.13}$$

In addition, (3.8) will take on the form

$$\begin{cases} -\lambda\,\mathrm{sn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m)\displaystyle\sum_{q=0}^{2n-1}\tilde{h}_{1,q}\,\mathrm{cn}^q(\lambda\xi, m) = 0, \\ -\lambda\,\mathrm{sn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m)\displaystyle\sum_{q=0}^{2n-1}\tilde{h}_{2,q}\,\mathrm{cn}^q(\lambda\xi, m) = 0, \end{cases} \tag{3.14}$$

where $\tilde{h}_{1,q} = h_{1,q}|_{c=0}$ and $\tilde{h}_{2,q} = h_{2,q}|_{c=0}$.

As $\tilde{h}_{1,q} = 0$ and $\tilde{h}_{2,q} = 0$ for all q, the following theorem is established. (Recall that j_r and k_r are the coefficients of cn^r as given in (3.5) for $r \in \mathbb{Z}_{\geq 0}$.)

Theorem 2. *Suppose $c = 0$, then the periodic traveling-wave ansatz* (3.5) *will take on one of the following forms depending on the values of b and d:*

1. *If $b^2 + d^2 \neq 0$, then $j_r = 0$ for all integers $r \geq 5$, and $k_r = 0$ for all integers $r \geq 3$. Thus,* (3.5) *reduces to*

$$\eta(\xi) = j_0 + j_1\,\mathrm{cn}(\lambda\xi, m) + j_2\,\mathrm{cn}^2(\lambda\xi, m) + j_3\,\mathrm{cn}^3(\lambda\xi, m) + j_4\,\mathrm{cn}^4(\lambda\xi, m)$$
$$\text{and} \quad w(\xi) = k_0 + k_1\,\mathrm{cn}(\lambda\xi, m) + k_2\,\mathrm{cn}^2(\lambda\xi, m)\,.$$

2. *If $b^2 + d^2 = 0$, then $j_r = k_r = 0$ for all integers $r \geq 1$. Consequently, only the trivial solution exists in this case.*

Proof. (1.) Let $\rho(\eta')$ denote the largest power of cn in η'. Then,

$$\rho(\eta') = \rho(w') = n - 1, \quad \rho(\eta''') = \rho(w''') = n + 1, \quad \text{and} \quad \rho((\eta w)') = \rho(w w') = 2n - 1.$$

Since $n > 2$ it follows that $2n - 1 > n + 1$, which implies the coefficient of cn^{2n-1} in the second equation of (3.14) will solely come from the ww'-term. Thus, $\tilde{h}_{2,2n-1} = n k_n^2$. Setting this coefficient equal to zero gives $k_n = 0$. Recalculating the above orders yields

$$\rho(\eta') = n-1, \quad \rho(w') = n-2, \quad \rho(\eta''') = n+1, \quad \rho(w''') = n, \quad \rho((\eta w)') = 2n-2, \quad \text{and} \quad \rho(ww') = 2n-3.$$

After i iterations of this argument, the orders become

$$\rho(\eta') = n-1, \quad \rho(w') = n-2-i, \quad \rho(\eta''') = n+1, \quad \rho(w''') = n-i, \quad \rho((\eta w)') = 2n-2-i, \quad \text{and} \quad \rho(ww') = 2n-3-2i.$$

In the proof of Theorem 1, it was shown that this argument can be repeated until $\rho(ww') \leq \rho(\eta''')$, but for the $c = 0$ case, the η'''-term is no longer present in the second equation of (3.7). Thus, the argument will break when $\rho\big((\eta w)'\big) \leq \rho(\eta')$. Now, consider the two possibilities for n.

- If n is even: let $z \in \mathbb{Z}_{\geq 1}$ such that $n = 2z$. Then $\rho(ww') = 2n-3-2i = 4z-3-2i$ and $\rho(\eta') = n-1 = 2z-1$. Since $\rho(ww')$ and $\rho(\eta')$ are both odd, there exists an integer i such that $\rho(ww') = \rho(\eta')$. Solving for i gives

$$4z-3-2i = 2z-1.$$

  Thus, $\rho(ww') = \rho(\eta')$ when $i = z-1$. Recalculating the orders yields

$$\rho(\eta') = 2z-1, \quad \rho(w') = z-1, \quad \rho(\eta''') = 2z+1, \quad \rho(w''') = z+1, \quad \rho((\eta w)') = 3z-1, \quad \text{and} \quad \rho(ww') = 2z-1.$$

- If n is odd: let $z \in \mathbb{Z}_{\geq 1}$ such that $n = 2z+1$. Then $\rho(ww') = 2n-3-2i = 4z-1-2i$ and $\rho(\eta') = n-1 = 2z$. Since $\rho(ww')$ is odd and $\rho(\eta')$ is even, there will not be an integer i such that $\rho(ww') = \rho(\eta')$. So this argument will break when $\rho(ww') < \rho(\eta')$, which will happen when

$$4z-1-2i < 2z,$$

  which yields $i > z - \frac{1}{2}$. Thus, $i = z$ will be the first iteration such that the above inequality is achieved. From here, the orders become

$$\rho(\eta') = 2z, \quad \rho(w') = z-1, \quad \rho(\eta''') = 2z+2, \quad \rho(w''') = z+1, \quad \rho((\eta w)') = 3z, \quad \text{and} \quad \rho(ww') = 2z-1.$$

  However, $\rho(\eta') > \rho(ww')$, so the coefficient of cn^{2z} in the second equation will solely come from the η'-term, which implies $\tilde{h}_{2,2z} = (2z+1)j_{2z+1}$. It then follows that $j_{2z+1} = 0$. Recalculating the orders yields

$$\rho(\eta') = 2z - 1, \quad \rho(w') = z - 1, \quad \rho(\eta''') = 2z + 1, \quad \rho(w''') = z + 1, \quad \rho((\eta w)') = 3z - 1,$$
$$\text{and} \quad \rho(ww') = 2z - 1.$$

So, regardless of whether n is even or odd, the same reduction of orders is achieved. Now the orders of the second equation are balanced, but the orders in the first equation are not. Specifically, $\rho((\eta w)')$ is greater than any other term in the first equation. This means that the coefficient of cn^{3z-1} in the first equation of (3.8) will solely come from the $(\eta w)'$-term, which implies $\tilde{h}_{1,3z-1} = 3zj_{2z}k_z$. Demanding $\tilde{h}_{1,3z-1} = 0$ reveals that either $j_{2z} = 0$ or $k_z = 0$.

- If $k_z = 0$: recalculating the orders

  $$\rho(\eta') = 2z-1, \quad \rho(w') = z-2, \quad \rho(\eta''') = 2z+1, \quad \rho(w''') = z, \quad \rho((\eta w)') = 3z-2, \quad \text{and}$$
  $$\rho(ww') = 2z - 3.$$

  Like in the odd case above, $\rho(\eta') > \rho(ww')$, so the coefficients of cn^{2z-1} and cn^{2z-2} are solely given by the η'-term. This implies that $\tilde{h}_{2,2z-1} = 2zj_{2z}$, so it follows that $j_{2z} = 0$. Similarly, since $\tilde{h}_{2,2z-2} = (2z-1)j_{2z-1}$, the same logic yields $j_{2z-1} = 0$. Thus,

  $$\rho(\eta') = 2z-3, \quad \rho(w') = z-2, \quad \rho(\eta''') = 2z-1, \quad \rho(w''') = z, \quad \rho((\eta w)') = 3z-4, \quad \text{and}$$
  $$\rho(ww') = 2z - 3.$$

- If $j_{2z} = 0$: recalculating the orders

  $$\rho(\eta') = 2z-2, \quad \rho(w') = z-1, \quad \rho(\eta''') = 2z, \quad \rho(w''') = z+1, \quad \rho((\eta w)') = 3z-2, \quad \text{and}$$
  $$\rho(ww') = 2z - 1.$$

  Now, $\rho(ww') > \rho(\eta')$, so it is similarly deduced that $\tilde{h}_{2,2z-1} = zk_z^2$ which implies $k_z = 0$. Therefore,

  $$\rho(\eta') = 2z-2, \quad \rho(w') = z-2, \quad \rho(\eta''') = 2z, \quad \rho(w''') = z, \quad \rho((\eta w)') = 3z-3, \quad \text{and}$$
  $$\rho(ww') = 2z - 3.$$

  Again $\rho(\eta') > \rho(ww')$ so $j_{2z-1} = 0$. Consequently,

  $$\rho(\eta') = 2z-3, \quad \rho(w') = z-2, \quad \rho(\eta''') = 2z-1, \quad \rho(w''') = z, \quad \rho((\eta w)') = 3z-4, \quad \text{and}$$
  $$\rho(ww') = 2z - 3.$$

So, regardless of whether $j_{2z} = 0$ or $k_z = 0$, the same order reduction is achieved. After i iterations of this argument, the orders become

$$\rho(\eta') = 2z - 3 - 2i, \quad \rho(w') = z - 2 - i, \quad \rho(\eta''') = 2z - 1 - 2i, \quad \rho(w''') = z - i,$$
$$\rho((\eta w)') = 3z - 4 - 3i, \quad \text{and} \quad \rho(ww') = 2z - 3 - 2i.$$

By balancing the orders, there are two instances in which this argument breaks. From the first equation, the argument breaks when $\rho((\eta w)') \leq \rho(\eta''')$, and from the second equation, the argument will break when $\rho(ww') = \rho(\eta') \leq \rho(w''')$.

$$\rho((\eta w)') \leq \rho(\eta''') : \quad 3z - 4 - 3i \leq 2z - 1 - 2i \quad \Longleftrightarrow \quad z - 3 \leq i;$$
$$\rho(ww') = \rho(\eta') \leq \rho(w''') : \quad 2z - 3 - 2i \leq z - i \quad \Longleftrightarrow \quad z - 3 \leq i.$$

So this argument will break after $(z-3)$ iterations for both cases. Plugging $i = z-3$ and recalculating the orders a final time yields

$$\rho(\eta') = 3, \quad \rho(w') = 1, \quad \rho(\eta''') = 5, \quad \rho(w''') = 3, \quad \rho((\eta w)') = 5, \quad \text{and} \quad \rho(ww') = 3.$$

Since $c \neq 0$ and $b^2 + d^2 \neq 0$, the highest order terms in both equations are balanced. Furthermore, since $\rho(\eta') = 3$ and $\rho(w') = 1$, the identities in (3.6) imply that $\rho(\eta) = 4$ and $\rho(w) = 2$. Therefore, if $c \neq 0$ and $b^2 + d^2 \neq 0$, then $j_r = 0$ for all $r \geq 5$ and $k_r = 0$ for all $r \geq 3$.

(2.) If $b^2 + d^2 = 0$, then $b = d = 0$. Recall that the parameters a, b, c, and d must satisfy the three conditions given in (3.3). Since $b = c = d = 0$, it follows that $a = \frac{1}{3}$. With this, system (3.13) reduces to

$$\begin{cases} -\sigma\eta' + w' + (\eta w)' + \dfrac{1}{3}w''' = 0\,, \\ -\sigma w' + \eta' + ww' = 0\,. \end{cases}$$

Now, $\tilde{h}_{1,5}$ and $\tilde{h}_{2,3}$ are given by

$$\tilde{h}_{1,5} = 6j_4k_2 \quad \text{and} \quad \tilde{h}_{2,3} = 4j_4 + 2k_2^2.$$

Requiring these both to be zero implies that $j_4 = k_2 = 0$. Substituting this into the rest of the coefficients, it follows that $\tilde{h}_{2,2} = 3j_3$ which implies $j_3 = 0$. Now, $\tilde{h}_{1,2}$ and $\tilde{h}_{2,1}$ are given by

$$\tilde{h}_{1,2} = -k_1(2\lambda^2m^2 - 3j_2) \quad \text{and} \quad \tilde{h}_{2,1} = {k_1}^2 + 2j_2.$$

Requiring $\tilde{h}_{2,1} = 0$ yields $j_2 = -\frac{1}{2}{k_1}^2$, and substituting this into $\tilde{h}_{1,2}$ gives

$$\tilde{h}_{1,2} = -\frac{1}{2}k_1(4\lambda^2m^2 + 3{k_1}^2).$$

Since $\lambda, m > 0$ and $k_1 \in \mathbb{R}$, it follows that $4\lambda^2 m^2 + 3{k_1}^2 > 0$. Thus, $k_1 = 0$, and similarly, $j_2 = 0$. Finally, $\tilde{h}_{2,0} = j_1$, forcing $j_1 = 0$.

Therefore, when $c \neq 0$ and $b^2 + d^2 = 0$, it is deduced that $j_r = k_r = 0$ for all integers $r \geq 1$ which implies both η and w are constant and consequently, only the trivial solution exists. □

Remark. The above theorem will still hold if η and w in (3.5) are truncated at different values of n_1 and n_2, so long as $n_1 > 4$ and $n_2 > 2$. However, the proof would need case splits for when $n_1 > n_2$ and $n_1 < n_2$, and inside these additional case splits there would be further splitting for when the difference of n_1 and n_2 is even or odd. For this reason, we have only considered the case when $n_1 = n_2 = n$.

The above theorem shows that periodic traveling-wave solutions to (3.13) may only occur when $c = 0$ and $b^2 + d^2 \neq 0$, so we will focus on this case. Substituting $j_r = 0$ for all $r \geq 5$ and $k_r = 0$ for all $r \geq 3$ into (3.8) yields

$$\begin{cases} -\lambda\,\mathrm{sn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m) \displaystyle\sum_{q=0}^{5} \tilde{h}_{1,q}\,\mathrm{cn}^q(\lambda\xi, m) = 0, \\ -\lambda\,\mathrm{sn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m) \displaystyle\sum_{q=0}^{3} \tilde{h}_{2,q}\,\mathrm{cn}^q(\lambda\xi, m) = 0. \end{cases}$$

Next, by demanding $\tilde{h}_{1,q} = 0$ and $\tilde{h}_{2,q} = 0$, we obtain the following system of 10 equations and 11 unknowns, λ, m, σ, j_i for $i = 0, 1, \ldots, 4$, and k_i for $i = 0, 1, 2$:

$$\begin{cases} \tilde{h}_{1,5} = -6\, j_4 \left(20\, b\lambda^2 m^2 \sigma - k_2\right) , \\ \tilde{h}_{1,4} = -60\, b\lambda^2 m^2 \sigma\, j_3 + 5\, j_3 k_2 + 5\, j_4 k_1 , \\ \tilde{h}_{1,3} = -24\, b\lambda^2 m^2 \sigma\, j_2 + 128\, b\lambda^2 m^2 \sigma\, j_4 - 24\, a\lambda^2 m^2 k_2 - 64\, b\lambda^2 \sigma\, j_4 - 4\, \sigma\, j_4 + 4\, j_2 k_2 \\ \qquad +4\, j_3 k_1 + 4\, j_4 k_0 , \\ \tilde{h}_{1,2} = -6\, b\lambda^2 m^2 \sigma\, j_1 + 54\, b\lambda^2 m^2 \sigma\, j_3 - 6\, a\lambda^2 m^2 k_1 - 27\, b\lambda^2 \sigma\, j_3 - 3\, \sigma\, j_3 + 3\, j_1 k_2 \\ \qquad +3\, j_2 k_1 + 3\, j_3 k_0 , \\ \tilde{h}_{1,1} = 16\, b\lambda^2 m^2 \sigma\, j_2 - 24\, b\lambda^2 m^2 \sigma\, j_4 + 16\, a\lambda^2 m^2 k_2 - 8\, b\lambda^2 \sigma\, j_2 + 24\, b\lambda^2 \sigma\, j_4 \\ \qquad -8\, a\lambda^2 k_2 - 2\, \sigma\, j_2 + 2\, j_0 k_2 + 2\, j_1 k_1 + 2\, j_2 k_0 + 2\, k_2 , \\ \tilde{h}_{1,0} = 2\, b\lambda^2 m^2 \sigma\, j_1 - 6\, b\lambda^2 m^2 \sigma\, j_3 + 2\, a\lambda^2 m^2 k_1 - b\lambda^2 \sigma\, j_1 + 6\, b\lambda^2 \sigma\, j_3 - a\lambda^2 k_1 - \sigma\, j_1 \\ \qquad +j_0 k_1 + j_1 k_0 + \sigma k_1^2\, k_2 , \\ \tilde{h}_{2,3} = -24\, d\lambda^2 m^2 \sigma\, k_2 + 2\, k_2^2 + 4\, j_4 , \\ \tilde{h}_{2,2} = -6\, d\lambda^2 m^2 \sigma\, k_1 + 3\, k_1 k_2 + 3\, j_3 , \\ \tilde{h}_{2,1} = 16\, d\lambda^2 m^2 \sigma\, k_2 - 8\, d\lambda^2 \sigma\, k_2 - 2\, \sigma\, k_2 + 2\, k_0 k_2 + k_1^2 + 2\, j_2 , \\ \tilde{h}_{2,0} = 2\, d\lambda^2 m^2 \sigma\, k_1 - d\lambda^2 \sigma\, k_1 - \sigma\, k_1 + k_0 k_1 + j_1 . \end{cases} \tag{3.15}$$

Numerical computation shows that if $j_1 \neq 0$ or $j_3 \neq 0$, then there does not exist a solution to (3.15). Similarly, numerical computation shows if $k_1 \neq 0$, then the only solution to (3.15) is one such that $\sigma = 0$, a contradiction. So, we will proceed under the assumption that $j_1 = j_3 = k_1 = 0$. In particular, the only non-trivial periodic traveling-wave solutions to (3.13) are of the form

$$\eta(\xi) = j_0 + j_2 \operatorname{cn}^2(\lambda\xi, m) + j_4 \operatorname{cn}^4(\lambda\xi, m) \quad \text{and} \quad w(\xi) = k_0 + k_2 \operatorname{cn}^2(\lambda\xi, m) . \tag{3.16}$$

Moreover, this reduces (3.15) to the following system of 5 equations with 8 unknowns:

$$\begin{cases} \bar{h}_{1,5} = -6\,j_4\left(20\,b\lambda^2 m^2\sigma - k_2\right)\,, \\ \bar{h}_{1,3} = -24\,b\lambda^2 m^2\sigma\, j_2 + 128\,b\lambda^2 m^2\sigma\, j_4 - 24\,a\lambda^2 m^2 k_2 - 64\,b\lambda^2\sigma\, j_4 - 4\,\sigma\, j_4 + 4\,j_2 k_2 \\ \qquad +4\,j_4 k_0\,, \\ \bar{h}_{1,1} = 16\,b\lambda^2 m^2\sigma\, j_2 - 24\,b\lambda^2 m^2\sigma\, j_4 + 16\,a\lambda^2 m^2 k_2 - 8\,b\lambda^2\sigma\, j_2 + 24\,b\lambda^2\sigma\, j_4 \\ \qquad -8\,a\lambda^2 k_2 - 2\,\sigma\, j_2 + 2\,j_0 k_2 + 2\,j_2 k_0 + 2\,k_2\,, \\ \bar{h}_{2,3} = -24\,d\lambda^2 m^2\sigma\, k_2 + 2\,{k_2}^2 + 4\,j_4\,, \\ \bar{h}_{2,1} = 16\,d\lambda^2 m^2\sigma\, k_2 - 8\,d\lambda^2\sigma\, k_2 - 2\,\sigma\, k_2 + 2\,k_0 k_2 + 2\,j_2\,, \\ \bar{h}_{1,4} = \bar{h}_{1,2} = \bar{h}_{1,0} = \bar{h}_{2,2} = \bar{h}_{2,0} = 0\,, \end{cases} \tag{3.17}$$

where $\bar{h}_{1,q} = \tilde{h}_{1,q}|_{j_1=j_3=k_1=0}$ and $\bar{h}_{2,q} = \tilde{h}_{2,q}|_{j_1=j_3=k_1=0}$.

The exact periodic traveling-wave solutions to (3.13) could then be established by solving the system of nonlinear equations (3.17) with the help of the computer software Maple. As there are three degrees of freedom, any combination of the 8 unknowns could be chosen as free parameters. For consistency, we will choose λ, m, and σ as free parameters, like in the previous section.

3.4 Exact Jacobi Elliptic Solutions

Finally, explicit solutions to the $abcd$-system (3.2) are given for the cases $c \neq 0$ and $c = 0$. Discussion of these solutions, and how they limit to solitary-wave solutions, will be left for the next section. In addition to the explicit solutions being stated, graphs of these solutions will also be provided.

3.4.1 The case when $c \neq 0$

Solving (3.12), our solution will take the form as stated by (3.9), where the parameters are found to be given by one of the following sets.

1. When $ac(b-6d)(3b-2d)$, $c(2m^2-1)(b+2d)(3b+2d) < 0$ and $(2m^2-1)(3b+2d)(b-d) \geq 0$

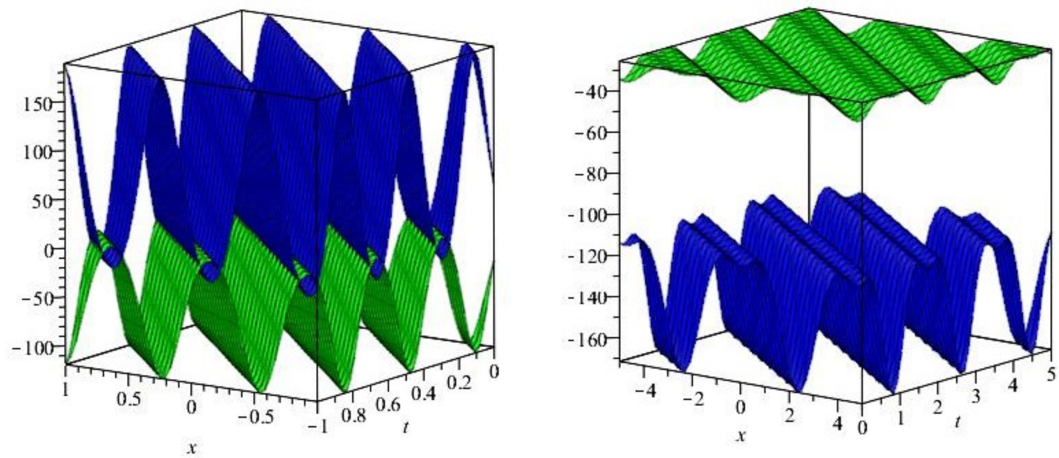
with equality only when $b = d$:

$$
\begin{cases}
j_0 = -\dfrac{a\,(3\,b+2\,d)\,(21\,b-46\,d)+2\,c\,(3\,b-2\,d)\,(b-6\,d)}{2c\,(3\,b-2\,d)\,(b-6\,d)}\,, \\[2ex]
j_1 = -\tau_1\tau_2\dfrac{12\,a\,m\,(3\,b+2\,d)\,\sqrt{(2\,m^2-1)\,(3\,b+2\,d)\,(b-d)}}{c\,(b-6\,d)\,(3\,b-2\,d)\,(2\,m^2-1)}\,, \\[2ex]
j_2 = \dfrac{9\,a\,m^2\,(3\,b+2\,d)}{c\,(3\,b-2\,d)\,(2\,m^2-1)}\,, \\[2ex]
k_0 = \tau_1\dfrac{(21\,b+8\,c+14\,d)\,\sqrt{-2\,ac\,(b-6\,d)\,(3\,b-2\,d)}}{2\,c\,(b-6\,d)\,(3\,b-2\,d)}\,, \\[2ex]
k_1 = \tau_2\dfrac{6\,m\sqrt{-2\,ac\,(b-6\,d)\,(3\,b-2\,d)\,(2\,m^2-1)\,(3\,b+2\,d)\,(b-d)}}{c\,(2\,m^2-1)\,(b-6\,d)\,(3\,b-2\,d)}\,, \\[2ex]
k_2 = -\tau_1\dfrac{9\,m^2\,(3\,b+2\,d)\,\sqrt{-2\,ac\,(b-6\,d)\,(3\,b-2\,d)}}{c\,(2\,m^2-1)\,(b-6\,d)\,(3\,b-2\,d)}\,, \\[2ex]
\lambda = \dfrac{1}{2}\sqrt{\dfrac{-6\,(3\,b+2\,d)}{c\,(2\,m^2-1)\,(b+2\,d)}}\,, \\[2ex]
\sigma = \tau_1\dfrac{4\sqrt{-2\,ac\,(b-6\,d)\,(3\,b-2\,d)}}{(b-6\,d)\,(3\,b-2\,d)}\,, \\[2ex]
m \in (\,0, 1\,]\,.
\end{cases}
\tag{3.18}
$$

Finding this solution requires taking the square root of two terms, which leads to two independent plus or minuses. To avoid confusion, we define

$$\tau_1 = \{\,1,\,-1\,\} \quad \text{and} \quad \tau_2 = \{\,1,\,-1\,\},$$

to denote these independent plus or minuses. Two graphs of this solution can be found below in Figures 3.1a and 3.1b, in which η is graphed in blue and w in green. The explicit parameters chosen for each graph are given in the captions.



(a) $\{m=\frac{3}{4},\ a=-\frac{5}{6},\ b=1,\ c=-\frac{5}{6},\ d=1,\ \tau_1=1,\ \tau_2=1\}$

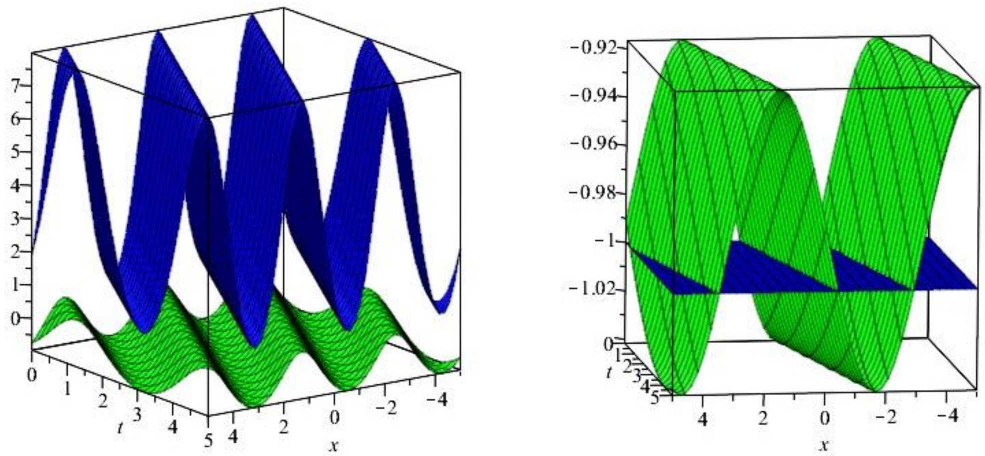
(b) $\{m=\frac{1}{4},\ a=-7,\ b=2,\ c=\frac{4}{3},\ d=4,\ \tau_1=-1,\ \tau_2=1\}$

Figure 3.1: Graphs of solution (3.18)

2. When $8ac+\sigma^2(b-2d)^2>0$:

$$
\begin{cases}
j_1=k_1=0\,,\\
j_0=\dfrac{1}{4\,c^2\left(4\,a\,c+\sigma^2\,(b^2+4\,d^2)\pm\sigma\,(b+2\,d)\sqrt{8\,a\,c+\sigma^2\,(b-2\,d)^2}\right)}\Big(-8\,b^4c\,\lambda^2m^2\sigma^4\\
\quad+16\,b^3c\,d\,\lambda^2m^2\sigma^4-64\,a\,b^2c^2\lambda^2m^2\sigma^2-32\,a\,b\,c^2d\,\lambda^2m^2\sigma^2-64\,a\,c^2d^2\lambda^2m^2\sigma^2\\
\quad+4\,b^4c\,\lambda^2\sigma^4-8\,b^3c\,d\,\lambda^2\sigma^4-64\,a^2c^3\lambda^2m^2+32\,a\,b^2c^2\lambda^2\sigma^2+16\,a\,b\,c^2d\,\lambda^2\sigma^2\\
\quad+32\,a\,c^2d^2\lambda^2\sigma^2+b^4\sigma^4-4\,b^3d\,\sigma^4+4\,b^2d^2\sigma^4+32\,a^2c^3\lambda^2+8\,a\,b^2c\,\sigma^2-8\,a\,b\,c\,d\,\sigma^2\\
\quad-4\,b^2c^2\sigma^2-16\,c^2d^2\sigma^2+8\,a^2c^2-16\,a\,c^3\mp\sigma\sqrt{8\,a\,c+\sigma^2\,(b-2\,d)^2}\Big(8\,b^3c\,\lambda^2m^2\sigma^2\\
\quad+32\,a\,b\,c^2\lambda^2m^2+32\,a\,c^2d\lambda^2m^2-4\,b^3c\,\lambda^2\sigma^2-16\,a\,b\,c^2\lambda^2-16\,a\,c^2d\,\lambda^2-b^3\sigma^2\\
\quad+2\,b^2d\,\sigma^2-4\,a\,b\,c+4\,b\,c^2+8\,c^2d\Big)\Big)\,,\\
j_2=\dfrac{3\,\lambda^2m^2}{2\,c}\left(4\,a\,c+b\,\sigma^2\,(b-2\,d)\pm b\,\sigma\sqrt{8\,a\,c+\sigma^2\,(b-2\,d)^2}\right),\\
k_0=\dfrac{1}{2\,c\left(\sigma\,(b+2\,d)\pm\sqrt{8\,a\,c+\sigma^2\,(b-2\,d)^2}\right)}\Big(-8\,b^2c\,\lambda^2m^2\sigma^2-32\,c\,d^2\lambda^2m^2\sigma^2\\
\quad-32\,a\,c^2\lambda^2m^2+4\,b^2c\,\lambda^2\sigma^2+16\,c\,d^2\lambda^2\sigma^2+16\,a\,\lambda^2c^2-b^2\sigma^2+2\,b\,c\,\sigma^2+2\,b\,d\,\sigma^2\\
\quad+4\,c\,d\,\sigma^2-4\,a\,c\mp\sigma\sqrt{8\,a\,c+\sigma^2\,(b-2\,d)^2}\Big(8\,b\,c\,\lambda^2m^2+16\,c\,d\,\lambda^2m^2-4\,b\,c\,\lambda^2\\
\quad-8\,c\,d\,\lambda^2+b-2\,c\Big)\Big)\,,\\
k_2=3\,\lambda^2m^2\left(\sigma\,(b+2\,d)\pm\sqrt{8\,a\,c+\sigma^2\,(b-2\,d)^2}\right)\,,\\
\lambda>0,\ \sigma\neq0,\ m\in(0,1]\,.
\end{cases}
\tag{3.19}
$$

Above, the $\pm$ and $\mp$ should be chosen such that the top or bottom signs are chosen together. Two graphs of this solution can be found below in Figures 3.2a and 3.2b, in which η is graphed in blue and w in green. The explicit parameters chosen for each graph are given in the captions.



(a) $\{\lambda = 1,\ m = \frac{1}{\sqrt{2}},\ \sigma = 1,\ a = 1,\ b = -\frac{8}{3},\ c = 1,\ d = 1\}$ and the top signs of $\pm$ and $\mp$

(b) $\{\lambda = \frac{1}{2},\ m = \frac{1}{4},\ \sigma = -\frac{1}{3},\ a = 0,\ b = -1,\ c = -\frac{2}{3},\ d = 2\}$ and the bottom signs of $\pm$ and $\mp$

Figure 3.2: Graphs of solution (3.19)

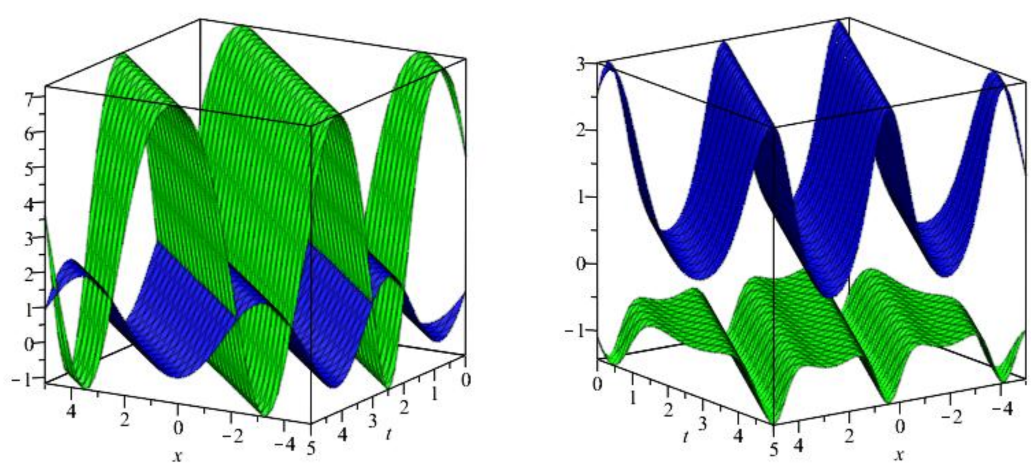
3.4.2 The case when $c = 0$

Solving (3.17), our solution will take the form as stated by (3.16), where the parameters are found to be given by one of the following sets.

1. When $4b - d \neq 0$:

$$\begin{cases} j_0 = \dfrac{1}{9\,\sigma^2\,(4\,b-d)^2}\Big(-32\,b\,\lambda^4\sigma^4\,(4\,b-d)^2\,(5\,b-3\,d)\,\big(11\,m^4-11\,m^2-4\big) \\ \qquad +3\,\sigma^2\,(4\,b-d)\,\big[3\,d-4\,b\,\big(3+5\,a\,\lambda^2\,\big(2\,m^2-1\big)\big)\,\big]+9\,a^2\Big)\,, \\ j_2 = \dfrac{20\,b\,\lambda^2 m^2\,\big(3\,a+4\,\lambda^2\sigma^2\,(4\,b-d)\,(5\,b-3\,d)\,\big(2\,m^2-1\big)\big)}{3(4\,b-d)}\,, \\ j_4 = -40\,b\,\lambda^4 m^4\sigma^2\,(5\,b-3\,d)\;, \\ k_0 = \dfrac{-3\,a+\sigma^2\,(4\,b-d)\,\big(3-20\,b\,\lambda^2\,\big(2m^2-1\big)\big)}{3\,\sigma\,(4\,b-d)}\,, \\ k_2 = 20\,b\,\lambda^2 m^2\sigma\,, \\ \lambda>0,\ \sigma\neq 0,\ m\in(\,0,1\,]\,. \end{cases} \tag{3.20}$$

Two graphs of this solution can be found below in Figures 3.3a and 3.3b, in which η is graphed in blue and w in green. The explicit parameters chosen for each graph are given in the captions.



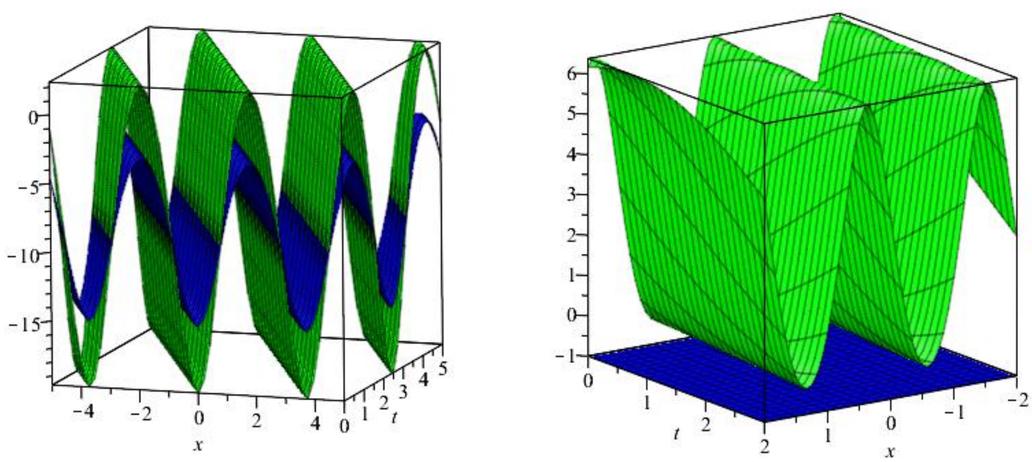
(a) $\{\lambda=\frac{1}{2},\ m=\frac{1}{2},\ \sigma=-2,\ a=1, b=-1,\ d=\frac{1}{3}\}$ (b) $\{\lambda=1,\ m=\frac{9}{10},\ \sigma=1,\ a=0, b=\frac{1}{6},\ d=\frac{1}{6}\}$

Figure 3.3: Graphs of solution (3.20)

2. When $b - 2d \neq 0$:

$$\begin{cases} j_0 = \dfrac{a^2 - \sigma^2 (b - 2d)\left(b - 2d\left(1 + 2a\lambda^2 (2m^2 - 1)\right)\right)}{\sigma^2 (b - 2d)^2}, \\ j_2 = -12\,\dfrac{a d \lambda^2 m^2}{b - 2d}, \\ k_0 = \dfrac{a + \sigma^2 (b - 2d)\left(1 - 4d\lambda^2 (2m^2 - 1)\right)}{\sigma (b - 2d)}, \\ k_2 = 12 d \lambda^2 m^2 \sigma, \\ \lambda > 0,\ \sigma \neq 0,\ m \in (0, 1]. \end{cases} \tag{3.21}$$

Two graphs of this solution can be found below in Figures 3.4a and 3.4b, in which η is graphed in blue and w in green. The explicit parameters chosen for each graph are given in the captions.



(a) $\{\lambda = 1,\ m = \frac{1}{\sqrt{2}},\ \sigma = -1, a = -\frac{11}{3},\ b = 2,\ d = 2\}$

(b) $\{\lambda = 2,\ m = \frac{3}{4},\ \sigma = \frac{1}{8},\ a = 0,\ b = -\frac{5}{3},\ d = 2\}$

Figure 3.4: Graphs of solution (3.21)

3.4.3 Semi-trivial Solutions

Substituting $a = b = 0$ into (3.12) and solving, a solution of the form (3.10) is found where

$$\begin{cases} j_0 = -1\,, \\ k_0 = -8\,d\,\lambda^2 m^2 \sigma + 4\,d\,\lambda^2 \sigma + \sigma\,, \\ k_1 = 0\,, \\ k_2 = 12\,d\,\lambda^2 m^2 \sigma\,, \\ \lambda > 0,\ \sigma \neq 0,\ m \in (\,0, 1\,]\,. \end{cases} \tag{3.22}$$

One may notice that Figures 3.2b and 3.4b are graphs in which $\eta = -1$. In fact, these are exactly graphs of solution (3.22), so additional graphs need not be provided here. The relationship between solution (3.22) and solutions (3.19) and (3.21) is made clear in the next section.

3.5 Conclusion

As stated at the end of Section 3.4.3, there is a distinct relationship between solutions (3.19), (3.21), and (3.22). In fact, if $\sigma(b - 2d) > 0$, then solution (3.21) can be retrieved by taking the limit as c approaches zero of solution (3.19) while choosing the bottom signs for $\pm$ and $\mp$. Similarly, solution (3.22) can be recovered from solutions (3.19) or (3.21) in any of the following ways

- if $d\sigma > 0$, then evaluating solution (3.19) at $a = b = 0$ and choosing the top signs for $\pm$ and $\mp$;
- if $\sigma(b - 2d) > 0$, then evaluating solution (3.19) at $a = 0$ and choosing the bottom signs for $\pm$ and $\mp$;
- if $b - 2d \neq 0$, then evaluating solution (3.21) at $a = 0$.

The semi-trivial solution (3.22) is also of importance. When we evaluate system (3.2) at $\eta = -1$ and $a = 0$, it collapses to a single equation. This equation is

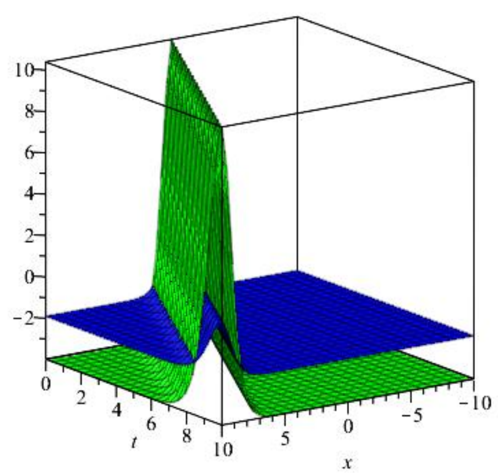
$$w_t + w w_x - d w_{xxt} = 0\,,$$

which is exactly the BBM equation after a transformation. This equation has been shown to have cnoidal wave solutions [1], so it is especially important that we recovered a cnoidal solution for this case.

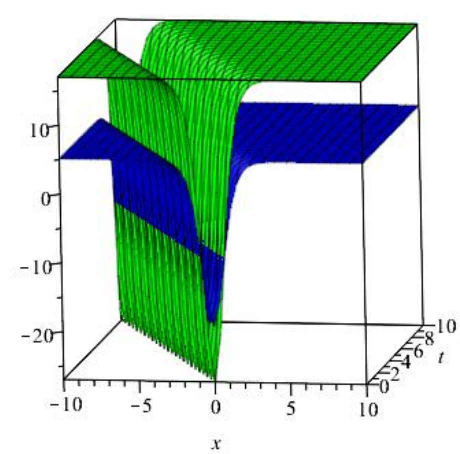
Another fundamental property of the Jacobi cnoidal function is that as m approaches one, the cnoidal function limits to the hyperbolic secant function. This is important because solitary-wave solutions are often given in terms of hyperbolic secant functions. Thus, in principle, taking the limit of our solutions as m approaches one may retrieve solitary-wave solutions. Indeed, for (3.19) and (3.21) we get solitary-wave solutions of the form

$$\eta = \bar{j}_0 + \bar{j}_2 \operatorname{sech}^2(\bar{\lambda}(x - \bar{\sigma}t)) \quad \text{and} \quad w = \bar{k}_0 + \bar{k}_2 \operatorname{sech}^2(\bar{\lambda}(x - \bar{\sigma}t), \tag{3.23}$$

where the bar indicates the evaluation at $m = 1$. This straightforward computation is left to the reader. Some solitary-wave solutions for these solutions are graphed in Figures 3.5a and 3.5b, in which η is graphed in blue and w in green. The exact values chosen for the parameters are given in the captions of each figure. In the paper [17, 18], solitary-wave solutions were shown to be given exactly by the same form in (3.23) for cases $\{a = 0,\ c \neq 0\}$, $\{b = d = 0,\ a = c\}$ and $\{b = c = 0,\ d \neq 0\}$ where the coefficients solve certain equations.



(a) Solution (3.19) when $\{\lambda = 1,\ m = 1,\ \sigma = 1,\ a = 1,\ b = -\frac{8}{3},\ c = 1,\ d = 1\}$ and the top signs of $\pm$ and $\mp$ are chosen



(b) Solution (3.21) when $\{\lambda = 1,\ m = 1,\ \sigma = -1,\ a = -\frac{11}{3},\ b = 2,\ d = 2\}$

Figure 3.5: Graphs of solitary-wave solutions of form (3.23)

From solution (3.20), we get a solitary-wave solution of the form

$$\eta = \bar{j}_0 + \bar{j}_2 \operatorname{sech}^2(\lambda(x - \sigma t)) + \bar{j}_4 \operatorname{sech}^4(\lambda(x - \sigma t)) \quad \text{and} \quad w = \bar{k}_0 + \bar{k}_2 \operatorname{sech}^2(\lambda(x - \sigma t)), \tag{3.24}$$

where again, the bar indicates the evaluation at $m = 1$. This is graphed in Figure 3.6a, in which η is graphed in blue and w in green. The form found in this paper (3.24) again matches the form of solitary-wave solutions found in [17, 18] for cases $\{a = c = 0\}$ and $\{c = d = 0,\, b \neq 0\}$. This also re-emphasizes exactly what was found in this paper, that solutions with a cn^4 term in η can only occur when $c = 0$.

Similarly, looking at solution (3.22) as m approaches one, we get a solitary-wave solution of the form

$$\eta = -1 \quad \text{and} \quad w = \bar{k}_0 + \bar{k}_2 \operatorname{sech}^2(\lambda(x - \sigma t)),$$

which was also found in [17, 18] for the case $\{a = 0\}$.

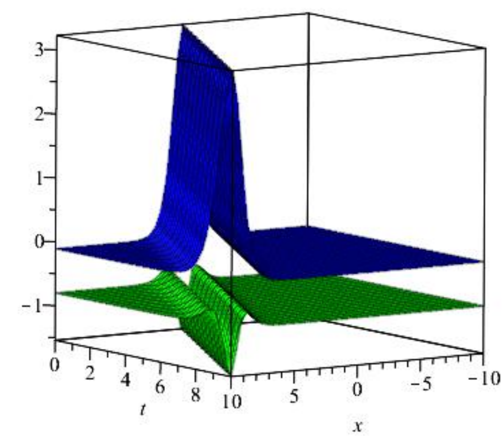
Of course, there is no one-to-one correspondence between cnoidal solutions and solitary-wave solutions. Thus, one should not expect to establish *all* solitary-wave solutions by taking the limit as the elliptic modulus m approaches one. One such instance of this is found in (3.18), where a solution of the form

$$\eta = \bar{j}_0 + \bar{j}_1 \operatorname{sech}(\bar{\lambda}(x - \bar{\sigma} t)) + \bar{j}_2 \operatorname{sech}^2(\bar{\lambda}(x - \bar{\sigma} t)) \quad \text{and} \quad w = \bar{k}_0 + \bar{k}_1 \operatorname{sech}(\bar{\lambda}(x - \bar{\sigma} t)) + \bar{k}_2 \operatorname{sech}^2(\bar{\lambda}(x - \bar{\sigma} t)), \tag{3.25}$$

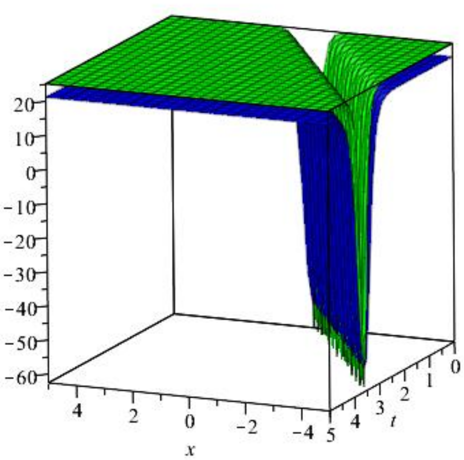
is achieved as m approaches one. This is graphed in Figure 3.6b, in which η is graphed in blue and w in green. This form of solution was not found in [17, 18], showing the lack of correspondence. In turn, a form of solitary-wave solutions was found in [17, 18] that was not recovered in this paper. This solution is found when $\{b = c = d = 0,\, a > 0\}$, and takes the form

$$\eta = j_0 + j_2 \operatorname{sech}^2(\lambda(x - \sigma t)) \quad \text{and} \quad w = k_0 + k_1 \tanh(\lambda(x - \sigma t)).$$

This solution was not established in this paper because hyperbolic tangent is not a limiting case of the Jacobi cnoidal function.



(a) Solution (3.20) when $\{\lambda = 1,\ m = 1,\ \sigma = 1,\ a = 0,\ b = \frac{1}{6},\ d = \frac{1}{6}\}$



(b) Solution (3.18) when $\{m = 1,\ a = -\frac{5}{6},\ b = 1,\ c = -\frac{1}{6},\ d = \frac{1}{3},\ \tau_1 = 1,\ \tau_2 = -1\}$

Figure 3.6: Graphs of solitary-wave solutions of form (3.24) and (3.25)

CHAPTER 4
CONCLUSION

In conclusion, it has been shown that periodic traveling-wave solutions given by the Jacobi cnoidal function exist for both the four systems and the $abcd$-system. In this final chapter, future research directions will be discussed.

With explicit formulae for the periodic traveling-wave solutions, a variety of applications in numerical analysis are now possible. One such example is the study of stability of these solutions. This study is the most practical application of these newly found solutions.

Certain solitary-wave solutions have been presented in this paper, but there is no one-to-one correspondence between Jacobi cnoidal solutions and solitary-wave solutions. For the $abcd$-system, all solitary-wave solutions have been calculated in [17, 18]. In Chapter 3.5, the relationship between solitary-wave solutions found in [17, 18] and the solutions recovered in this paper was discussed. For the four systems, synchronized solitary-wave solutions have been found in [33], but in general, all solitary-wave solutions have not been explicitly found. So, although certain solitary-wave solutions were found as limiting cases here, it is not expected that all solitary-wave solutions were found. An interesting future research direction would be to explicitly calculate all solitary-wave solutions, and compare them to the solitary-wave solutions presented here.

In this thesis, it was assumed that both waves were traveling in the same direction, and with the same speed. A fundamental property of the $abcd$-system is it allows for bidirectional waves. It would be interesting to see if cnoidal solutions could be found in the setting where the waves are traveling in opposite directions. That is, solutions of the form $\big(u(x,t), v(x,t)\big)$ where

$$u(x,t) = f(x - \sigma t) \quad \text{and} \quad v(x,t) = g(x + \sigma t).$$

This is not a feature of the four systems, so it is not expected that a solution like this would exist. However, for both systems, it would also be interesting to study the existence of periodic traveling-wave solutions that travel at different speeds, i.e. solutions of the form

$$u(x,t) = f(x - \sigma_1 t) \quad \text{and} \quad v(x,t) = g(x - \sigma_2 t),$$

when $\sigma_1 \neq \sigma_2$.

Topological degree theory for positive operators [29, 30] is another approach to proving the existence of periodic traveling-wave solutions. This theory has been used to prove the existence of periodic traveling-wave solutions for the four systems [12]. For the $abcd$-system, this approach was used in [15] to prove the existence of periodic traveling-wave solutions when $b, d > 0$ and $a, c \leq 0$, but has not been used to prove existence in general. A future research direction would be to use topological degree theory to prove the existence of periodic traveling-wave solutions for the $abcd$-system in a general setting, rather than the approach used in this thesis.

Another future research direction would be to study the conservation laws of these systems. Nguyen *et al.* have found three conservation laws for the four systems [36], and similarly, it was shown that for certain values of a, b, c, and d the $abcd$-system also has three conservation laws [5]. However, whether there exists infinitely many conservation laws for these systems is still an open problem. More generally, one can ask if these systems are integrable. Many other properties of these systems associated with integrability like recursion operators, bi-Hamiltonian structures, Lax pairs operators, and Bäcklund transformations, have also not been studied. These would all be important future research projects, although they are not directly related to the results of this thesis.

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APPENDICES

APPENDIX A

The coefficients $k_{j,q}$ from (2.15) for the four systems (2.1)–(2.4)

For the Schrödinger KdV-KdV system (2.1), the $k_{j,q}$ in (2.15) are:

$$
\begin{cases}
k_{1,3} = 4\lambda^2 a_0\, d_2\, m^2 - \frac{2}{3}\, d_2\, h_2\,, \\
k_{1,2} = \lambda^2 a_0\, d_1\, m^2 - \frac{1}{2}\, d_1\, h_2 - \frac{1}{2}\, d_2\, h_1\,, \\
k_{1,1} = d_2\, a_0\, B^2 - \frac{8}{3}\lambda^2 a_0\, d_2\, m^2 + \frac{2}{3}\, d_2\, bB + \frac{4}{3}\, d_2\, a_0\, \lambda^2 - \frac{1}{3}\, d_0\, h_2 - \frac{1}{3}\, d_1\, h_1 - \frac{1}{3}\, d_2\, h_0 \\
\qquad - \frac{1}{3}\, d_2\, \mu_0 + \frac{1}{3}\, d_2\, \sigma\,, \\
k_{1,0} = \frac{1}{2}\, B^2 a_0\, d_1 - \frac{1}{3}\lambda^2 a_0\, d_1\, m^2 + \frac{1}{3}\, Bb d_1 + \frac{1}{6}\lambda^2 a_0\, d_1 - \frac{1}{6}\, d_0\, h_1 - \frac{1}{6}\, d_1\, h_0 - \frac{1}{6}\, d_1\, \mu_0 + \frac{1}{6}\, d_1\, \sigma\,, \\
k_{2,4} = -18\, B a_0\, d_2\, \lambda^2 m^2 - 6\, b d_2\, \lambda^2 m^2 + B d_2\, h_2 + d_2\, h_2\, \mu_1\,, \\
k_{2,3} = -6\, B a_0\, d_1\, \lambda^2 m^2 - 2\, b d_1\, \lambda^2 m^2 + B d_1\, h_2 + B d_2\, h_1 + d_1\, h_2\, \mu_1 + d_2\, h_1\, \mu_1\,, \\
k_{2,2} = -B^3 a_0\, d_2 + 24\, B a_0\, d_2\, \lambda^2 m^2 - B^2 b d_2 - 12\, B a_0\, d_2\, \lambda^2 + 8\, b d_2\, \lambda^2 m^2 + B d_0\, h_2 + B d_1\, h_1 \\
\qquad + B d_2\, h_0 + B d_2\, \mu_0 - B d_2\, \sigma - 4\, b d_2\, \lambda^2 + d_0\, h_2\, \mu_1 + d_1\, h_1\, \mu_1 + d_2\, h_0\, \mu_1 + d_2\, \omega\,, \\
k_{2,1} = -B^3 a_0\, d_1 + 6\, B a_0\, d_1\, \lambda^2 m^2 - B^2 b d_1 - 3\, B a_0\, d_1\, \lambda^2 + 2\, b d_1\, \lambda^2 m^2 + B d_0\, h_1 \\
\qquad + B d_1\, h_0 + B d_1\, \mu_0 - B d_1\, \sigma - b d_1\, \lambda^2 + d_0\, h_1\, \mu_1 + d_1\, h_0\, \mu_1 + d_1\, \omega\,, \\
k_{2,0} = -B^3 a_0\, d_0 - 6\, B a_0\, d_2\, \lambda^2 m^2 - B^2 b d_0 + 6\, B a_0\, d_2\, \lambda^2 - 2\, b d_2\, \lambda^2 m^2 + B d_0\, h_0 \\
\qquad + B d_0\, \mu_0 - B d_0\, \sigma + 2\, b d_2\, \lambda^2 + d_0\, h_0\, \mu_1 + d_0\, \omega\,, \\
k_{3,3} = \frac{1}{12}\, {d_2}^2 + \frac{1}{12}\, {h_2}^2 - \lambda^2 c h_2\, m^2\,, \\
k_{3,2} = -\frac{1}{4}\lambda^2 c h_1\, m^2 + \frac{1}{8}\, d_1\, d_2 + \frac{1}{8}\, h_1\, h_2\,, \\
k_{3,1} = -\frac{1}{3}\lambda^2 c h_2 + \frac{2}{3}\lambda^2 c h_2\, m^2 + \frac{1}{12}\, d_0\, d_2 + \frac{1}{12}\, h_0\, h_2 - \frac{1}{12}\, h_2\, \sigma + \frac{1}{24}\, {d_1}^2 + \frac{1}{24}\, {h_1}^2 + \frac{1}{12} h_2\,, \\
k_{3,0} = \frac{1}{12}\lambda^2 c h_1\, m^2 + \frac{1}{24}\, d_0\, d_1 + \frac{1}{24}\, h_0\, h_1 - \frac{1}{24}\, h_1\, \sigma - \frac{1}{24}\lambda^2 c h_1 + \frac{1}{24} h_1\,.
\end{cases}
\tag{A.1}
$$

For the Schrödinger BBM-BBM system (2.2), the $k_{j,q}$ in (2.15) are:

$$\begin{cases} k_{1,3} = -\frac{2}{3}\, d_2\, h_2 + 4\,\lambda^2\, a_1\, d_2\, m^2\sigma\,, \\ k_{1,2} = -\frac{1}{2}\, d_1\, h_2 - \frac{1}{2}\, d_2\, h_1 + \lambda^2\, a_1\, d_1\, m^2\sigma\,, \\ k_{1,1} = \frac{2}{3}\, Bb d_2 - \frac{1}{3}\, d_0\, h_2 - \frac{1}{3}\, d_1\, h_1 - \frac{1}{3}\, d_2\, h_0 - \frac{1}{3}\, d_2\, \mu_0 + \frac{1}{3}\, d_2\, \sigma + \frac{4}{3}\,\lambda^2\, a_1\, d_2\, \sigma + B^2 a_1\, d_2\, \sigma \\ \qquad - \frac{2}{3}\, B a_1\, d_2\, \omega - \frac{8}{3}\,\lambda^2\, a_1\, d_2\, m^2\sigma\,, \\ k_{1,0} = \frac{1}{3}\, Bb d_1 - \frac{1}{6}\, d_0\, h_1 - \frac{1}{6}\, d_1\, h_0 - \frac{1}{6}\, d_1\, \mu_0 + \frac{1}{6}\, d_1\, \sigma + \frac{1}{2}\, B^2 a_1\, d_1\, \sigma - \frac{1}{3}\, B a_1\, d_1\, \omega \\ \qquad - \frac{1}{3}\,\lambda^2\, a_1\, d_1\, m^2\sigma + \frac{1}{6}\,\lambda^2\, a_1\, d_1\, \sigma\,, \\ k_{2,4} = -18\, B a_1\, d_2\, \lambda^2\, m^2\sigma + 6\, a_1\, d_2\, \lambda^2\, m^2\omega - 6\, b d_2\, \lambda^2\, m^2 + B d_2\, h_2 + d_2\, h_2\, \mu_1\,, \\ k_{2,3} = -6\, B a_1\, d_1\, \lambda^2\, m^2\sigma + 2\, a_1\, d_1\, \lambda^2\, m^2\omega - 2\, b d_1\, \lambda^2\, m^2 + B d_1\, h_2 + B d_2\, h_1 \\ \qquad + d_1\, h_2\, \mu_1 + d_2\, h_1\, \mu_1\,, \\ k_{2,2} = 6\, B a_1\, d_2\, \lambda^2\, m^2\sigma + 6\, \left(-B\left(-m^2+1\right) + B m^2\right) a_1\, d_2\, \lambda^2\, \sigma - 6\, B\left(-m^2+1\right) a_1\, d_2\, \lambda^2\, \sigma \\ \qquad - 2\, a_1\, d_2\, \lambda^2\, m^2\omega - B^3 a_1\, d_2\, \sigma + B^2 a_1\, d_2\, \omega - 2\,\left(2\, m^2-1\right) a_1\, d_2\, \lambda^2\, \omega + 2\,\left(-m^2+1\right) a_1\, d_2\, \lambda^2\, \omega \\ \qquad + 2\, b d_2\, \lambda^2\, m^2 + B d_2\, \mu_0 + B d_0\, h_2 + d_1\, h_1\, \mu_1 + d_2\, h_0\, \mu_1 - B^2 b d_2 + d_2\, \omega - B d_2\, \sigma \\ \qquad + d_0\, h_2\, \mu_1 + B d_1\, h_1 + B d_2\, h_0 + 2\,\left(2\, m^2-1\right) b d_2\, \lambda^2 - 2\,\left(-m^2+1\right) b d_2\, \lambda^2\,, \\ k_{2,1} = 3\, B a_1\, d_1\, \lambda^2\, m^2\sigma - 3\, B\left(-m^2+1\right) a_1\, d_1\, \lambda^2\, \sigma - a_1\, d_1\, \lambda^2\, m^2\omega + B^2 a_1\, d_1\, \omega \\ \qquad - B^3 a_1\, d_1\, \sigma + \left(-m^2+1\right) a_1\, d_1\, \lambda^2\, \omega + b d_1\, \lambda^2\, m^2 - B^2 b d_1 + B d_0\, h_1 \\ \qquad + B d_1\, h_0 + B d_1\, \mu_0 - B d_1\, \sigma + d_0\, h_1\, \mu_1 + d_1\, h_0\, \mu_1 + d_1\, \omega - \left(-m^2+1\right) b d_1\, \lambda^2\,, \\ k_{2,0} = -B^3 a_1\, d_0\, \sigma + B^2 a_1\, d_0\, \omega + d_0\, \omega - B^2 b d_0 + B d_0\, h_0 + B d_0\, \mu_0 - B d_0\, \sigma \\ \qquad + d_0\, h_0\, \mu_1 + 6\, B\left(-m^2+1\right) a_1\, d_2\, \lambda^2\, \sigma - 2\,\left(-m^2+1\right) a_1\, d_2\, \lambda^2\, \omega + 2\,\left(-m^2+1\right) b d_2\, \lambda^2\,, \\ k_{3,3} = 24\, c h_2\, \lambda^2\, m^2\sigma - 2\, {d_2}^2 - 2\, {h_2}^2\,, \\ k_{3,2} = 6\,\lambda^2\, c h_1\, m^2\sigma - 3\, d_1\, d_2 - 3\, h_1\, h_2\,, \\ k_{3,1} = -16\, c h_2\, \lambda^2\, m^2\sigma + 8\, c h_2\, \lambda^2\, \sigma - 2\, d_0\, d_2 - {d_1}^2 - 2\, h_0\, h_2 - {h_1}^2 + 2\, h_2\, \sigma - 2\, h_2\,, \\ k_{3,0} = -2\,\lambda^2\, c h_1\, m^2\sigma + \lambda^2\, c h_1\, \sigma - d_0\, d_1 - h_0\, h_1 + h_1\, \sigma - h_1\,. \end{cases} \tag{A.2}$$

For the Schrödinger KdV-BBM system (2.3), the $k_{j,q}$ in (2.15) are:

$$\begin{cases} k_{1,3} = 4\lambda^2 a_0\, d_2\, m^2 - \frac{2}{3}\, d_2\, h_2\,, \\ k_{1,2} = \lambda^2 a_0\, d_1\, m^2 - \frac{1}{2}\, d_1\, h_2 - \frac{1}{2}\, d_2\, h_1\,, \\ k_{1,1} = d_2\, a_0\, B^2 - \frac{8}{3}\lambda^2 a_0\, d_2\, m^2 + \frac{2}{3}\, d_2\, bB + \frac{4}{3}\, d_2\, a_0\, \lambda^2 - \frac{1}{3}\, d_0\, h_2 - \frac{1}{3}\, d_1\, h_1 - \frac{1}{3}\, d_2\, h_0 \\ \qquad - \frac{1}{3}\, d_2\, \mu_0 + \frac{1}{3}\, d_2\, \sigma\,, \\ k_{1,0} = \frac{1}{2}\, B^2 a_0\, d_1 - \frac{1}{3}\lambda^2 a_0\, d_1\, m^2 + \frac{1}{3}\, Bb d_1 + \frac{1}{6}\lambda^2 a_0\, d_1 - \frac{1}{6}\, d_0\, h_1 - \frac{1}{6}\, d_1\, h_0 - \frac{1}{6}\, d_1\, \mu_0 + \frac{1}{6}\, d_1\, \sigma\,, \\ k_{2,4} = -18\, B a_0\, d_2\, \lambda^2 m^2 - 6\, b d_2\, \lambda^2 m^2 + B d_2\, h_2 + d_2\, h_2\, \mu_1\,, \\ k_{2,3} = -6\, B a_0\, d_1\, \lambda^2 m^2 - 2\, b d_1\, \lambda^2 m^2 + B d_1\, h_2 + B d_2\, h_1 + d_1\, h_2\, \mu_1 + d_2\, h_1\, \mu_1\,, \\ k_{2,2} = -B^3 a_0\, d_2 + 24\, B a_0\, d_2\, \lambda^2 m^2 - B^2 b d_2 - 12\, B a_0\, d_2\, \lambda^2 + 8\, b d_2\, \lambda^2 m^2 + B d_0\, h_2 + B d_1\, h_1 \\ \qquad + B d_2\, h_0 + B d_2\, \mu_0 - B d_2\, \sigma - 4\, b d_2\, \lambda^2 + d_0\, h_2\, \mu_1 + d_1\, h_1\, \mu_1 + d_2\, h_0\, \mu_1 + d_2\, \omega\,, \\ k_{2,1} = -B^3 a_0\, d_1 + 6\, B a_0\, d_1\, \lambda^2 m^2 - B^2 b d_1 - 3\, B a_0\, d_1\, \lambda^2 + 2\, b d_1\, \lambda^2 m^2 + B d_0\, h_1 \\ \qquad + B d_1\, h_0 + B d_1\, \mu_0 - B d_1\, \sigma - b d_1\, \lambda^2 + d_0\, h_1\, \mu_1 + d_1\, h_0\, \mu_1 + d_1\, \omega\,, \\ k_{2,0} = -B^3 a_0\, d_0 - 6\, B a_0\, d_2\, \lambda^2 m^2 - B^2 b d_0 + 6\, B a_0\, d_2\, \lambda^2 - 2\, b d_2\, \lambda^2 m^2 + B d_0\, h_0 \\ \qquad + B d_0\, \mu_0 - B d_0\, \sigma + 2\, b d_2\, \lambda^2 + d_0\, h_0\, \mu_1 + d_0\, \omega\,, \\ k_{3,3} = \frac{1}{3}\, {d_2}^2 + \frac{1}{3}\, {h_2}^2 - 4\lambda^2 c h_2\, m^2 \sigma\,, \\ k_{3,2} = \frac{1}{2}\, d_1\, d_2 + \frac{1}{2}\, h_1\, h_2 - \lambda^2 c h_1\, m^2 \sigma\,, \\ k_{3,1} = \frac{1}{3}\, d_0\, d_2 + \frac{1}{3}\, h_0\, h_2 - \frac{1}{3}\, h_2\, \sigma - \frac{4}{3}\lambda^2 c h_2\, \sigma + \frac{1}{6}\, {d_1}^2 + \frac{1}{6}\, {h_1}^2 + \frac{1}{3} h_2 + \frac{8}{3}\lambda^2 c h_2\, m^2 \sigma\,, \\ k_{3,0} = \frac{1}{6} h_1 - \frac{1}{6}\lambda^2 c h_1\, \sigma + \frac{1}{6}\, d_0\, d_1 + \frac{1}{6}\, h_0\, h_1 - \frac{1}{6}\, h_1\, \sigma + \frac{1}{3}\lambda^2 c h_1\, m^2 \sigma\,. \end{cases} \tag{A.3}$$

For the Schrödinger BBM-KdV system (2.4), the $k_{j,q}$ in (2.15) are:

$$
\begin{cases}
k_{1,3} = -\frac{2}{3}\, d_2\, h_2 + 4\,\lambda^2\, a_1\, d_2\, m^2\sigma\,, \\
k_{1,2} = -\frac{1}{2}\, d_1\, h_2 - \frac{1}{2}\, d_2\, h_1 + \lambda^2\, a_1\, d_1\, m^2\sigma\,, \\
k_{1,1} = \frac{2}{3}\, Bb d_2 - \frac{1}{3}\, d_0\, h_2 - \frac{1}{3}\, d_1\, h_1 - \frac{1}{3}\, d_2\, h_0 - \frac{1}{3}\, d_2\, \mu_0 + \frac{1}{3}\, d_2\, \sigma \\
\qquad + \frac{4}{3}\,\lambda^2\, a_1\, d_2\,\sigma + B^2 a_1\, d_2\,\sigma - \frac{2}{3}\, B a_1\, d_2\,\omega - \frac{8}{3}\,\lambda^2\, a_1\, d_2\, m^2\sigma\,, \\
k_{1,0} = \frac{1}{3}\, Bb d_1 - \frac{1}{6}\, d_0\, h_1 - \frac{1}{6}\, d_1\, h_0 - \frac{1}{6}\, d_1\,\mu_0 + \frac{1}{6}\, d_1\,\sigma + \frac{1}{2}\, B^2 a_1\, d_1\,\sigma \\
\qquad - \frac{1}{3}\, B a_1\, d_1\,\omega - \frac{1}{3}\,\lambda^2\, a_1\, d_1\, m^2\sigma + \frac{1}{6}\,\lambda^2\, a_1\, d_1\,\sigma\,, \\
k_{2,4} = -18\, B a_1\, d_2\,\lambda^2\, m^2\sigma + 6\, a_1\, d_2\,\lambda^2\, m^2\omega - 6\, b d_2\,\lambda^2\, m^2 + B d_2\, h_2 + d_2\, h_2\,\mu_1\,, \\
k_{2,3} = -6\, B a_1\, d_1\,\lambda^2\, m^2\sigma + 2\, a_1\, d_1\,\lambda^2\, m^2\omega - 2\, b d_1\,\lambda^2\, m^2 + B d_1\, h_2 + B d_2\, h_1 \\
\qquad + d_1\, h_2\,\mu_1 + d_2\, h_1\,\mu_1\,, \\
k_{2,2} = 6\, B a_1\, d_2\,\lambda^2\, m^2\sigma + 6\,\left(-B\left(-m^2+1\right) + Bm^2\right) a_1\, d_2\,\lambda^2\,\sigma - 6\, B\left(-m^2+1\right) a_1\, d_2\,\lambda^2\,\sigma \\
\qquad - 2\, a_1\, d_2\,\lambda^2\, m^2\omega - B^3 a_1\, d_2\,\sigma + B^2 a_1\, d_2\,\omega - 2\,\left(2\, m^2 - 1\right) a_1\, d_2\,\lambda^2\,\omega \\
\qquad + 2\,\left(-m^2+1\right) a_1\, d_2\,\lambda^2\,\omega + 2\, b d_2\,\lambda^2\, m^2 + B d_2\,\mu_0 + B d_0\, h_2 + d_1\, h_1\,\mu_1 + d_2\, h_0\,\mu_1 \\
\qquad - B^2 b d_2 + d_2\,\omega - B d_2\,\sigma + d_0\, h_2\,\mu_1 + B d_1\, h_1 + B d_2\, h_0 + 2\,\left(2\, m^2 - 1\right) b d_2\,\lambda^2 \\
\qquad - 2\,\left(-m^2+1\right) b d_2\,\lambda^2\,, \\
k_{2,1} = 3\, B a_1\, d_1\,\lambda^2\, m^2\sigma - 3\, B\left(-m^2+1\right) a_1\, d_1\,\lambda^2\,\sigma - a_1\, d_1\,\lambda^2\, m^2\omega + B^2 a_1\, d_1\,\omega \\
\qquad - B^3 a_1\, d_1\,\sigma + \left(-m^2+1\right) a_1\, d_1\,\lambda^2\,\omega + b d_1\,\lambda^2\, m^2 - B^2 b d_1 + B d_0\, h_1 \\
\qquad + B d_1\, h_0 + B d_1\,\mu_0 - B d_1\,\sigma + d_0\, h_1\,\mu_1 + d_1\, h_0\,\mu_1 + d_1\,\omega - \left(-m^2+1\right) b d_1\,\lambda^2\,, \\
k_{2,0} = -B^3 a_1\, d_0\,\sigma + B^2 a_1\, d_0\,\omega + d_0\,\omega - B^2 b d_0 + B d_0\, h_0 + B d_0\,\mu_0 - B d_0\,\sigma \\
\qquad + d_0\, h_0\,\mu_1 + 6\, B\left(-m^2+1\right) a_1\, d_2\,\lambda^2\,\sigma - 2\,\left(-m^2+1\right) a_1\, d_2\,\lambda^2\,\omega + 2\,\left(-m^2+1\right) b d_2\,\lambda^2\,, \\
k_{3,3} = \frac{1}{12}\, {d_2}^2 + \frac{1}{12}\, {h_2}^2 - \lambda^2\, c h_2\, m^2\,, \\
k_{3,2} = -\frac{1}{4}\,\lambda^2\, c h_1\, m^2 + \frac{1}{8}\, d_1\, d_2 + \frac{1}{8}\, h_1\, h_2\,, \\
k_{3,1} = -\frac{1}{3}\,\lambda^2\, c h_2 + \frac{2}{3}\,\lambda^2\, c h_2\, m^2 + \frac{1}{12}\, d_0\, d_2 + \frac{1}{12}\, h_0\, h_2 - \frac{1}{12}\, h_2\,\sigma + \frac{1}{24}\, {d_1}^2 \\
\qquad + \frac{1}{24}\, {h_1}^2 + \frac{1}{12} h_2\,, \\
k_{3,0} = \frac{1}{12}\,\lambda^2\, c h_1\, m^2 + \frac{1}{24}\, d_0\, d_1 + \frac{1}{24}\, h_0\, h_1 - \frac{1}{24}\, h_1\,\sigma - \frac{1}{24}\,\lambda^2\, c h_1 + \frac{1}{24} h_1\,.
\end{cases}
\tag{A.4}
$$

APPENDIX B

Permission to Use Letter from Bruce Brewer

March 30, 2024

Jacob Daniels
3900 Old Main Hill, Logan, UT 84322, USA

Dear Bruce Brewer,

I am in the process of preparing my thesis in the mathematics department at Utah State University. I hope to complete my degree program in 2024.

I am requesting your permission to include the attached material as shown. I will include acknowledgments and/or appropriate citations to your work as shown and copyright and reprint rights information in a special appendix. The bibliographic citation will appear at the end of the manuscript as shown. Please advise me of any changes you require.

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Jacob Daniels
(208) 809-0241
jake.daniels@usu.edu

I hereby give permission to Jacob Daniels to reprint the following material in his thesis.

Bruce Brewer, Jake Daniels, Nghiem V. Nguyen. "Exact Jacobi elliptic solutions of some models for the interaction of long and short waves". In: *AIMS Mathematics* 9.2 (2024), pp. 2854-2873.

Signed: Bruce Brewer
Date: 3/30/24