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Tournament Directed Graphs

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TOURNAMENT DIRECTED GRAPHS

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Abstract

Paired comparison is the process of comparing objects two at a time. A *tournament* in Graph Theory is a representation of such paired comparison data. Formally, an *n-tournament* is an oriented complete graph on n vertices; that is, it is the representation of a paired comparison, where the winner of the comparison between objects x and y (x and y are called *vertices*) is depicted with an arrow or *arc* from the winner to the other.

In this thesis, we shall prove several results on tournaments. In Chapter 2, we will prove that the maximum number of vertices that can beat exactly m other vertices in an n-tournament is $\min\{2m+1, 2n-2m-1\}$. The remainder of this thesis will deal with tournaments whose arcs have been colored. In Chapter 3 we will define what it means for a k-coloring of a tournament to be k-primitive. We will prove that the maximum k such that some strong n-tournament can be k-colored to be k-primitive lies in the interval $\left[\binom{n-1}{2}, \binom{n}{2} - \lceil \frac{n}{4} \rceil\right)$. In Chapter 4, we shall prove special cases of the following 1982 conjecture of Sands, Sauer, and Woodrow from [14]: Let T be a 3-arc-colored tournament containing no 3-cycle C such that each arc in C is a different color. Then T contains a vertex v such that for any other vertex x, x has a monochromatic path to v.

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Chapter 1

Introduction

Paired comparison is the process of comparing objects two at a time. For example, a voter could compare all candidates in a race pairwise ("I would vote for A over B, for C over B, and for A over C."). A *tournament* in Graph Theory is a representation of such paired comparison data. Formally, an *n-tournament* is an oriented complete graph on n vertices; that is, it is the representation of a paired comparison, where the winner of the comparison between objects x and y (x and y are called *vertices*) is depicted with an arrow or *arc* from the winner to the other. We write $x \to y$ to denote that x beat y. The *score* of a vertex x in an n-tournament is the number of wins it enjoys in the paired comparison; equivalently, it is the number of outward going arcs from x in the depiction of the tournament. For example, Figure 1.1 is the drawing of an 8-tournament, in which vertex i has score $8 - i$. (Typically drawings are not used for serious analysis of data sets consisting of a large number of objects.)

We denote the set of all vertices in a tournament T by $V(T)$ and the set of all arcs by $A(T)$. A *subtournament* of T is a tournament T' such that $V(T') \subseteq V(T)$ and $A(T') \subseteq A(T)$. A subtournament T' is *induced* if for all $x, y \in V(T)$ such that $x \to y$ in T and $x, y \in V(T')$, we have $x \to y$ in T'. We let $T - x$ denote the induced subtournament of T resulting from deleting x from T (and consequently all arcs with x as an endpoint).

In this thesis, we shall prove several results on tournaments. In Chapter 2, we will prove that the

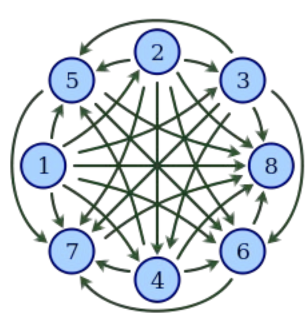

Figure 1.1: An 8-tournament

maximum number of vertices that an n-tournament can have of score m is $\min\{2m+1, 2n-2m-1\}$. We will also prove formulas for the maximum number of possible vertices of score m or greater (or less) in an n-tournament. The remainder of this thesis will deal with tournaments whose arcs have been colored. In Chapter 3 we will define what it means for a k-coloring of a tournament to be k-primitive. We will prove that the maximum k such that some strong n-tournament can be k-colored to be k-primitive lies in the interval $\left[\binom{n-1}{2}, \binom{n}{2} - \lceil \frac{n}{4} \rceil\right)$. In Chapter 4, we shall prove special cases of the following 1982 conjecture of Sands, Sauer, and Woodrow from [14].

Conjecture 1.0.1. *Let T be a 3-arc-colored tournament containing no 3-cycle C such that each arc in C is a different color. Then T contains a monochromatic sink. That is, T contains a vertex v such that for any other vertex x, x has a monochromatic path to v.*

Furthermore, we shall prove that two seemingly distinct methods used in the literature (for example see [5] and [11]) to prove that a k-colored tournament has a monochromatic sink are identical.

Chapter 2

Maximum number of vertices of score m in an n-tournament

The *score* of a vertex x in an n-tournament T, denoted $s_T(x)$, is the number of wins x enjoys in T. When it is clear, we write $s(x)$, instead of $s_T(x)$. The *score sequence* of an n-tournament is the list of the n scores of its vertices in nondecreasing order. In a tournament, at most one vertex can have score zero and at most one vertex can have score $n-1$. In this chapter we will prove much more by answering the following questions.

1. What is the maximum number of vertices possible of score m in an n-tournament?
2. What is the maximum number of vertices that can have score m or greater in an n-tournament? m or less?

Let m and n be integers satisfying $0 \le m \le n-1$, and define $f(m,n)$ to be the maximum k such that there is an n-tournament with k vertices of score m. In Section 2.1, we prove that

$$f(m,n) = \begin{cases} 2m+1 & \text{if } m \le \frac{n-1}{2} \\ 2n-2m-1 & \text{if } m > \frac{n-1}{2}. \end{cases}$$

An n-tournament is said to *realize* $f(m,n)$ if it has $f(m,n)$ vertices of score m. In Section 2.2, we characterize tournaments that realize $f(m,n)$. In Section 2.3 we apply the results of Section 2.1 to form a conjecture on the minimum number of rounds that must be played to determine a winner in a series of round-robin tournaments, if to continue to the next round-robin tournament, a player must beat at least a fixed proportion of its opponents.

2.1 Determining $f(m,n)$, the maximum vertices of score m in an n-tournament

To prove our main result, we make use of Landau's Theorem [9] and require the development of a few lemmas of our own.

Theorem 2.1.1. *[**Landau's Theorem**] A sequence of integers $S = (s_1, s_2, \ldots, s_{n-1}, s_n)$, where $s_1 \le s_2 \le \ldots \le s_{n-1} \le s_n$, is the score sequence of some n-tournament if and only if*

$$\sum_{i=1}^{k} s_i \ge \binom{k}{2} \text{ for } k \in \{1, 2, \ldots, n-1\} \text{ and } \sum_{i=1}^{n} s_i = \binom{n}{2}. \tag{2.1}$$

A *path* in a n-tournament T is a sequence of distinct vertices $v_1 \ldots v_\ell$ in T such that between consecutive vertices v_i and v_{i+1} there is an arc from v_i to v_{i+1}. Tournament T is *strong* if there is some path from x to y and from y to x, for any pair of distinct vertices x and y. A result in [7] similar to Landau's Theorem characterizes the score sequences of strong tournaments.

Theorem 2.1.2. *S as in Theorem 2.1.1 is the score sequence of some strong n-tournament if and only if* (2.1) *holds with the inequality replaced by a strict inequality.*

Lemma 2.1.3. *If j is an integer such that $j \ge 2$, then $\binom{2m+j}{2} = \binom{2m+1}{2} + \sum_{i=1}^{j-1} 2m + i$.*

Proof. Observe that $\binom{2m+2}{2} = \binom{2m+1}{2} + 2m + 1$. Inductively, suppose that $\binom{2m+\ell}{2} = \binom{2m+1}{2} + \sum_{i=1}^{\ell-1} 2m + j$. It follows that
$\binom{2m+\ell+1}{2} = \binom{2m+\ell}{2} + 2m + \ell = \binom{2m+1}{2} + \left(\sum_{i=1}^{\ell-1} 2m + i\right) + 2m + \ell = \binom{2m+1}{2} + \sum_{i=1}^{\ell} 2m + i.$ □

Lemma 2.1.4. *Let m and n be integers such that $0 \le m \le n-1$. Then $f(m,n) = f(n-m-1,n)$.*

Proof. Let T be a tournament that realizes $f(m,n)$. Let T' be the dual graph of T generated by reversing all arcs in T. Then T' has $f(m,n)$ vertices of score $n-m-1$, so $f(n-m-1,n) \ge f(m,n)$. Suppose $f(n-m-1,n) > f(m,n)$, and let T^* be a tournament that realizes $f(n-m-1,n)$. Then the dual graph of T^* has $f(m-n-1,n)$ vertices of score m. But this is a contradiction as $f(m,n)$ is the maximum number of vertices of score m possible in an n-tournament. Therefore, $f(m,n) = f(n-m-1,n)$. □

We now have developed the tools necessary to prove our main result, but first we give an example application. In the 2010 Vancouver Olympics, placement in curling semi-finals was based on a 10-team tournament, in which each of 10 teams played each other team once (a paired comparison), and the top four records moved on to the semi-finals. A team was considered to be a contender for the gold medal if they won 7 of their 9 match ups. In this system, how many different teams can all end the round-robin tournament with a record of 7 wins? Our result will easily give 5 as the answer to this and any problem of this nature; that is, where 10 is replaced with n and 7 with m.

Theorem 2.1.5. *Let m and n be integers such that $0 \le m \le n-1$. Then*

$$f(m,n) = \begin{cases} 2m+1 & \text{if } m \le \frac{n-1}{2} \\ 2n-2m-1 & \text{if } m > \frac{n-1}{2}. \end{cases}$$

Proof. Suppose that $m \leq \frac{n-1}{2}$. Let π be the score sequence of a n-tournament T with exactly $f(m,n)$ vertices of score m. Then π is of the form

$$\pi = \underbrace{(\ldots, \underbrace{m, m, \ldots, m}_{f(m,n)}}_{k}, \ldots).$$

Let s_i be the i^{th} integer in π. By Theorem 2.1.1, it must be that $\sum_{i=1}^{k} s_i \geq \binom{k}{2} = \frac{k(k-1)}{2}$. Furthermore, since score sequences are nondecreasing it follows that $km \geq \sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2}$, which implies $k \leq 2m+1$.

Since $k \geq f(m,n)$, it follows that

$$f(m,n) \leq 2m+1. \tag{2.2}$$

Now consider the sequence $\pi^* = (\underbrace{m, m, \ldots, m}_{2m+1}, 2m+1, 2m+2, \ldots, n-1)$. Let t be an integer such that $t \geq 1$. Then by Lemma 2.1.3, $\sum_{i=1}^{2m+t} s_i^* = \binom{2m+t}{2}$, where s_i^* is the i^{th} integer in π^*.

By way of contradiction, suppose that π^* is not a score sequence. Then by Theorem 2.1.1, it must be that $\sum_{i=1}^{\ell} s_i^* = \ell m < \binom{\ell}{2}$ for some $\ell < 2m+1$. But this implies that $\ell > 2m+1$, a contradiction. Therefore, π^* is a score sequence. It follows that

$$f(m,n) \geq 2m+1. \tag{2.3}$$

By Equations (2.2) and (2.3), it follows that

$$f(m,n) = 2m+1.$$

Now suppose that that $m \geq \frac{n-1}{2}$. Then $n-m-1 \leq \frac{n-1}{2}$. It follows from Lemma 2.1.4 that $f(m,n) = f(n-m-1,n) = 2(n-m-1)+1 = 2n-2m-1$.

□

Theorem 2.1.5 may be of little use in certain situations. Suppose players compete in a round robin tournament. It would not make sense to award all players who have exactly ℓ wins, but rather all who have score ℓ or more. The maximum players that could have score ℓ or greater (or less) thus may be of interest. To this end, we make the following definitions.

Definition 2.1.6. *Let m and n be integers such that $0 \leq m \leq n-1$. Define $L(m,n)$ to be the maximum k such that there exists an n-tournament with k vertices of score m or less. Define $G(m,n)$ to be the maximum k such that there exists an n-tournament with k vertices of score m or greater.*

The following theorems give formulas for $L(m,n)$ and $G(m,n)$. It should be no surprise that $L(m,n)$ and $G(m,n)$ are intimately related to $f(m,n)$. Note that a *regular* tournament is one where every vertex has the same score.

Theorem 2.1.7. *Let m and n be integers such that $0 \leq m \leq n-1$. Then*

$$L(m,n) = \begin{cases} 2m+1 & \text{if } m \leq \frac{n-1}{2} \\ n & \text{if } m > \frac{n-1}{2}. \end{cases}$$

Proof. Let $m \leq \frac{n-1}{2}$. By Theorem 2.1.5, $L(m,n) \geq 2m+1$. Consider the score sequence $(s_1, \ldots, s_n)$ of an n-tournament with exactly $L(m,n)$ vertices of score m or less. Using Theorem 2.1.1 and the fact that score sequences are nondecreasing, it must be that $mL(m,n) \geq \sum_{i=1}^{L(m,n)} s_i \geq \binom{L(m,n)}{2}$, which implies $L(m,n) \leq 2m+1$. Therefore, $L(m,n) = 2m+1$.

Now consider the case where $m > \frac{n-1}{2}$. If n is odd, a regular n-tournament, which has n vertices of score $\frac{n-1}{2}$, proves the claim. If n is even, construct a regular tournament T on $n-1$ vertices. Take another vertex, called v_n, and connect it to T such that v_n beats m vertices in T and loses to all others. In this tournament, $n-m-1$ vertices have score $n/2$, m vertices have score $(n-2)/2$ and one vertex has score m, so that n vertices have score m or less. Therefore, $L(m,n) = n$. □

Theorem 2.1.8. *Let m and n be integers such that $0 \leq m \leq n-1$. Then*

$$G(m,n) = \begin{cases} n & \text{if } m \leq \frac{n-1}{2} \\ 2n-2m-1 & \text{if } m > \frac{n-1}{2}. \end{cases}$$

Proof. Let $m > \frac{n-1}{2}$. By Theorem 2.1.5, $G(m,n) \geq 2n-2m-1$. Now take an n-tournament T with $G(m,n)$ vertices of score m or greater. Reverse all arcs in T and observe that the resulting tournament has $G(m,n)$ vertices of score $n-m-1$ or less. So $G(m,n) \leq L(n-m-1,n) = 2n-2m-1$ by Theorem 2.1.7. Therefore, $G(m,n) = 2n-2m-1$.

Now suppose $m \leq (n-1)/2$. If n is odd, a regular n-tournament proves the claim. If n is even, construct a regular tournament on $n+1$ vertices called T. Delete one arbitrary vertex (and any arcs adjacent to it) from T and call the resulting tournament T'. Observe that T' has n vertices, all of which have score $\frac{n-1}{2}$ or $\frac{n+1}{2}$. Since $\frac{n-1}{2}, \frac{n+1}{2} \geq m$, we are done. □

2.2 Constructing $f(m,n)$ realizing tournaments

Let n and m be integers satisfying $0 \leq m \leq n-1$. An tournament T *realizes* $f(m,n)$ if T is an n-tournament containing exactly $f(m,n)$ vertices of score m. In this section we discuss the structure of $f(m,n)$-realizing tournaments.

The following corollary is a direct result of Theorems 2.1.1 and 2.1.5.

Corollary 2.2.1. *Tournaments that realize $f(m,n)$ are not strong, with $m = \frac{n-1}{2}$ being the only exception.*

We can say much more. The following theorems characterize $f(m,n)$-realizing tournaments.

Theorem 2.2.2. *Let n and m be integers such that $0 \leq m \leq n-1$. Then there exists an $f(m,n)$-realizing tournament T such that the subtournament T' induced on the $f(m,n)$ vertices of score m in T is regular, and if $m \leq \frac{n-1}{2}$, then every vertex in T' is beat by every vertex in $T - T'$ and if $m > \frac{n-1}{2}$, then every vertex in T' beats every vertex in $T - T'$.*

Proof. Suppose $m \leq \frac{n-1}{2}$. Construct a regular tournament T' on $2m+1$ vertices. Each vertex in T' has score m. Take any tournament T'' on $n-2m-1$ vertices. Now form an $f(m,n)$-realizing tournament T by letting all vertices in T'' beat all vertices in T'.

Now suppose $m > \frac{n-1}{2}$. Construct a regular tournament T' on $2n-2m-1$ vertices. Each vertex in T' has score $n-m-1$. Take any tournament T'' on $2m-n+1$ vertices. Create an $f(m,n)$-realizing tournament T by letting all vertices in T' beat all vertices in T''. □

In fact, it turns out that all $f(m,n)$-realizing tournaments have the structure described in Theorem 2.2.2.

Theorem 2.2.3. *Let n and m be integers such that $0 \leq m \leq n-1$. Then the following are true of an $f(m,n)$-realizing tournament T:*

1. *The subtournament induced on the $f(m,n)$ vertices of score m in T is regular.*
2. *If $m > \frac{n-1}{2}$ then every vertex of score m in T beats every vertex in T who does not have score m. If $m \leq \frac{n-1}{2}$, then every vertex of score m in T is beaten by every vertex in T who does not have score m.*

Proof. Suppose $m \leq \frac{n-1}{2}$. Suppose an n-tournament T realizes $f(m,n)$ and let T' be the $2m+1$ vertex subtournament of T containing the $f(m,n)$ vertices of T of score m. For a contradiction, suppose that T' is not regular. Then for some vertex $v \in V(T')$, we have $s_{T'}(v) > \binom{2m+1}{2}/(2m+1) = m$. But all vertices in T' have score m in T, so T' must be regular after all. Since $s_{T'}(v) = m$, it must be that v is beaten by every vertex in T that is not in T'.

Now suppose $m > \frac{n-1}{2}$. Suppose an n-tournament T realizes $f(m,n)$ and let T' be the $2n-2m-1$ vertex subtournament of T containing the $f(m,n)$ vertices of T of score m. For a contradiction, suppose that T' is not regular. Then for some vertex $v \in V(T')$, we have $s_{T'}(v) < \binom{2n-2m-1}{2}/(2n-2m-1) = n-m-1$. Then

$$s_T(v) \leq s_{T'}(v) + |V(T)| - |V(T')| < n-m-1+n-(2n-2m-1) = m.$$

But all vertices in T' have score m in T, so T' must be regular after all. Since $s_{T'}(v) = n-m-1$ for $v \in V(T')$, it must be that v beats every vertex in T that is not in T'.

□

2.3 Maximum rounds needed to determine a winner in a series of tournaments

Suppose n players compete in a round-robin tournament and let α be a constant in the interval $(0,1)$. All players who beat fewer than $\lfloor \alpha n \rfloor$ players will be eliminated. The remaining k players will compete in another round-robin tournament, in which all players who beat fewer than $\lfloor \alpha k \rfloor$ players will be eliminated. This process of round-robin tournaments will continue, perhaps infinitely, until only one player remains (we call this player, if it exists, the *winner*). Let $r(\alpha, n)$ denote the minimum number of rounds that must occur to find a winner. In each round of play, we will assume that a maximum number of players advance to the next round. This will allow us to utilize the machinery developed in Section 2.1.

Throughout, we restrict our discussion to values of α greater or equal to 1/2. For if $\alpha < 1/2$, then $\lfloor \alpha n \rfloor \leq \frac{n-1}{2}$ so Theorem 2.1.8 implies that it is possible that in each round no players are eliminated, and thus no winner is ever decided.

We begin by using Theorem 2.1.8 to formulate $r(\alpha, n)$ as a recursive problem. Let n_i denote the number of players remaining to compete in the i^{th} tournament. Then

$$n_i = 2n_{i-1} - 2\lfloor \alpha n_{i-1} \rfloor - 1, \qquad \text{where } n_1 = n. \tag{2.4}$$

Now if a closed formula for n_i were found, call it $N(i)$, then $r(\alpha, n)$ would be one less than the solution to $N(i) = 1$. However, the floor function in Equation 2.4 makes it difficult to solve. When the floor is ignored, Mathematica gives

$$N(i) = \frac{2 - 2\alpha - (2 - 2\alpha)^{i-1}(1 - n + 2n\alpha)}{2\alpha - 1}.$$

This approximation for $N(i)$, yields the following approximation for $r(\alpha, n)$:

$$r(\alpha, n) \approx \frac{\log \frac{4(1-2\alpha+\alpha^2)}{1-n+2n\alpha}}{\log(2 - 2\alpha)} - 1.$$

We now state a conjecture for an exact value of $r(\alpha, n)$ for particular values of α. In particular, we conjecture about proportions of the form $\frac{2^k-1}{2^k}$, where $k \in \{2, 3, \ldots\}$.

Conjecture 2.3.1. $r\left(\frac{2^k-1}{2^k}, n\right) = \left\lceil \frac{1}{k-1}(\log_2 n - 1) \right\rceil$, *where* $k \in \{2, 3, \ldots\}$.

Conjecture 2.3.1 has been computer verified for k and n such that $2 \leq k \leq 53$ and $n \leq 10,000,000$. The code used for verification can be found in Appendix B.

Chapter 3

K-primitive digraphs

3.1 Introduction to primitivity and k-primitivity

In this chapter, we will deal with directed graphs, which are similar to tournaments except that it is not necessary that between every pair of vertices there is an arc. Let D be a directed graph (digraph) on n vertices and let $V(D)$ denote the vertex set and $A(D)$ the arc set of D. A *walk* is a sequence of not necessarily distinct vertices $v_1 \dots v_\ell$ in D such that between consecutive vertices v_i and v_{i+1} there is an arc from v_i to v_{i+1}. The *length* of a walk $v_1 \dots v_\ell$ is $\ell - 1$. The digraph D is said to be *primitive* if for some m, between any ordered pair of vertices of D there is a walk of length m from the first vertex to the other.

For a simple example, consider the digraph D in Figure 3.1. A walk from v_1 to v_2 is of length $1 + 3k$ for some nonnegative integer k, and a walk from v_1 to v_3 is of length $2 + 3\ell$ for some nonnegative integer ℓ. Because we cannot choose nonnegative integers k and ℓ such that $1 + 3k = 2 + 3\ell$, it follows that D is not primitive.

It is well known that a digraph is primitive if and only if it is strongly connected (given any two vertices there is a walk between them in both directions) and the lengths of the cycles in D are relatively prime (see [4, Lemma 3.4.1]).

For every statement about the primitivity of a digraph, a parallel statement can be made about its adjacency matrix. Let $\mathbb{B} = \{0, 1\}$ and $\mathcal{M}_n(\mathbb{B})$ be the set of all $n \times n$ $(0,1)$-matrices. The matrix $A \in \mathcal{M}_n(\mathbb{B})$ is *primitive* if A^m has all nonzero entries for some positive m. It is easily shown that D is a primitive digraph if and only its adjacency matrix is primitive. Because for each $A \in \mathcal{M}_n(\mathbb{B})$, there is a unique digraph D whose adjacency matrix is A, studying the primitivity of digraphs is equivalent to studying the primitivity matrices in $\mathcal{M}_n(\mathbb{B})$.

A generalization of primitivity is k-primitivity. Suppose that the arcs of D are colored by

$$
\begin{tikzcd}
 & v_1 \arrow[dr] & \\
v_3 \arrow[ur] & & v_2 \arrow[ll]
\end{tikzcd}
$$

Figure 3.1: A digraph that is not primitive

colors $c_1, c_2, \dots, c_k$ and let $(m_1, m_2, \dots, m_k)$ be a k-tuple of nonnegative integers. Then there is an $(m_1, m_2, \dots, m_k)$-*walk* from vertex v_i to vertex v_j if there is a walk of length $m_1 + m_2 + \cdots + m_k$ from v_i to v_j that contains exactly m_ℓ arcs of color c_ℓ for all $\ell \in \{1, \dots, k\}$. In the walk, the location of the m_ℓ arcs colored c_ℓ is not restricted for any ℓ.

A k-coloring of a digraph D is *k-primitive* if there exists a k-tuple of positive integers $(m_1, m_2, \dots, m_k)$ such that between any two vertices $v_i, v_j \in V(D)$ there is an $(m_1, m_2, \dots, m_k)$-walk from v_i to v_j. A digraph D is *k-primitive* if there exists a k-coloring of D that is k-primitive.

The concept of a k-primitive matrix, which relates to a k-primitive digraph in the same way that a primitive matrix relates to a primitive digraph, will be defined in detail in the next section.

In this chapter we investigate digraphs that are known to be primitive and give conditions to assure that they are k-primitive, where k is a function of the number of vertices in the digraph. In Section 3.2 we give formal definitions and the preexisting results pertaining to k-primitivity. In Section 3.3, we give results concerning the k-primitivity of general digraphs, and in Section 3.4, we have results about the k-primitivity of tournaments. We end with Section 3.5, a section about game colorings of primitive tournaments.

Throughout the article, we assume that $n \geq 2$ and that all digraphs are simple (no multiple edges or loops).

3.2 Preliminaries

We begin this section with some basic definitions. For a matrix $A \in \mathcal{M}_n(\mathbb{B})$, note that we write $A > 0$ ($A \geq 0$) to denote that A has only strictly positive (nonnegative) entries.

Definition 3.2.1. *Let $A \in \mathcal{M}_n(\mathbb{B})$. Then A is said to be* primitive *if there exists m such that $A^m > 0$. A digraph D is* primitive *if there exists an m such that for every $v_i, v_j \in V(D)$ there is a walk from v_i to v_j of length m.*

It is easily shown that a digraph D is primitive if and only if its adjacency matrix is primitive.

A generalization of the concept of primitivity is that of k-primitivity, where k-colorings of digraphs are considered.

Let $A_1, A_2, \dots, A_k \in \mathcal{M}_n(\mathbb{B})$ be such that their sum, A, is also in $\mathcal{M}_n(\mathbb{B})$. That is, each nonzero entry of A corresponds to exactly one nonzero entry in some A_i. Then we say that A is the *disjoint* sum of $A_1, A_2, \dots, A_k$. This decomposition of A corresponds in a natural way to a *k-coloring* of the digraph D.

Definition 3.2.2. *Let $A_1, A_2, \dots, A_k \in \mathcal{M}_n(\mathbb{B})$ and let $m_1, \dots, m_k$ be positive integers. The sum of all possible products containing m_i A_i's for all $i \in \{1, \dots, k\}$ is called the $(m_1, \dots, m_k)$-*Hurwitz product *of $A_1, \dots, A_k$ and is denoted $(A_1, \dots, A_k)^{(m_1, \dots, m_k)}$.*

For example,

$$(A, B)^{(2,2)} = AABB + BBAA + BABA + ABAB + ABBA + BAAB.$$

Definition 3.2.3. *Let $A \in \mathcal{M}_n(\mathbb{B})$. The matrix A is said to be k-*primitive *if there exists a k-tuple $(A_1, \dots, A_k) \in (\mathcal{M}_n(\mathbb{B}))^k$ whose components sum to A, and positive integers $m_1, \dots, m_k$ such that $(A_1, \dots, A_k)^{(m_1, \dots, m_k)} > 0$.*

Corresponding to the definition of a k-primitive matrix we define k-primitivity of digraphs. This was done in the introduction and is repeated here for emphasis.

Definition 3.2.4. *Let D be a digraph whose arcs are colored with colors $c_1, c_2, \ldots, c_k$. This coloring of D is* k-primitive *if there exists a k-tuple of positive integers $(m_1, m_2, \ldots, m_k)$ such that between any two vertices $v_i, v_j \in V(D)$ there is an $(m_1, m_2, \ldots, m_k)$-walk from v_i to v_j. A digraph D is* k-primitive *if there exists a k-coloring of D that is k-primitive.*

As in the case for primitive matrices and primitive digraphs, it is straight forward to show that the matrix $A \in \mathcal{M}_n(\mathbb{B})$ is k-primitive if and only if the corresponding digraph D is k-primitive. This fact will be used throughout without reference.

A fundamental theorem about primitive digraphs is:

Theorem 3.2.5. [4, Lemma 3.4.1] *A digraph D is primitive if and only if it is strongly connected and the greatest common divisor of its cycle lengths is* 1.

To state a similar characterization of k-primitive digraphs requires the following definition.

Definition 3.2.6. *Let D be a digraph whose arcs are colored with colors $c_1, c_2, \ldots, c_k$ and let $\{\mathbb{C}_1, \ldots, \mathbb{C}_\ell\}$ be the set of cycles of D. The* color-cycle matrix *of D is the $k \times \ell$ nonnegative integer matrix $M_D = (m_{r,s}^{(D)})$, where $m_{r,s}^{(D)}$ is the number of arcs of color c_r in the cycle $\mathbb{C}_s$.*

The following theorem, due to D. Olesky, B. Shader, and P. van den Driessche, is critical to our study of k-primitivity.

Theorem 3.2.7. [12] *A k-coloring of D is k-primitive if and only if the $k \times k$ minors of the color-cycle matrix of D are relatively prime.*

Theorem 2.1 of [1] states:

Theorem 3.2.8. [1] *If D is a primitive directed graph, then there is a 2-coloring of D that is 2-primitive.*

3.3 K-primitivity of digraphs

As Theorem 3.2.8 states, every primitive digraph is 2-primitive. That result can be improved. Given a digraph D, let $\delta_D^+(v)$ denote the outdegree or score in D of the vertex v, and let $\delta_D^-(v)$ denote the indegree in D of the vertex v (assuming of course that $v \in V(D)$). Further let $\Delta_D(v) = \delta_D^+(v) + \delta_D^-(v)$, the total degree in D of the vertex v.

Theorem 3.3.1. *Let D be a primitive digraph and let $H = D - v$, where $v \in V(D)$. If H is k-primitive, then D is $(k + \Delta_D(v) - 1)$-primitive.*

Proof. Consider a coloring of H with $c_1, \ldots, c_k$ such that the resulting coloring is k-primitive. We will extend this coloring to a $(k + \Delta_D(v) - 1)$ coloring of D. Since D is primitive, $\delta_D^+(v) \geq 1$ and $\delta_D^-(v) \geq 1$. Label the vertices of H so that $v \to v_i$ for $i \in \{1, \ldots, \delta_D^+(v)\}$ and $v_j \to v$ for

$j \in \{\delta_D^+(v)+1, \ldots, \delta_D^+(v)+\delta_D^-(v) = \Delta_D(v)\}$ are the arcs in D that are not in H. For convenience, let $\Delta = \Delta_D(v)$. Color arc $v \to v_i$ color c_{k+i} for $i \in \{1, \ldots, \delta_D^+(v)\}$ and arc $v_j \to v$ color c_{k+j} for $j \in \{\delta_D^+(v)+1, \ldots, \Delta-1\}$ and color arc $v_\Delta \to v$ color c_1.

Since H is primitive, there is a path, P_i in H from vertex v_i to vertex v_Δ for each $i \in \{1, \ldots, \delta_D^+(v)\}$ in H and a path P_j in H from vertex v_1 to vertex v_j for each $j \in \{\delta_D^+(v)+1, \ldots, \Delta-1\}$. For $x, y \in V(D)$, let $x \to^P y$ denote the path from x to y following path P.

Assuming that M_H is $k \times \ell$, let $\mathbb{C}_{\ell+i}$ be the cycle $v \to v_i \to^{P_i} v_\Delta \to v$ for $i \in \{1, \ldots, \delta_D^+(v)\}$ and $\mathbb{C}_{\ell+j}$ be the cycle $v \to v_1 \to^{P_j} v_j \to v$, for $j \in \{\delta_D^+(v)+1, \ldots, \Delta-1\}$. Then

$$M_D = \begin{bmatrix} M_H & * & * \\ O & B & * \end{bmatrix},$$

where

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & & & & \\ 0 & & & 1 & 0 & \cdots & 0 & 0 \\ 0 & & & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \ddots & & \\ 0 & 0 & & & \cdots & & 1 & 0 \\ 0 & 0 & & & \cdots & & 0 & 1 \end{bmatrix}_{(\Delta-1)\times(\Delta-1)}.$$

Clearly the $(k+\Delta-1)\times(k+\Delta-1)$ minors of M_D are relatively prime since the $k \times k$ minors of M_H are. Therefore, D is $(k+\Delta-1)$-primitive. □

Our next result shows that there is no gap in the set of k such that a digraph is k-primitive. In Section 3.4, we will address the question "What is the maximum k such that some primitive n-tournament is k-primitive?"

Theorem 3.3.2. *Let D be a digraph. If D is k-primitive, $k \geq 2$, then D is $(k-1)$-primitive.*

Proof. Consider a k-primitive coloring of a digraph D. Let M be the $k \times \ell$ color-cycle matrix of the k-primitive digraph D, where ℓ is the number of cycles in D. Then, since D is k-primitive, the set of $k \times k$ minors of M are relatively prime. Thus, there are integers, α_q such that $\sum_{q \in Q_{\ell,k}} \alpha_q \det M[1, \ldots, k \mid q] = 1$, where $Q_{\ell,k}$ denotes the set of strictly increasing sequences of length k from the set $\{1, 2, \ldots, \ell\}$. Let $B = E_{1,k}(1)M$, where left multiplication by $E_{1,k}(1)$ adds the k^{th} row of M to the first row. Then, for $q \in Q_{\ell,k}$,

$$\det B[1, \ldots, k \mid q] = \det M[1, \ldots, k \mid q].$$

But

$$\det B[1, \ldots, k \mid q] = \sum_{i=1}^{k} (-1)^{i+k} m_{k,q_i} \det B[1, \ldots, k-1 \mid \{q_1, \ldots, q_k\} \setminus q_i].$$

Thus,

$$\sum_{q \in Q_{\ell,k}} \alpha_q \left(\sum_{i=1}^{k} (-1)^{i+k} m_{k,q_i} \det B[1,\dots,k-1 \mid \{q_1,\dots,q_k\} \setminus q_i] \right) =$$

$$\sum_{q \in Q_{\ell,k}} \sum_{i=1}^{k} ((-1)^{i+k} \alpha_q m_{k,q_i}) \det B[1,\dots,k-1 \mid \{q_1,\dots,q_k\} \setminus q_i] = 1.$$

That is, the $(k-1) \times \ell$ matrix $B[1,\dots,k-1 \mid 1,\dots \ell]$ has relatively prime $(k-1) \times (k-1)$-minors. Further, $B[1,\dots,k-1 \mid 1,\dots,\ell]$ is the color-cycle matrix for the $(k-1)$-coloring of D where each edge with color k is recolored color 1. Thus, D is $(k-1)$-primitive. $\square$

3.4 K-primitivity of tournaments

In this section we examine tournaments. Recall that an n-tournament T is an orientation of the complete loopless graph on n vertices. We say T is a *strong* tournament if it is strongly connected. Further all strong tournaments are pancyclic (see [8]). Thus, every strong tournament of order at least 4 is primitive by Theorem 3.2.5. However, Theorem 3.3.1 can be used to prove much more.

Theorem 3.4.1. *Let T be a strong n-tournament, where $n \geq 4$. Then T is $\binom{n-1}{2}$-primitive.*

Proof. Clearly the result is true for $n = 4$. Inductively, suppose the result is true for strong $(n-1)$-tournaments. Let T be a strong n-tournament, where $n > 4$. Then T contains a strong induced $(n-1)$-subtournament. So by Theorem 3.3.1, T is $\binom{n-2}{2} + (n-1) - 1 = \binom{n-1}{2}$- primitive. Thus, by induction, the result is true for all $n \geq 4$.

$\square$

Let $k_{\max}(n)$ be the largest k such that some n-tournament is k-primitive. Theorem 3.4.1 says that $k_{\max}(n) \geq \binom{n-1}{2}$. Let U_6 be the upset 6-tournament whose vertices are labeled with $v_1,\dots,v_6$ such that $v_i \to v_j$ is an arc in U_6 for every pair (i,j) with $i < j$, except $v_n \to v_1$. If all 15 arcs of U_6 are colored differently, the 15×15 color-cycle matrix of U_6 has determinant 0. So by Theorem 3.2.7, at least some strong tournaments are not $\binom{n}{2}$-primitive. In fact, more is true.

Definition 3.4.2. *Let D be a digraph with arcs colored $c_1, c_2, \dots, c_k$. Let $v \in V(D)$, and let H be a subdigraph of D. Define*

$$C(v) = \{c_i \ : \ c_i \textit{ is the color of some arc incident with } v\},$$

and

$$C(H) = \{c_i \ : \ c_i \textit{ is the color of some arc in } H\}.$$

Lemma 3.4.3. *Let D be a k-colored digraph with arcs colored with colors $c_1,\dots,c_k$. If there exists a vertex $v \in V(D)$ such that $C(v) \cap C(D-v) = \emptyset$, then this coloring is not k-primitive.*

Proof. Without loss of generality, we may assume that the arcs incident with v are colored with colors $c_1, \ldots, c_i$ and the cycles containing v are $\mathbb{C}_1, \ldots, \mathbb{C}_j$. Since $C(v) \cap C(D-v) = \emptyset$, the color-cycle matrix for D is

$$M_D = \begin{bmatrix} B & O \\ * & M_{D-v} \end{bmatrix},$$

where B is $i \times j$. Further any nonzero $k \times k$ minor of M_D is a determinant of the form $\begin{bmatrix} B_1 & O \\ * & M_1 \end{bmatrix}$, where B_1 is an $i \times i$ submatrix of B, and M_1 is a $(k-i) \times (k-i)$ submatrix of M_{D-v}. Now, if $\mathbb{C}_q$ is a cycle containing v $(1 \leq q \leq j)$ the number of arcs colored with colors $c_1, \ldots, c_i$ is exactly two. That is every column sum of B_1 is two. Thus, 2 is an eigenvalue of B_1, and hence, 2 divides $\det(B_1)$. Consequently 2 divides $\det \begin{bmatrix} B_1 & O \\ * & M_1 \end{bmatrix}$, and hence 2 divides every $k \times k$ minor of M_D. By Theorem 3.2.7, this coloring of D is not k-primitive. $\square$

Let D be a k-colored digraph, and let $D^\star$ be the subdigraph of D with the same vertex set as D resulting from deleting all arcs a from D such that there is no arc colored the same color as a in D except a itself. Let $|D^\star|$ denote the number of arcs in $D^\star$.

Theorem 3.4.4. *Let D be an n-tournament. If $k \geq \binom{n}{2} - \lceil \frac{n}{4} \rceil$, then D is not k-primitive.*

Proof. Consider a $(\binom{n}{2} - \ell)$-coloring of the digraph D. Then, $|D^\star| \leq 2\ell$. Thus, the number of isolated vertices in $D^\star$ is at least $n - 4\ell$. Thus, if $\ell < \frac{n}{4}$, there is a vertex in D that satisfies the hypothesis of Lemma 3.4.3, and consequently this coloring of D is not $(\binom{n}{2} - \ell)$-primitive. Thus, if an $(\binom{n}{2} - \ell)$-coloring is $(\binom{n}{2} - \ell)$-primitive, then $\ell \geq \frac{n}{4}$, and since ℓ is an integer, $\ell \geq \lceil \frac{n}{4} \rceil$. Thus, if $k \geq \binom{n}{2} - \lceil \frac{n}{4} \rceil$, then D is not k-primitive. $\square$

Corollary 3.4.5. *Let $n \geq 4$. Then $\binom{n-1}{2} \leq k_{\max}(n) < \binom{n}{2} - \lceil \frac{n}{4} \rceil$.*

Proof. This is an application of Theorems 3.4.1 and 3.4.4. $\square$

3.5 Game colorings of primitive tournaments

The results in Section 3.4 discuss the existence of k-colorings of tournaments such that the result is k-primitive. In this section, we address the notion of purposeful colorings. That is, is it possible to color the arcs of a tournament such that the coloring satisfies certain properties and the resulting coloring is k-primitive? Specifically, we will examine game colorings, which we define formally below.

Definition 3.5.1. *Let D be an uncolored digraph and $\ell_1, \ldots, \ell_k$ be positive integers. Suppose k players $P_1, \ldots, P_k$ take turns coloring arcs. P_1 begins and colors ℓ_1 uncolored arcs c_1, P_2 then colors ℓ_2 uncolored arcs c_2, etc. This process repeats until there are no arcs left to color. Note that the last player may not be able to finish their turn. The resulting coloring is a $(\ell_1, \ldots, \ell_k)$-game coloring of D.*

In "Properties of 2-Primitive Tournament Digraphs", Beasley and Neal prove the following.

Theorem 3.5.2. [2] *Given a strong n-tournament T, where $n \geq 4$, a $(1,1)$-game coloring of T exists such that the result is 2-primitive.*

We extend this result and prove the following.

Theorem 3.5.3. *Let T be a strong n-tournament, where $n \geq 4$, and let $m \geq 1$. Then a $(2,m)$-game coloring of T exists such that the result is 2-primitive. If $m \leq \frac{n(n-1)}{2} - 2$, then a $(1,m)$-coloring of T exists such that the result is 2-primitive.*

Proof. Let T be a strong n-tournament and let C be a 4-cycle in T, say $v_1 \to v_2 \to v_3 \to v_4 \to v_1$. Without loss of generality, assume $v_1 \to v_3$ and color this arc R. Color $v_2 \to v_3$ color R and all other arcs in C color B. Then the cycle matrix of T will have a submatrix of the form $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so by Theorem 3.2.7 regardless of how we color the remaining arcs in T, the coloring will be 2-primitive.

Clearly the remaining arcs of T can be colored so that T is $(2,m)$-game colored (if m is 1 or 2, assume player 1 is coloring with B and player 2 is coloring with R, otherwise assume the opposite).

Let $m \leq \frac{n(n-1)}{2} - 2$. To obtain a $(1,m)$-primitive coloring of T, color the subtournament induced on v_1, v_2, v_3, v_4 as above. Because there are $\frac{n(n-1)}{2}$ arcs in T, there are $\frac{n(n-1)}{2} - 2$ arcs that are either uncolored or colored B in T. Thus, because $m \leq \frac{n(n-1)}{2} - 2$, the remaining arcs in T can be colored such that the result is $(1,m)$-primitive (assume player 1 is coloring in R).

□

Theorem 3.5.4. *Let T be a strong n-tournament, with $n \geq 4$. Then there exists a $(1,1,1)$-game coloring of T such that the result is 3-primitive.*

Proof. Let T be a strong n-tournament and let C be a 4-cycle in T, say $v_1 \to v_2 \to v_3 \to v_4 \to v_1$. Without loss of generality, assume $v_1 \to v_3$ and $v_4 \to v_2$. Now let $v_1 \xrightarrow{B} v_2$, $v_2 \xrightarrow{G} v_3$, $v_3 \xrightarrow{R} v_4$, $v_4 \xrightarrow{R} v_1$, $v_1 \xrightarrow{B} v_3$, and $v_4 \xrightarrow{G} v_2$, where the letter above the arrow represents the color of the arc.

Then the color-cycle matrix of T will have a submatrix of the form $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, which has determinant 1. By Theorem 3.2.7, regardless of how we color the remaining arcs in T, the coloring will be 3-primitive. Clearly the remaining arcs of T can be colored so that T is $(1,1,1)$-game colored. □

Chapter 4

Monochromatic sinks in arc-colored tournaments

4.1 What is a monochromatic sink?

In this chapter, we will deal exclusively with tournaments whose arcs have been colored with some set of colors $\{c_1, c_2, \ldots, c_k\}$. Let T be a tournament. A vertex $v \in V(T)$ is a *sink* if for every vertex $x \in V(T)\backslash v$, we have $x \to v$. A monochromatic sink is a generalization of a sink in an arc-colored tournament. Formally, a monochromatic sink is defined as follows.

Definition 4.1.1. *Let T be a k-arc-colored tournament. A vertex $v \in V(T)$ is a* monochromatic sink *in T if for every vertex $x \in V(T) \setminus \{v\}$ there exists a monochromatic path from x to v.*

We note that while a sink of a k-arc-colored tournament is also a monochromatic sink, a monochromatic sink is not necessarily a sink. Indeed, every 1-arc-colored tournament has a monochromatic sink, regardless of the presence of a sink, as we shall see below.

Theorem 4.1.2 (Sands, Sauer,Woodrow [14])**.** *Every arc-colored tournament whose arcs are colored with at most two colors contains a monochromatic sink.*

Theorem 4.1.2 is easily seen to be true for 1-arc-colored tournaments, and we give a simple proof below. Note that for a vertex v in a tournament T, we define

$$I(v) = \{x : x \in V(T) \text{ and } x \to v\} \quad \text{and} \quad O(v) = \{x : x \in V(T) \text{ and } v \to x\}.$$

Proof. Let T be a tournament and let $v \in V(T)$ be a vertex of minimum score in T. Clearly all vertices in $I(v)$ have a path to v. Now consider a vertex $x \in O(v)$. If $I(v) \cap O(x) = \emptyset$, then the score of x is less than the score of v, but v is a vertex of minimum score. Thus, $I(v) \cap O(x) \neq \emptyset$ and so there is a path from x to v. Thus, v is a monochromatic sink. $\square$

It is natural to ask if Theorem 4.1.2 is true for 3-arc-colored tournaments. In [14] Sands, *et al.* showed through counterexample that it is not (see Figure 4.1).

Notice that the tournament in Figure 4.1 contains a 3-cycle with each arc a different color. We call such a 3-cycle a *rainbow 3-cycle*. In 1982 Sands, Sauer, and Woodrow in [14] formulated the following conjecturing, noting that the conjecture was also independently posed by Erdős.

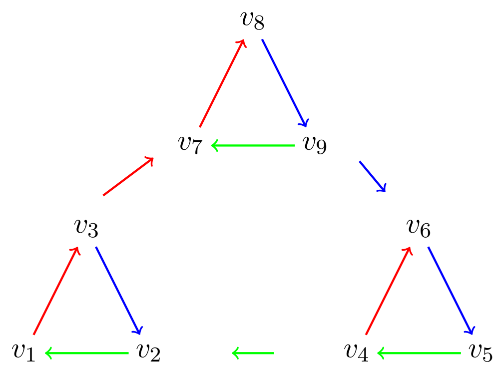

Figure 4.1: A 3-arc-colored tournament with no monochromatic sink (see [14])

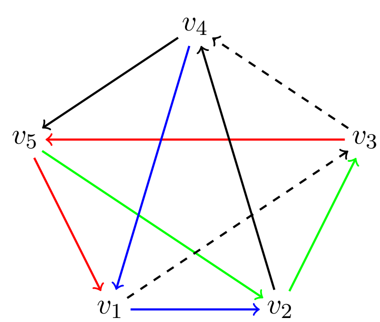

Figure 4.2: 5-arc-colored tournament with no rainbow 3-cycles and no monochromatic sink (see [15])

Conjecture 4.1.3. *If T is a 3-arc-colored tournament with no rainbow 3-cycles, then T contains a monochromatic sink.*

In 1988, Shen [15] showed via counterexample that Conjecture 4.1.3 is false if T is colored with 5 or more colors (see Figure 4.2). In 2004, Galeana-Sńchez and Rojas-Monroy showed that Conjecture 4.1.3 is false if T is 4-arc-colored by providing the counterexample in Figure 4.3. By adding vertices one at a time to the tournament in Figure 4.3 who beat everything that is already there, arbitrarily large 4-arc-colored tournaments with no rainbow 3-cycles and no monochromatic sink can be constructed.

Conjecture 4.1.3 remains open for 3-arc-colored tournaments. Since it was posed, many partial results have been proven in support of Conjecture 4.1.3, a few of which we list below.

Theorem 4.1.4 (Galeana-Sanchez [5])**.** *Let T be a k-arc-colored tournament such that every 3-cycle in T is monochromatic. Then T has a monochromatic sink.*

Theorem 4.1.5 (Shen [15])**.** *Let T be a k-arc-colored tournament with no rainbow triples (no rainbow 3-cycles or rainbow transitive triples). Then T has a monochromatic sink.*

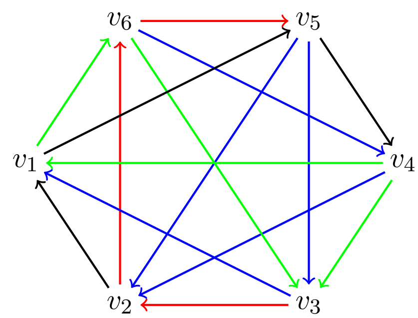

Figure 4.3: 4-arc-colored tournament with no rainbow 3-cycles and no monochromatic sink (see [6])

In this chapter, we will prove more partial results in support of Conjecture 4.1.3. Most of our proofs will rely on Lemma 4.1.6 below from [11], whose statement requires the following definitions. A property P of tournaments is *hereditary* if given any tournament T with property P, every induced subtournament of T has property P. A property P is *1-sinkable* if every arc-colored tournament with property P has at least one monochromatic sink.

Lemma 4.1.6 (Melcher, Reid [11] **Smallest Counterexample Lemma**). *Let P be a hereditary property of arc-colored tournaments. If P is not a 1-sinkable property and T is a smallest arc-colored tournament such that P is a property of T and T contains no monochromatic sink, then the following are true of T:*

1. *$T - v$ contains exactly one monochromatic sink and that monochromatic sink dominates v for every vertex $v \in V(T)$,*
2. *T contains a Hamiltonian cycle $v_0 \to v_1 \to v_2 \to \cdots \to v_{n-1} \to v_0$ such that v_i is the monochromatic sink of $T - v_{(i+1) mod\ n}$, and*
3. *The vertex $v_{(i+1) mod\ n}$ is the only vertex in T that does not have a monochromatic path to v_i.*

Note that here we take vertex indices modulo n, and will do so throughout this chapter without explicit mention.

In Section 4.2, we explore the existence of monochromatic sinks in tournaments that have been colored in particular ways. In Section 4.3, we will explore the existence of monochromatic sinks in tournaments with specific structural properties (tournaments with a vertex of score one and tournaments such that $T - x - y$ is transitive for some $x, y \in V(T)$). In Section 4.4, we will compare two methods commonly used to determine if arc-colored tournaments with some property P have monochromatic sinks, namely the smallest counterexample (see Lemma 4.1.6) and kernel perfect methods. We will show that these methods are equivalent.

4.2 Monochromatic sinks in tournaments with color restrictions

Theorem 4.1.2 tells us that 1- and 2-arc-colored tournaments contain monochromatic sinks. What if we have a tournament that is colored with three colors, but most of its arcs are a single color?

This question leads us to the following definition. If $n \geq 3$ and T is an n-tournament that is k-arc-colored by $\{1, 2, \ldots, k\}$ such that for each $i \in \{1, 2, \ldots, k-1\}$, exactly one arc in T is colored with i, then T is *nearly k-monochromatic*. We shall prove in two ways a result on the existence of monochromatic sinks in nearly 3-monochromatic tournaments. But first, some more notation.

Let T be a k-arc-colored tournament whose arcs have been colored with $\{1, \ldots, k\}$. Let $x, y \in V(T)$. If there is an arc from x to y of color i, we write $x \overset{i}{\to} y$ to denote the colored arc from x to y, and if there is a monochromatic path from x to y of color i we write $x \overset{i}{\Rightarrow} y$ to denote the monochromatic path. If there exists a monochromatic path from x to y whose color we do not wish to specify, we write $x \Rightarrow y$. Let

$$I^i(x) = \{y : y \in V(T) \text{ and } y \overset{i}{\to} x\}.$$

Theorem 4.2.1. *Let T be nearly 3-monochromatic such that T contains no rainbow 3-cycles. Then T contains a monochromatic sink.*

First, a proof of Theorem 4.2.1 by induction.

Proof. The only 3-tournament satisfying the hypotheses of Theorem 4.2.1 is transitive, and therefore, has a monochromatic sink.

Let $n \geq 4$ and suppose every nearly 3-monochromatic $(n-1)$-tournament with no rainbow 3-cycles contains a monochromatic sink. Let T be a nearly 3-monochromatic n-tournament with no rainbow 3-cycles colored with $\{R, B, G\}$. Without loss of generality assume there is exactly one arc a_1 colored R and exactly one arc a_2 colored B in T (and thus all other arcs are colored G). Fix $v \in V(T)$. Consider $T - v$. If neither a_1 or a_2 has v as an endpoint, then by the induction hypothesis $T - v$ contains a monochromatic sink. If a_1 or a_2 has v as an endpoint, $T - v$ is either 1- or 2-arc-colored, and thus by Theorem 4.1.2, $T - v$ contains a monochromatic sink. So in either case $T - v$ has a monochromatic sink, say v_0. If $v \to v_0$, then v_0 is a monochromatic sink of v and we are done. So throughout we will assume $v_0 \to v$. We consider two possibilities.

First, suppose $v_0 \overset{G}{\to} v$. Further, suppose $I^R(v_0) = I^B(v_0) = \emptyset$. Then for all $x \in V(T)\backslash\{v_0, v\}$, we have $x \overset{G}{\Rightarrow} v_0 \overset{G}{\to} v$. Thus, v is a monochromatic sink of T.

Now suppose $I^R(v_0) \neq \emptyset$ and $I^B(v_0) \neq \emptyset$. Let $q_1 \in I^R(v_0)$ and $q_2 \in I^B(v_0)$. If $q_1 \to v$ and $q_2 \to v$, then v is a monochromatic sink of T. So suppose $v \overset{G}{\to} q_1$ and $v \overset{G}{\to} q_2$. Now if $q_1 \overset{G}{\to} q_2$, q_2 is a monochromatic sink of T and if $q_2 \overset{G}{\to} q_1$, q_1 is a monochromatic sink of T. So, without loss of generality, assume $q_2 \overset{G}{\to} v$ and $v \overset{G}{\to} q_1$. Then q_1 is a monochromatic sink of T.

Now without loss of generality, suppose $I^R(v_0) \neq \emptyset$ and $I^B(v_0) = \emptyset$. We will denote the only element in $I^R(v_0)$ by q_1. Note that $q_1 \overset{R}{\to} v_0$. Now if $q_1 \to v$, then v is a monochromatic sink of T. So, assume that $v \to q_1$. Now it cannot be that $v \overset{B}{\to} q_1$, else T would have a rainbow 3-cycle given by qv_0vq. Moreover, since we have already used our only color R arc, it must be that $v \overset{G}{\to} q_1$. Therefore, for all $x \in V(T)\backslash\{v_0, v, q_1\}$, we have $x \overset{G}{\Rightarrow} v_0 \overset{G}{\to} v \overset{G}{\to} q_1$, making q_1 a monochromatic sink of T.

To complete the proof suppose the arc from v_0 to v is not colored G. Without loss of generality, suppose $v_0 \overset{R}{\to} v$. This implies that $I^R(v_0) = \emptyset$. Now suppose $I^B(v_0) \neq \emptyset$. Let $q \in I^B(v_0)$. Now the arc between q and v must be color G. If $v \to q$, v_0vqv_0 is a rainbow 3-cycle. So it must be that $q \overset{G}{\to} v$. Now if v beats a vertex $x \in V(T)\backslash\{q, v, v_0\}$, $v \overset{G}{\to} x$, and thus $v \overset{G}{\to} x \overset{G}{\Rightarrow} v_0$. Thus, v_0 is a monochromatic sink in T. Therefore, assume that for all $x \in V(T)\backslash\{q, v, v_0\}$, $x \to v$. Then v is a monochromatic sink of T. To complete the proof of this case, suppose now that $I^B(v_0) = \emptyset$. Now suppose for some $x \in I^G(v_0)$, $v \to x$. It cannot be that $v \overset{B}{\to} x$, because then vxv_0v would be a rainbow 3-cycle. Thus, it must be that $v \overset{G}{\to} x$. So $v \overset{G}{\to} x \overset{G}{\to} v_0$, and thus v_0 is a monochromatic sink of T.

Therefore, we see in any case that T has a monochromatic sink. This completes the induction step so we are done.

□

We now offer another proof of Theorem 4.2.1, which utilizes the Smallest Counterexample Lemma (SCL), Lemma 4.1.6.

Proof. Suppose Theorem 4.2.1 is false, and let T be a tournament of smallest order such that T is a 3-arc-colored tournament colored by $\{R, B, G\}$ such that exactly one arc is colored R and exactly one arc in colored B, and T contains no rainbow 3-cycles, but contains no monochromatic sink. Now having no rainbow 3-cycles and having at most one arc of color R and at most one arc of color B is a hereditary property, thus we can apply the SCL. Label the vertices of T with $v_0, v_1, \ldots, v_{n-1}$ as in the SCL. Now, the cycle $v_0v_1 \ldots v_{n-1}v_0$ cannot be monochromatic otherwise every vertex in T would be a monochromatic sink (a contradiction). Thus, without loss of generality we will assume $v_0 \overset{G}{\to} v_1 \overset{R}{\to} v_2$. Now by the SCL, v_2 has a monochromatic path to v_0. If this path is color B, then it must be length one since T has just one arc of color B. But then $v_0v_1v_2v_0$ is a rainbow 3-cycle. Since T has no rainbow 3-cycles, it must be that $v_2 \overset{G}{\Rightarrow} v_0$.

Now suppose that $v_2 \overset{B}{\to} v_3$. By the SCL, we know there is a monochromatic path from v_1 to v_{n-1}. Since all color R and B arcs have already been used, it must be that $v_1 \overset{G}{\Rightarrow} v_{n-1}$. Again no R or B arcs remain, so $v_{n-1} \overset{G}{\to} v_0$. Thus $v_1 \overset{G}{\Rightarrow} v_{n-1} \overset{G}{\to} v_0$, so that v_1 has a monochromatic path to v_0, a contradiction.

Therefore, it must be that $v_2 \overset{G}{\to} v_3$. Now v_3 has a monochromatic path to v_1. If the path is colored B, it is length one, which means $v_1v_2v_3v_1$ is a rainbow 3-cycle. Thus it must be that $v_3 \overset{G}{\Rightarrow} v_1$. But now $v_2 \overset{G}{\to} v_3 \overset{G}{\Rightarrow} v_1$, so v_2 has a monochromatic path to v_1, a contradiction.

By the SCL this shows that no smallest counterexample to Theorem 4.2.1 exists. Therefore, Theorem 4.2.1 is true. □

Note that Theorem 4.2.1 is false if the hypothesis "no rainbow 3-cycles" is removed. For example, take a rainbow 3-cycle and another vertex v. Let v beat everything in the rainbow 3-cycle in color 1. The resulting tournament has no monochromatic sink. Larger examples can be built by adding vertices one at a time such that every added vertex beats everything added before it in color 1, a construction similar to that used by Galeana-Sńchez and Rojas-Monroy in [6] to disprove Conjecture 4.1.3 for four colors.

We now prove a stronger result than Theorem 4.2.1. Once Theorem 4.2.2 is proven, we will have a third proof of Theorem 4.2.1 and will have proved much more. Let T be a tournament

k-arc-colored with $\{1, \dots, k\}$. Let T_i be the directed graph resulting from deleting every arc from T that is not color i. The diameter of T_i, denoted $\mathrm{diam}(T_i)$, is the length of the longest path in T_i.

Theorem 4.2.2. *Let T be a 3-arc-colored tournament colored with $\{R, G, B\}$ such that T contains no rainbow 3-cycles. If $diam(T_B) = diam(T_G) = 1$, then G contains a monochromatic sink.*

Proof. Being 3-arc-colored with no rainbow 3-cycles and $\mathrm{diam}(T_B)$, $\mathrm{diam}(T_G) \le 1$ is a hereditary property, thus we can apply the SCL. Suppose that Theorem 4.2.2 is false. Let T be a tournament of smallest order satisfying the hypotheses of Theorem 4.2.2 and label the vertices of T with $v_0, \dots, v_{n-1}$ according to the SCL. Suppose there exists some i such that $v_i \stackrel{R}{\to} v_{i+1}$. Now there is a monochromatic path from v_{i+1} to v_{i-1}. If $v_{i+1} \stackrel{R}{\Rightarrow} v_{i-1}$, then $v_i \stackrel{R}{\Rightarrow} v_{i-1}$, a contradiction. Thus, without loss of generality suppose that $v_i \stackrel{B}{\Rightarrow} v_{i-1}$. Now since $\mathrm{diam}(T_B) = 1$, it must be that $v_{i+1} \stackrel{B}{\to} v_{i-1}$. Now if $v_{i-1} \stackrel{B}{\to} v_i$, then $v_{i+1} \stackrel{B}{\Rightarrow} v_i$, a contradiction, and if $v_{i-1} \stackrel{G}{\to} v_i$, then $v_{i-1}v_iv_{i+1}$ is a rainbow 3-cycle. Thus, it must be that $v_{i-1} \stackrel{R}{\to} v_i$. Inductively, it follows that the cycle $v_0v_1 \dots v_{n-1}v_0$ is monochromatic, which means T has a monochromatic sink. But this cannot be as T has no monochromatic sink.

Thus, there are no arcs colored R on the cycle $v_0v_1 \dots v_{n-1}v_0$. Without loss of generality, suppose that $v_0 \stackrel{B}{\to} v_1$. Now since $\mathrm{diam}(T_B) = \mathrm{diam}(T_G) = 1$, the arcs of the cycle $v_0v_1 \dots v_{n-1}v_0$ are alternating B and G. That is, for i even $v_i \stackrel{B}{\to} v_{i+1}$ and for i odd, $v_i \stackrel{G}{\to} v_{i+1}$.

Suppose for some odd i, $v_{i+1} \to v_{i-1}$. If $v_{i+1} \stackrel{B}{\to} v_{i-1}$, then $v_{i+1} \stackrel{B}{\Rightarrow} v_i$, if $v_{i+1} \stackrel{G}{\to} v_{i-1}$, then $v_i \stackrel{G}{\Rightarrow} v_{i-1}$, and if $v_{i+1} \stackrel{R}{\to} v_{i-1}$, then $v_{i-1}v_iv_{i+1}$ is a rainbow 3-cycle. A similar argument for i even shows that for all i it must be that $v_{i-1} \to v_{i+1}$.

Recalling that $\mathrm{diam}(T_B) = \mathrm{diam}(T_G) = 1$, it must be true for all i that $v_{i-1} \stackrel{R}{\to} v_{i+1}$.

Now suppose that $v_1 \to v_{n-2}$. If $v_1 \stackrel{R}{\to} v_{n-2}$, then $v_1 \stackrel{R}{\to} v_{n-2} \stackrel{R}{\to} v_0$, a contradiction. If $v_1 \stackrel{B}{\to} v_{n-2}$, then $v_0v_1v_{n-2}$ is a B path with more than one arc, a contradiction. If $v_1 \stackrel{G}{\to} v_{n-2}$, then $v_0v_1v_{n-2}$ is a rainbow 3-cycle. Thus it must be that $v_{n-2} \to v_1$.

Now it cannot be that $v_{n-2} \stackrel{G}{\to} v_1$, as $v_{n-2}v_1v_2$ would be a G path with two arcs. Suppose $v_{n-2} \stackrel{R}{\to} v_1$. Then $v_{n-2} \stackrel{R}{\to} v_1 \stackrel{R}{\to} v_3 \stackrel{R}{\to} v_5 \stackrel{R}{\to} \dots \stackrel{R}{\to} v_{n-5} \stackrel{R}{\to} v_{n-3}$, which makes v_{n-3} a monochromatic sink of T, a contradiction. Thus, it must be that $v_{n-2} \stackrel{B}{\to} v_1$.

A similar argument shows that $v_i \stackrel{B}{\to} v_1$ for all even i except 2. In fact, $v_4 \stackrel{B}{\to} v_1$, so $v_4v_1v_2$ is a rainbow 3-cycle. Therefore, there exists no smallest counter example to Theorem 4.2.2 and therefore Theorem 4.2.2 is true. □

We now prove results on tournaments that contain a vertex v such that all arcs with v as an endpoint are colored B and other arcs are colored either R of G. First we introduce some more notation.

Let T be a tournament colored with $C = \{1, \dots, k\}$ and let $x \in V(T)$. Define

$$\xi^+(x) = \{i : i \in C \text{ and } y \stackrel{i}{\to} x \text{ for some } y \in V(T)\},$$

and

$$\xi^-(x) = \{i : i \in C \text{ and } x \stackrel{i}{\to} y \text{ for some } y \in V(T)\}.$$

Finally, define

$$\xi(x) = \xi^+(x) \cup \xi^-(x).$$

Theorem 4.2.3. *Let T be a 3-arc-colored tournament colored with $\{R, G, B\}$ with a vertex $x \in V(T)$ such that $\xi(x) = \{B\}$ and the arcs of $T - x$ are colored R and G. Then T contains a monochromatic sink.*

Proof. Let T and x be as in the problem statement. If $O(x) = \emptyset$, then x is a monochromatic sink of T. Thus, assume that $O(x) \neq \emptyset$. Then the subtournament induced on $O(x)$ is 2-arc-colored, and thus contains a monochromatic sink, say v. Consider $y \in I(x)$. Since $\xi(x) = \{B\}$, it follows that $y \xrightarrow{B} x \xrightarrow{B} v$. Therefore, v is a monochromatic sink of T. □

4.3 Monochromatic sinks in tournaments with structural restrictions

In this section, we discuss the existence of monochromatic sinks in tournaments that have certain structural properties. First we will explore tournaments with a vertex of score one, and then we will explore tournaments that have large transitive subtournaments.

Recall from Chapter 2 that the score of a vertex v in a tournament T is the number of arcs outgoing from v. If a tournament T is colored with any number of colors and has a vertex of score zero, then T has a sink, and thus a monochromatic sink. In this section, we address the existence of a monochromatic sink in tournaments that have a vertex of score near zero, specifically a vertex of score one. We begin with the following conjecture.

Conjecture 4.3.1. *Let T be a k-arc-colored tournament such that T contains a vertex of score 1 and no rainbow 3-cycles. Then T contains a monochromatic sink.*

Theorem 2.1.5 in Chapter 2 tells us that the maximum possible number of vertices of score one in a tournament T is three. Thus, Conjecture 4.3.1 can be broken into three parts: assume T has exactly three vertices of score one, exactly two vertices of score one, and exactly one vertex of score one. We shall address the first scenario below. (Note that because having a vertex of score one is not a hereditary property, we cannot use the SCL.)

Theorem 4.3.2. *Let T be a k-arc-colored tournament such that T contains three vertices of score 1 and no rainbow 3-cycles. Then T contains a monochromatic sink.*

Proof. Let T be k-arc-colored and let v_0, v_1, and v_2 be the vertices in T with score 1. It is easy to see that $v_0 \to v_1 \to v_2 \to v_0$. Without loss of generality, because T contains no rainbow 3-cycles, v_1 and v_2 must have monochromatic paths to v_0. Now since $v_0 \to v_1$ and v_0 has score one, it must be that for all $x \in V(T) \backslash \{v_0, v_1, v_2\}$, we have $x \to v_0$. Thus, v_0 is a monochromatic sink of T. □

When T has less than three vertices of score one, the situation becomes more complicated. We have not yet been able to prove Conjecture 4.3.1 for this scenario. Instead, we provide now a few partial results for the case where T has exactly two vertices of score one. We will let $s(v)$ denote the score of a vertex $v \in V(T)$.

Theorem 4.3.3. *Let T be a k-arc-colored tournament with no rainbow 3-cycles such that T contains two vertices v_0 and v_1 such that $s(v_0) = s(v_1) = |\xi^-(v_0)| = |\xi^-(v_1)| = 1$. Then T contains a monochromatic sink.*

Proof. Let T, v_0, and v_1 be as above. Without loss of generality suppose that $v_0 \to v_1$. Since $s(v_0) = 1$, it must be for all $x \in V(T)\backslash\{v_0, v_1\}$ that $x \to v_0$, and since $s(v_1) = 1$, there exists exactly one vertex $w \in V(T)\backslash\{v_0, v_1\}$ such that $v_1 \to w$.

Now if $v_0 \to v_1$ and $w \to v_0$ are colored alike, then $w \Rightarrow v_1$ and since for all $x \in V(T)\backslash\{w, v_1\}$ we have $x \to v_1$, it follows that T has a monochromatic sink, v_1. Thus, we will assume without loss of generality that $v_0 \overset{R}{\to} v_1$ and $w \overset{G}{\to} v_0$. Now wv_0v_1w is a 3-cycle, and since T does not have any rainbow 3-cycles, it must be that either $v_1 \overset{R}{\to} w$ or $v_1 \overset{G}{\to} w$. Suppose $v_1 \overset{G}{\to} w$. Then $v_1 \overset{G}{\to} w \overset{G}{\to} v_0$ and since for all $x \in V(T)\backslash\{v_0, v_1\}$ we know $x \to v_0$, it follows that v_0 is a monochromatic sink of T. Therefore, we will now assume that $v_1 \overset{R}{\to} w$.

Now since $|\xi^-(v_1)| = 1$ and $v_0 \overset{R}{\to} v_1$, it follows for all $x \in V(T)\backslash\{v_1, w\}$ that $x \overset{R}{\to} v_1$. Thus for all $x \in V(T)\backslash\{v_1, w\}$ we have $x \overset{R}{\to} v_1 \overset{R}{\to} w$. Therefore, w is a monochromatic sink of T and we are done. ☐

Theorem 4.3.4. *Let T be a k-arc-colored tournament with no rainbow 3-cycles and two vertices of score one and two vertices of score two. Then T contains a monochromatic sink. In fact, one of the vertices of score 1 or of score 2 is a monochromatic sink of T.*

Proof. Let T be a k-arc-colored tournament such that T contains no rainbow 3-cycles and contains two vertices v_0 and v_1 such that $s(v_0) = s(v_1) = 1$ and two vertices v_2 and v_3 such that $s(v_2) = s(v_3) = 2$. Without loss of generality suppose that $v_0 \to v_1$. Now $x \to v_0$ for all $x \in V(T)\backslash\{v_0, v_1\}$, and there exists exactly one vertex $w \in V(T)\backslash\{v_0, v_1\}$ such that $v_1 \to w$ (that is, for all $x \in V(T)\backslash\{v_1, w\}$ we have $x \to v_1$).

Now for a contradiction, suppose that $s(w) \neq 2$. Now $s(v_0) = s(v_1) = 1$ and for all $y \in I(w)\backslash\{v_1\}$, we have $d^+(y) \geq 3$ since $y \to w$, $y \to v_0$ and $y \to v_1$. Therefore, it must be that $v_2, v_3 \in O(w)\backslash\{v_0\}$. Without loss of generality, suppose that $v_2 \to v_3$. But then $s(v_2) \geq 3$, as we also know that $v_2 \to v_0$ and $v_2 \to v_1$, which contradicts $s(v_2) = 2$. Therefore, it must be that $s(w) = 2$. Without loss of generality, say that $w = v_2$.

Now if $v_0 \to v_1$ and $w \to v_0$ are colored alike, then T has a monochromatic sink, v_1. Thus, we will assume without loss of generality that $v_0 \overset{R}{\to} v_1$ and $w \overset{G}{\to} v_0$. Now wv_0v_1w is a 3-cycle and since T does not have any rainbow 3-cycles, it must be that either $v_1 \overset{R}{\to} w$ or $v_1 \overset{G}{\to} w$. Suppose $v_1 \overset{G}{\to} w$. Then v_0 is a monochromatic sink of T. Therefore, we will now assume that $v_1 \overset{R}{\to} w$.

Let $x \in O(w)\backslash\{v_0\}$, and suppose $w \overset{G}{\to} x$. If $x \overset{G}{\to} v_1$, then $w \overset{G}{\to} x \overset{G}{\to} v_1$, making v_1 a monochromatic sink of T. If $x \overset{B}{\to} v_1$, then v_1wxv_1 is a rainbow 3-cycle. If $x \overset{R}{\to} v_1$, then for all $x \in O(w)$, we have $x \overset{R}{\to} v_1 \overset{R}{\to} w$, making w a monochromatic sink of T. A similar argument shows that if $w \overset{B}{\to} x$, then w or v_1 is a monochromatic sink of T. Thus, we will now assume that $w \overset{R}{\to} x$.

For all $y \in V(T)\backslash\{w, v_0, v_1, x\}$, we have $y \to w, y \to v_0$, and $y \to w$. Since T has two vertices of score two, it must be that $s(x) = 2$. That is $x = v_3$. Furthermore, since $x \to v_0$ and $x \to v_1$, it must be for all $y \in V(T)\backslash\{v_0, v_1\}$ that $y \to x$. Since $v_0 \overset{R}{\to} v_1 \overset{R}{\to} w \overset{R}{\to} x$, it follows that x is a monochromatic sink of T and we are done.

□

We now shift our discussion to tournaments with large transitive subtounaments. If a tournament is colored with any number of colors and is transitive, then it certainly contains a monochromatic sink. In [11], Melcher and Reid prove results on the existence of a monochromatic sink in tournaments that are almost transitive. Specifically, they prove the following.

Theorem 4.3.5 (Melcher,Reid [10]). *Let T be a tournament that contains a subset F of at most two arcs whose direction reversal yields a transitive tournament. Then if T is arc-colored with any number of colors so that T contains no rainbow 3-cycles, then T contains a monochromatic sink.*

Theorem 4.3.6 (Melcher,Reid [11]). *Let T be a tournament that contains a vertex $x \in V(T)$ such that $T-x$ is transitive. If T is arc-colored with any number of colors so that T contains no rainbow 3-cycles, then T contains a monochromatic sink.*

They remark that this result is best possible in the sense that given $n \geq 5$, there exists an n-tournament T containing distinct vertices x and y such that $T - x - y$ is transitive, and T can be colored with five colors such that there are no rainbow 3-cycles, and T does not contain a monochromatic sink. However, it is not necessarily true that just because a k-colored tournament T is such that $T-x-y$ is transitive that T does not have a monochromatic sink. In the result that follows we place constraints on coloring of tournaments of this type so that they will be guaranteed to have a monochromatic sink.

Theorem 4.3.7. *Let T be a k-arc-colored tournament such that there exists vertices $x, y \in V(T)$ such that $T-x-y$ is transitive. Suppose $x \to y$ and $|\xi(y)| = 1$. Then T contains a monochromatic sink.*

Proof. Let T be a tournament and suppose that $T - x - y$ is transitive and $x \to y$. Let $v_{n-3} \to v_{n-2} \to \ldots \to v_0$ be the unique Hamiltonian path in $T-x-y$. Now if for all $i \in \{n-3, n-2, \ldots, 0\}$ we have $v_i \to y$, then y is a monochromatic sink.

So assume there exists some $i \in \{n-3, n-2, \ldots, 0\}$ such that $y \to v_i$. Let v_ℓ be the minimum such i. Then for all $j < \ell$ we know $v_j \to y$. Now since $|\xi(y)| = 1$, without loss of generality, assume that $y \overset{R}{\to} v_\ell$. Then $x \overset{R}{\to} y$ and $v_j \overset{R}{\to} y$ for all $j < \ell$. So for all $j < \ell$ we have $v_j \overset{R}{\Rightarrow} v_\ell$ and $x \overset{R}{\Rightarrow} v_\ell$. Moreover, since $T - x - y$ is transitive, we know for all $k > \ell$ that $v_k \to v_\ell$. Therefore, v_ℓ is a monochromatic sink of T. □

We remark that Theorem 4.3.7 is quite similar to Theorem 4.2.3. Both consider colorings with a vertex such that all incoming and outgoing arcs are the same color. However, Theorem 4.3.7 imposes less restrictions on the coloring (indeed any number of colors can be used and rainbow 3-cycles are allowable), but only applies to tournaments that have a certain structure.

4.4 The smallest counterexample lemma and kernel perfect method

In this section we will present a method, the Kernel Perfect Method (KPM), used in the literature (see for example [13]) to prove that arc-colored tournaments with certain properties have monochromatic sinks. We will discuss that while on the surface the KPM may seem different than using the Smallest Counterexample Lemma (SCL), which has been used throughout this chapter to obtain results, the methods are equivalent in a sense that will be made clear later. Before describing the KPM we require the following definitions and theorems.

Let D be a digraph. A set of independent vertices N in D is a *kernel* if for all $x \in V(D) \backslash N$ there exists a vertex $y \in N$ such that $x \to y$. Furthermore N is said to be a kernel by monochromatic paths if (i) for all $x \in V(D) \backslash N$ there exists a $y \in N$ such that there is a monochromatic path in D from x to y, and (ii) between distinct vertices in N there is no monochromatic path in D. We remark that if D is a tournament, then a kernel is simply a sink and a kernel by monochromatic paths is simply a monochromatic sink.

The closure of a digraph D, k-colored with $\{1, \ldots, k\}$, denoted $\mathbb{C}(D)$, is defined as follows:

- $V(\mathbb{C}(D) = V(D)$
- $A(\mathbb{C}(D)) = A(D) \cup \bigcup_{i=1}^{k} \{u \xrightarrow{i} v : u \overset{i}{\Rightarrow} v \text{ in } D\}$.

It is easily seen that D has a kernel by monochromatic paths if and only if $\mathbb{C}(D)$ has a kernel. Digraph D is *kernel perfect* if every induced subdigraph of D has a kernel. Let $A(D)$ denote the arc set of D. An arc $(v_i, v_j) \in A(D)$ is *symmetric* if $(v_j, v_i) \in A(D)$ and *asymmetric* if $(v_j, v_i) \notin A(D)$. We define the symmetric part of D, denoted $Sym(D)$, to be the spanning subdigraph of D whose arcs are the symmetric arcs of D, and define the asymmetric part of D, denoted $Asym(D)$, to be the spanning subdigraph of D whose arcs are the asymmetric arcs of D. In [3] the following is proven.

Theorem 4.4.1. *A complete digraph is kernel perfect if and only if every directed cycle has at least one symmetric arc.*

The following is an immediate corollary.

Corollary 4.4.2. *Let T be an arc-colored tournament. Then $\mathbb{C}(T)$ is kernel perfect if and only if every directed cycle in $\mathbb{C}(T)$ has at least one symmetric arc. That is, T has a monochromatic sink if and only if every directed cycle in $\mathbb{C}(T)$ has at least one symmetric arc.*

We now can state the KPM for proving the existence of a monochromatic sink in an arc-colored tournament T with property P:

1. Suppose T does not contain a monochromatic sink.
2. Find a cycle γ of minimum length in $\mathbb{C}(T)$ that contains no symmetric arc (this is possible by Corollary 4.4.2).
3. Establish that $\gamma \subseteq T$.

4. Show that the subtournament $T' = T[V(\gamma)]$, the subtournament of T induced on $V(\gamma)$, has the properties (1), (2), (3) of Theorem 4.4.3. This step as well as the previous two steps do not rely upon what property P is.

5. Find a symmetric arc in γ (because of step 4 above, this amounts to showing that T' has a monochromatic sink).

6. Conclude using Corollary 4.4.2 that T has a monochromatic sink.

We will compare the KPM with the Smallest Counterexample Lemma (SCL) proved in [11] and stated early as Lemma 4.1.6, which we state again here for easy reference.

Theorem 4.4.3. *[Smallest Counterexample Lemma (SCL)] Let P be a hereditary property of arc-colored tournaments. If P is not a 1-sinkable property and T is a smallest arc-colored tournament such that T has property P and T contains no monochromatic sink, the following is true of T:*

1. *$T - v$ contains exactly one monochromatic sink and that monochromatic sink dominates v for every $v \in V(T)$,*

2. *T contains a Hamiltonian cycle $v_0 \to v_1 \to v_2 \to \ldots \to v_{n-1} \to v_0$ such that v_i is the monochromatic sink of $T - v_{(i+1) \bmod n}$.*

3. *The vertex $v_{(i+1) \bmod n}$ is the only vertex in T that does not have a monochromatic path to v_i.*

Moreover, if T is any tournament satisfying (1),(2), and (3) above, then T does not contain a monochromatic sink.

At first it may seem the KPM can be applied in more situations that the SCL, which can only be used to prove results about tournaments with some *hereditary* property P. However, the KPM in practice is subject to the same limitations. To accomplish step 5 of the KPM, we have never seen an author consider any vertices outside of $T' = T[V(\gamma)]$. If P was not hereditary, and all that was known about T' is that it has properties (1), (2), and (3) of the SCL, then if step 5 could be accomplished without using vertices outside of T', it would be a proof that all 3-arc-colored tournaments have a monochromatic sink. However, as Sands, Sauer, and Woodrow show in [14], not all 3-arc-colored tournaments have a monochromatic sink. Thus, P must be hereditary if only vertices inside T' will be considered.

From what we have seen of KPM applications, the only difference between it and the SCL is work is done each time to set up a SCL like situation (i.e. steps 2-4, which are independent of P, are repeated each time the KPM is used). Thus the utility in the SCL appears to be that it takes care of all the set up.

To further demonstrate the similarity between the KPM and the SCL, we now prove that Corollary 4.4.2 can be used to prove Theorem 4.4.3 and vice versa.

Theorem 4.4.4. *Corollary 4.4.2 implies Theorem 4.4.3.*

Proof. Assume Corollary 4.4.2 is true. Let P be a hereditary property of arc-colored tournaments that is not a 1-sinkable property. Let T be an arc-colored tournament of smallest order having property P, but containing no monochromatic sink. Then $\mathbb{C}(T)$ is not a kernel perfect digraph. Thus, by Corollary 4.4.2 there exists some cycle in $\mathbb{C}(T)$ with no symmetric arcs. Let $\gamma = (v_0, v_1, v_2, \ldots, v_{n-1}, v_0)$ be a cycle of minimum length in $\mathbb{C}(T)$ such that γ has no symmetric arcs. That is

$$A(\gamma) \cap A(Sym(\mathbb{C}(T))) = \emptyset. \tag{4.1}$$

Thus, $A(\gamma) \subseteq A(Asym(\mathbb{C}(T)))$, which implies that $\gamma \subseteq T$. Consider $T' = T[V(\gamma)]$, the subtournament of T induced on $V(\gamma)$. We will show that $T' = T$. To the contrary, suppose that $T \neq T'$. Since T is a tournament of smallest order with hereditary property P and no monochromatic sink, it follows that every properly induced subtournament of T contains a monochromatic sink. Since $T \neq T'$ and $T' \subseteq T$, every subtournament of T' contains a monochromatic sink. That is, $\mathbb{C}(T')$ is kernel perfect. Since $\gamma \subseteq T'$, it follows from Corollary 4.4.2 that γ has a symmetric arc. However, this contradicts Equation 4.1. Therefore, $T = T'$.

The following are all true.

(a) For any distinct vertices $v_i, v_j \in V(T)$ such that $j \notin \{i-1, i+1\}$, it is true that $(v_i, v_j) \in A(Sym(\mathbb{C}(T)))$.

Let $v_i, v_j \in V(T)$ be such that $j \notin \{i-1, i+1\}$. With out loss of generality, assume that $v_i \to v_j$. Then $\gamma' = (v_i, v_j, v_{j+1}, \ldots, v_{i-1}, v_i)$ is a cycle with $\ell(\gamma') < \ell(\gamma)$. It follows by the definition of γ that $\gamma' \not\subseteq Asym(\mathbb{C}(T)$. Since $A(\gamma) \subseteq A(Asym(\mathbb{C}(T)))$, it follows that $(v_i, v_j) \in A(Sym(\mathbb{C}(T)))$.

(b) For every vertex $v \in V(T)$, $T - v$ contains exactly one monochromatic sink and that monochromatic sink dominates v.

Without loss of generality, let $v = v_0$. It follows from (a) that v_{n-1} is a monochromatic sink of $T - v_0$. To show that v_{n-1} is the only monochromatic sink of $T - v_0$, suppose for some $j \neq n-1$ that v_j is a monochromatic sink of $T - v_0$. Then $v_{j+1} \Rightarrow v_j$ and by the definition of γ, $v_j \to v_{j+1}$. Thus, $(v_i, v_{j+1}) \in A(Sym(\mathbb{C}(T)))$. But this contradicts Equation 4.1. Therefore, v_{n-1} is the only monochromatic sink of $T - v_0$.

(c) γ is a Hamiltonian cycle in T such that v_i is the monochromatic sink of $T - v_{(i+1) \bmod n}$ and the vertex $v_{(i+1) \bmod n}$ is the only vertex in T that does not have a monochromatic path to v_i.

This follows from the justification of (b) and the fact that T does not have a monochromatic sink.

Therefore, Theorem 4.4.3 is true.

□

Theorem 4.4.5. *Theorem 4.4.3 implies Corollary 4.4.2.*

Proof. ($\Leftarrow$) Let T be an arc-colored tournament. Suppose every directed cycle in $\mathbb{C}(T)$ has at least one symmetric arc. By way of contradiction, suppose that $\mathbb{C}(T)$ is not kernel perfect. Then there exists an induced subtournament T' of T of smallest order such that T' has no monochromatic sink. Now being an induced subtournament of T is a hereditary property and not 1-sinkable since $\mathbb{C}(T)$ is not kernel perfect. Thus we may apply Theorem 4.4.3 to T'. By Theorem 4.4.3 $V(T')$ can be labeled with $\{v_0, v_1, \ldots, v_{n-1}\}$ such that $\gamma = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is a Hamiltonian cycle in T' and there is no monochromatic path from v_{i+1} to v_i for all i such that $0 \leq i \leq n-1$ (indices taken modulo n). That is, $A(\gamma) \cap A(Sym(\mathbb{C}(T))) = \emptyset$. But $\gamma \subseteq T' \subseteq T \subseteq \mathbb{C}(T)$, so $A(\gamma) \cap A(Sym(\mathbb{C}(T))) = \emptyset$ contradicts "every directed cycle in $\mathbb{C}(T)$ has at least one symmetric arc". Thus, $C(T)$ is kernel perfect.

($\Rightarrow$) Let T be an arc-colored tournament and suppose $\mathbb{C}(T)$ is kernel perfect. By way of contradiction, suppose there exists a directed cycle in $\mathbb{C}(T)$ such that the cycle contains no symmetric arcs. Let γ be a cycle of smallest length in $\mathbb{C}(T)$ with no symmetric arcs. Note that $\gamma \subseteq T$. Consider the induced subtournament of T, $T' = T[V(\gamma)]$. Since $\mathbb{C}(T)$ is kernel perfect, T' has a monochromatic sink. Repeating arguments (a), (b), (c), and (d) in the proof of Theorem 4.4.4, we see that T' satisfies properties (1),(2), and (3) of Theorem 4.4.3. Thus, by Theorem 4.4.3, T' does not contain a monochromatic sink. This is a contradiction, as we have already shown T' has a monochromatic sink. Therefore, every directed cycle in $\mathbb{C}(T)$ contains at least one symmetric arc. $\square$

Appendix A

Ideas for further research

A.1 Chapter 2 extensions: possible variations of $f(m, n)$

Here we give a series of problems for future research to extend the results in Chapter 2. We list ideas for several new $f(m, n)$-like functions. The first two problems were suggested to us by David E. Brown and the last three are suggestions of K. Brooks Reid.

Problem A.1.1. *Determine the minimum n, denoted $g(m, k)$, such that there is an n-tournament with k vertices of score m.*

Problem A.1.2. *Determine the maximum score, denoted $h(k, n)$, which can be repeated k times in an n-tournament.*

Problem A.1.3. *Suppose that a, b, n are integers such that $0 \leq a < b < n$. What is the maximum number of vertices possible in an n-tournament, each of which has score greater than or equal to a and less than or equal to b (i.e., in the interval $[a, b]$)? Generalize to several disjoint intervals.*

Problem A.1.4. *Let m, n and c be positive integers such that n is odd, $0 < m < n$, and $0 < c < (1/24)(n^3 - n)$. What is the maximum number of vertices possible in an n-tournament, each of which has score greater than or equal to m, but the number of 3-cycles is less than or equal to c? (Note that $(1/24)(n^3 - n)$ is the maximum possible number of 3-cycles in any n-tournament, and that number is realized only for regular n-tournaments. Also, the number of 3-cycles in an n-tournament is $\binom{n}{3} - \sum_{i=1}^{n} \binom{s_i}{2}$, where $(s_1, \ldots, s_n)$ is the score sequence of T. A lot of scores near $(n-1)/2$ means a lot of 3-cycles.)*

Problem A.1.5. *Let m, n, and c be as in Problem A.1.4. What is the maximum number of vertices possible in an n-tournament, each of which has score less than or equal to m, but the number of 3-cycles is greater than or equal to c?*

A.2 Chapter 4 extensions: open monochromatic sink conjectures

Here we list several conjectures that can be explored for further research. For all the the below conjectures, we have partial proofs, which we do not include here. We begin by listing two conjectures that have already been discussed in this chapter.

Conjecture A.2.1. *Let T be a 3-arc-colored tournament with no rainbow 3-cycles and a vertex of score one. Then T contains a monochromatic sink.*

Conjecture A.2.2. *Let T be a 3-arc-colored tournament with no rainbow 3-cycles and two distinct vertices such that $T - x - y$ is transitive. Then T contains a monochromatic sink.*

For a directed graph D let

$$\Delta(D) = \max_{v \in V(D)} |I_D(v)| + |O_D(v)|.$$

A matching in D is a subset of the arcs in D such that no arcs in the set share an endpoint. Let T be a tournament k-arc-colored with $\{c_1, \dots, c_k\}$. Let T_{c_i} be the directed graph resulting from deleting every arc from T that is not color c_i. Thus, if $\Delta(T_{c_i}) \leq 1$, the arcs colored c_i in T form a matching.

Conjecture A.2.3. *Let T be a 3-arc-colored tournament colored with $\{R, G, B\}$ such that T contains no rainbow 3-cycles and $\Delta(T_B) \leq 1$. That is, suppose the arcs colored B in T form a matching. Then T contains a monochromatic sink.*

In [14], Sands, Sauer, and Woodrow prove the following.

Theorem A.2.4 (Sands, Sauer, Woodrow [14])**.** *Let T be a 3-arc-colored tournament whose vertices can be partitioned into disjoint blocks such that*

1. *two vertices in different blocks are always connected by a red arc.*

2. *two vertices in the same block are always connected by a blue or green arc.*

Then T contains a monochromatic sink.

The following problem was inspired by Theorem A.2.4.

Conjecture A.2.5. *Let T be a 3-arc-colored tournament colored with $\{R, B, G\}$ with no rainbow 3-cycles whose vertices can be partitioned into 2 disjoint blocks B_1 and B_2 such that*

1. *for all $a \in B_1$ and $b \in B_2$ the arc connecting a and b is either colored G or B, and*

2. *for all $a_1, a_2 \in B_1$ and $b_1, b_2 \in B_2$, the arc between a_1 and a_2 and the arc between b_1 and b_2 are R.*

Then T contains a monochromatic sink.

We remark that we have proved a special case of Conjecture A.2.5. Namely, if block A is transitive, then the conjecture is true. We do not include a proof here.

Appendix B

C++ code to support Conjecture 2.3.1

The following code written in C++ was used to verify Conjecture 2.3.1

```
using namespace std;
#include <iostream>
#include <cmath>
#include <math.h>

int RemainingAfterComp(int numPlayers, double proportion);

int main(){
   for(int k=1; k<54;k++){
      int numPlayers=4;
      double proportion=(pow(2.0,k)-1)/pow(2.0,k);
      char winner;

      for(numPlayers; numPlayers<10000000; numPlayers++){
         int TotalEliminated=0;
         int count=0;
         while(numPlayers-TotalEliminated>1){
            int eliminatedBeforeRound=TotalEliminated;
            TotalEliminated=numPlayers-RemainingAfterComp(numPlayers-TotalEliminated,p
            if(eliminatedBeforeRound==TotalEliminated){
               winner=’F’; //false
               break;
            }
            count++;
         }
         if(numPlayers-TotalEliminated==1) winner=’T’;
         int conjecture= ceil((log((double)numPlayers)/log((double)2)-1)/(k-1));
         if(winner==’F’) cout<<"No winner:  "<<numPlayers<<endl;
         if(count!=conjecture) cout<<"Doesn’t match conjecture."<<endl;
```

```
      cout<<"k= "<<k <<" and " <<" numplayers= "<< numPlayers <<endl;
    }
  }
  return 0;
}

// this function uses Theorem 2.1.8 in Chapter 2
int RemainingAfterComp(int numPlayers, double proportion){
  if(proportion>=1||proportion<=0){
    cout<<"You entered an invalid proportion."<<endl;
    return -1;
  }
  int MinScoreToStay=numPlayers*proportion; //will truncate.  "round down"
  if(MinScoreToStay <=(numPlayers-1)/2) return numPlayers;
  else return 2*numPlayers-2*MinScoreToStay-1;
}
```

Appendix C

Author's biography

Sarah Mousley graduated from Utah State University in May 2013 with a B.S. in Mathematics with an Actuarial Science Emphasis and departmental and university honors. In Fall 2013, she will begin pursuit of her PhD in Mathematics specializing in Graph Theory at the University of Illinois at Urbana-Champaign. She will do so with the support of a National Defense Science and Engineering Graduate Fellowship from the Department of Defense. Sarah hopes to become a professor where she can teach and do research. Sarah graduated Valedictorian from Riverton High school in Riverton, Utah in June 2009. She was a 2012 Barry M. Goldwater Scholar.

Bibliography

[1] L. Beasley and S. Kirkland, A note on k-primitive directed graphs, *Linear Algebra Appl.*, **373** (2003) 67-74. 13

[2] L. Beasley and C. Neal, Properties of 2-primitive tournament digraphs, *Congressus Numerantium*, **163**(2004) 33-40. 17

[3] C. Berge and P. Duchet, Recent problems and results about kernels in directed graphs, *Discrete Math.* **86** (1990), no. 1-3, 27-31. 27

[4] R. Brualdi and H. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York, 1991. 11, 13

[5] H. Galeana-Sánchez, On monochromatic paths and monochromatic cycles in edge-coloured tournaments, *Discrete Math.*, **156** (1996) 103-112. 4, 19

[6] H. Galeana-Sánchez and R. Rojas-Monroy, A counterexample to a conjecture on edge-coloured tournaments. *Discrete Math.* **282** (2004), no. 1-3, 275-276. 2, 20, 22

[7] F. Harary, F., Z. D. Norman, and D. Cartwright, Structural Models: An Introduction to the Theory of Directed Graphs, *John Wiley & Sons, Inc., New York-London-Sydney* 1965 ix+415 pp. 1965. 6

[8] G. R. T. Hendry, Extending cycles in graphs, *Discrete Mathematics*, **85**(1990) 59-72. 15

[9] H. G. Landau, On dominance relations and the structure of animal societies. III: The condition for a score structure, *Bull. Math. Biophys.* **15** (1953), 143-148. 5

[10] M. Melcher and K.B. Reid, Monochromatic sinks in arc-colored tournaments with small feedback arc sets. *Bull. Inst. Combin. Appl.* **62** (2011), 79-89. 26

[11] M. Melcher and K. B. Reid, Monochromatic sinks in nearly transitive arc-colored tournaments, *Discrete Math.* **310** (2010), no. 20, 2697-2704. 4, 20, 26, 28

[12] D. Olesky, B. Shader, and P. van den Driessche, Exponents of tuples of nonnegative matrices, *Linear Algebra Appl.*, **363** (2003) 275-293. 13

[13] L. Pastrana-Ramrez and M. Snchez-Lpez, Kernels by monochromatic directed paths in 3-colored tournaments and quasi-tournaments. *Int. J. Contemp. Math. Sci.* **5** (2010), no. 33-36, 1689-1704.

[14] B. Sands, N. Sauer, R. Woodrow, On monochromatic paths in edge-coloured digraphs, *J. Combin. Theory Ser. B*, **33** (1982) 271-275. 1, 2, 4, 18, 19, 28, 32

[15] M. G. Shen, On monochromatic paths in m-coloured tournaments, *J. Combin. Theory Ser. B*, **45** (1988) 108-111. 2, 19