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**ON INFINITE-HORIZON MINIMUM-COST HEDGING
UNDER CONE CONSTRAINTS**

by

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Kevin X.D. Huang

ABSTRACT

We prove there exists and analyze a strategy that minimizes the cost of hedging a liability stream in infinite-horizon incomplete security markets with a type of constraints that feasible portfolio strategies form a convex cone. We provide a theorem that extends Stiemke Lemma to over cone domains and we use the result to construct a series of primal-dual problems. Applying stochastic duality theory, dynamic programming technique and the theory of convex analysis to the dual formulation, we decompose the infinite-horizon dynamic hedging problem into one-period static hedging problems such that optimal portfolios in different events can be solved for independently.

JEL classification: C61, C63, G10, G20

Key words: infinite horizon; minimum-cost hedging; cone constraints

ON INFINITE-HORIZON MINIMUM-COST HEDGING UNDER CONE CONSTRAINTS¹

1. Introduction

A market participant often needs to provide for a stream of payments stemming from contingent liability claims. Failure to meet such a claim may cause financial distress and insolvent liquidation. Two recent such tragedies are the bankruptcy of Barings Bank and of Orange County, both resulting from non-covered speculations in security markets.

What can the market participant do to reduce the default risk? The answer is, hedging. Hedging is a portfolio strategy that generates a payoff stream at least as large as the liability stream, so that it offsets the default risk. In general, there may exist multiple portfolio strategies that can serve to hedge the given liability stream. In such case the market participant may wish to find the least expensive such strategy, which is referred to as a minimum-cost hedging strategy.

Cost minimization is often adopted in the literature as an optimality criterion. The main advantage of this criterion is that the optimal solutions are independent of preferences and of probability beliefs of market participants. Edirisinghe, Naik and Uppal (1993) and Naik and Uppal (1994) provide extensive discussions about other favorable attributes of the cost-minimization criterion and its relation to the utility-maximization approach.

In finite-horizon complete frictionless markets, a simple strategy of replicating the underlying liability stream provides the minimum-cost hedging at any security prices, as long as there are no arbitrage opportunities. Black and Scholes (1973), Merton (1973) and Cox, Ross and Rubinstein (1979) pioneer this approach in their classic work on hedging and valuation of call and put options.

Recent research work has relaxed the assumptions that markets are complete and frictionless. In such generalized environment a liability stream desired to be hedged may be not marketed and, even if it is marketed, exact replication may no longer provide the least expensive hedging. Aliprantis, Brown and Werner (2000)

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characterize a class of incomplete market structures in a two-period model of portfolio insurance in which minimum-cost portfolio insurance is price independent and can be obtained by replicating the insured payoff on a set of fundamental states. Jouini and Kallal (1995) and Luttmer (1996) analyze the minimum cost of replicating a marketed payoff in a two-period security trading model under a constraint that feasible portfolios form a convex cone. Naik and Uppal (1994) and Broadie, Cvitanic and Soner (1998) study minimum-cost hedging in multi-period complete markets in the presence of margin requirements on stocks and bonds. There is also a growing literature on finite-horizon minimum-cost hedging with transaction costs (e.g., Garman and Ohlson (1981), Bensaid, Lesne, Pages and Scheinkman (1991), Edirisinghe, Naik and Uppal (1993) and Jouini and Kallal (1995)). In these studies, two types of algorithms have been developed for solving the minimum-cost hedging problem in finite-horizon complete markets with transaction costs and/or margin requirements. One type of algorithms requires solving a system of simultaneous equations, one for each event, such that optimal portfolios in different events must be solved for simultaneously. The other type of algorithms involves a backward recursion procedure such that to solve for an optimal portfolio in an event requires finding first the optimal portfolios in all subsequent events.

More recently, the assumption that markets are of finite horizon is also relaxed. Santos and Woodford (1997) characterize the minimum cost of hedging a liability stream in infinite-horizon markets with a constraint that portfolio net worth be nonnegative. Huang and Werner (2000) provide an extension of their result to a broader class of constraints in markets with no uncertainty (see Huang (2000) in an uncertainty setting). Limited feasible arbitrage can exist in equilibrium with a constraint belonging to that class (see also LeRoy and Werner (2000)). In this paper we solve the minimum-cost hedging problem in infinite-horizon incomplete security markets in the presence of cone constraints on portfolio strategies. The

type of constraints considered here nests as special cases many portfolio restrictions often encountered, including margin requirements and target security proportions that are not considered by Huang and Werner (2000) or Huang (2000). With this type of constraints there cannot exist feasible arbitrage in equilibrium so that minimum-cost hedging at equilibrium prices is necessarily arbitrage-free hedging.

Our analysis is general enough to allow for an abstract convex cone constraint, general incomplete security markets, and an open-ended infinite horizon. The simultaneous equation approach and the backward recursion method developed in the previous studies for solving the minimum-cost hedging problem in finite-horizon markets are not applicable here because of such generality. Such method requiring differentiability as that of Santos and Woodford (1997) too becomes awkward. We take here an approach that combines stochastic duality technique, dynamic programming principle and the theory of convex analysis to establish the existence of a minimum-cost hedging strategy under the condition of no feasible arbitrage and to solve for the optimal portfolios in different events independently without differentiability.

Our approach relies on the extension of a mathematical result, Stiemke Lemma, to over cone domains. We provide a theorem that establishes such extension and we apply the result to derive admissible stochastic discount factors in infinite-horizon security markets with convex cone constraints. We use these admissible stochastic discount factors to construct a series of primal-dual problems, one pair for each event. Applying the aforementioned theory and technique and using a continuity argument, we decompose the original infinite-horizon dynamic hedging problem into independent one-period static hedging problems, one for each event. Independence means that optimal portfolios in different events can be obtained separately yet function together as a whole in forming a minimum-cost hedging strategy. The minimum hedging cost is shown to be equal to the greatest present

value of the liability stream with respect to the admissible stochastic discount factors.

The paper is organized as follows. In Section 2 we introduce basic concepts of sequential markets such as cone constraints and feasible arbitrage. In Section 3 we present our theorem that extends Stiemke Lemma to over cone domains and we apply the theorem to obtain admissible stochastic discount factors in infinite-horizon markets with cone constraints. In Section 4 we solve the minimum-cost hedging problem under a general convex cone constraint. In Section 5 we show how to apply our results to several portfolio constraints considered in the literature. Section 6 concludes. All proofs are contained in the Appendix.

2. Sequential Markets, Cone Constraints, and Feasible Arbitrage

Time is discrete with infinite horizon, begins at $t = 0$. Dynamic uncertainty is described by a set Ω of states of the nature and an increasing sequence $\{N_t\}_{t=0}^\infty$ of finite information partitions with $N_0 = \{\Omega\}$. This uncertainty environment can be interpreted as an event tree D where an event $s^t \in N_t$ identifies a node of the tree. For each s^t , we denote by s_-^t its unique immediate predecessor if $t \neq 0$, $\{s_+^t\}$ a finite set of its immediate successors, $D(s^t)$ a subtree with root s^t , and $D(s^t) \setminus \{s^t\}$ the subtree excluding the root.

In each event there is a finite number of securities that are traded in exchange for consumption in that event. We denote by $d_i(s^t)$ a dividend paid before trade at s^t to the holder of one share of a security i that is traded at s_-^t , $q_i(s^t)$ an ex-dividend price at s^t of i , and $R_i(s^{t+1})$ the payoff of holding one share of i from s^t into $s^{t+1} \in \{s_+^t\}$. It follows that $R_i(s^{t+1}) = q_i(s^{t+1}) + d_i(s^{t+1})$ if i continues to be traded at s^{t+1} for price $q_i(s^{t+1})$, and $R_i(s^{t+1}) = d_i(s^{t+1})$ if i is liquidated at s^{t+1} . In any event new securities can be issued while existing securities can be liquidated so that the price vector $q(s^t)$ and the payoff vector $R(s^t)$ may have

different dimensions. We allow negative dividends to allow for securities that are not of limited liability. As such, neither prices q nor payoffs R are presumed nonnegative.

A portfolio strategy is described by a vector-valued adapted process specifying the number of shares of securities to be held in each event after trade. Let $\Theta(s^t)$ be a set of feasible portfolios in event s^t . We assume that $\Theta(s^t)$ is a polyhedral cone, i.e., the intersection of a finite number of supporting half-spaces of an Euclidean space that is stable under addition and multiplication by nonnegative real numbers. The polar cone of $\Theta(s^t)$ is defined as $\Theta(s^t)^* = \{\vartheta(s^t) : \vartheta(s^t)' \theta(s^t) \geq 0, \forall \theta(s^t) \in \Theta(s^t)\}$, and thus is a polyhedral cone as well. The Cartesian product $\Theta = \prod_{s^t \in D} \Theta(s^t)$ is then a convex cone. A portfolio strategy θ is feasible with respect to Θ if and only if each portfolio $\theta(s^t)$ is feasible with respect to $\Theta(s^t)$. Examples of $\Theta(s^t)$ are

- short-sales constraint: $\theta_i(s^t) \geq 0$ for all i , and for all s^t ;
- nonnegativity of portfolio net worth: $q(s^t)' \theta(s^t) \geq 0$ for all s^t ;
- margin requirements: $q_i(s^t) \theta_i(s^t) \geq -m_i(s^t) q(s^t)' \theta(s^t)$
for some $m_i(s^t) \geq 0$, for all i , and for all s^t ;
- target security proportions: $q_i(s^t) \theta_i(s^t) \geq t_{ij}(s^t) q_j(s^t) \theta_j(s^t)$
for some $t_{ij}(s^t) > 0$, for all i and j , $i \neq j$, and for all s^t .

Margin requirements specify the maximum amount of each security that a market participant can short-sell as a percentage of its portfolio net worth. Compared to other types of constraints, margin requirements capture the participant's ability to increase short-sales or borrowing as a function of its creditworthiness, a prominent feature of security markets. Cox and Rubinstein (1985, p.98), Chance (1991, p.55), Smith, Proffitt and Stephens (1992, p.69) and Robertson (1990) provide in-depth discussions about margin requirements on stocks, bonds and futures contracts.

Target security proportions require that the ratio of the market value of a security to the value of another security in the portfolio be maintained within a desired range. Corporations, funds and financial institutions are typically required to retain certain security proportions. Taggart (1977) and Marsh (1982) provide such classical examples as target debt to equity ratios (see also Constantinides (1986), Dumas and Luciano (1991) and Leland (1996)).

The payoff stream of a feasible portfolio strategy θ is denoted z^θ and is given by $z^\theta(s^t) = R(s^t)' \theta(s^t_-) - q(s^t)' \theta(s^t)$ for all $s^t \neq s^0$. A feasible arbitrage is a feasible portfolio strategy θ such that, $q(s^0)' \theta(s^0) \leq 0$ and $z^\theta(s^t) \geq 0$ for all $s^t \neq s^0$, with at least one strict inequality.

With cone constraints on portfolio strategies and monotone preferences, there cannot exist feasible arbitrage in equilibrium. This is so since adding a feasible arbitrage on top of any feasible portfolio plan would create a feasible portfolio strategy, which provides a market participant with more consumption in some event without decreasing the consumption in any other event. Consequently, we can solve the minimum-cost hedging problem under the condition of the absence of feasible arbitrage without making explicit use of utility maximization or market equilibrium.

3. A Result on Stiemke Lemma over Cone Domains

The well-known Stiemke Lemma plays an important role in the theory of financial markets with a finite horizon and no trading frictions. The lemma implies that the absence of arbitrage is equivalent to the existence of strictly positive event prices, i.e., admissible stochastic discount factors, in a two-period frictionless security market (e.g., LeRoy and Werner (2000, p.69)).

This result is essential for deriving admissible stochastic discount factors under trading restrictions that feasible portfolios form a polyhedral cone. The following

theorem extends Stiemke Lemma to over cone domains.

Theorem 3.1. *Let k and m be two positive integers, let p be a k -dimensional vector, let G be a $k \times m$ matrix, let C be a polyhedral cone in $\mathbb{R}^k$, and let C^* be the polar cone of C . The following two conditions are equivalent: (i) there does not exist $x \in C$ such that $G'x \geq 0$ and $p'x \leq 0$, with at least one strict inequality; (ii) there exists $\alpha \in \mathbb{R}_{++}^m$ such that $p - G\alpha \in C^*$.*

The proof of Theorem 3.1 is contained in the Appendix. The proof involves an application of Tucker's Theorem of Alternatives (e.g., Tucker (1952)). In light of Theorem 3.1, in a two-period market with m possible events in the second date, k securities with prices p and payoffs G , and a portfolio constraint given by C , there is no feasible arbitrage if and only if there exist admissible stochastic discount factors α characterized by (ii). In the special case with $C = \mathbb{R}^k$, corresponding to the situation with no market frictions, the theorem reduces to Stiemke Lemma.

Theorem 3.1 can be applied to derive admissible stochastic discount factors in infinite-horizon markets with cone constraints. To do this in our present model, set in the theorem $C = \Theta(s^t)$, $G = [R(s^{t+1})]_{s^{t+1} \in \{s_+^t\}}$ and $p = q(s^t)$. Then (i) in the theorem is equivalent to the condition of no feasible arbitrage involving nonzero security holdings only at s^t , and (ii) is equivalent to the existence of strictly positive numbers $\{a(s^t), a(s^{t+1}), s^{t+1} \in \{s_+^t\}\}$ satisfying

$$\left\{ q(s^t) - \sum_{s^{t+1} \in \{s_+^t\}} [a(s^{t+1})/a(s^t)] R(s^{t+1}) \right\} \in \Theta(s^t)^*, \quad (1)$$

where $\Theta(s^t)^*$ is the polar cone of $\Theta(s^t)$. Note that the ratios $[a(s^{t+1})/a(s^t)]$ in (1) correspond to the admissible stochastic discount factors α in Theorem 3.1. Since only these ratios are restricted by (1), the absence of feasible arbitrage allows one to derive a system of admissible stochastic discount factors $\{a(s^t)\}_{s^t \in D}$ that are consistent with (1) in every event. We denote by A the set of all such systems.

4. Solving the Minimum-Cost Hedging Problem

In this section we solve the minimum-cost hedging problem under the condition of the absence of feasible arbitrage. Denote by $z \geq 0$ a liability stream for which there is a feasible portfolio strategy θ such that $z^\theta \geq z$. The objective here is to prove the existence of and to obtain a feasible portfolio strategy that solves the problem $m(z) \equiv \inf\{q(s^0)'\theta(s^0) : z^\theta \geq z, \theta \in \Theta\}$, and to determine $m(z)$. The following result is related to Theorem 4.1 in Huang (2000). The proof of the result is again given in the Appendix. The proof makes use of the admissible stochastic discount factors derived in Section 3, stochastic duality technique, dynamic programming principle and the theory of convex analysis.

Theorem 4.1. *If there is no feasible arbitrage, then there exists a minimum-cost hedging strategy that can be obtained by solving at each s^t the following one-period static hedging problem*

$$\min_{\theta(s^t)} q(s^t)'\theta(s^t) \tag{2}$$

$$\begin{aligned} \text{s.t.} \quad & R(s^{t+1})'\theta(s^t) \geq \sup_{a \in A} \sum_{s^\tau \in D(s^{t+1})} [a(s^\tau)/a(s^{t+1})] z(s^\tau), \\ & s^{t+1} \in \{s_+^t\}, \theta(s^t) \in \Theta(s^t). \end{aligned} \tag{3}$$

Moreover, the minimum hedging cost is given by

$$m(z) = \sup_{a \in A} \sum_{s^t \in D \setminus \{s^0\}} [a(s^t)/a(s^0)] z(s^t). \tag{4}$$

Therefore, for any liability stream that can be hedged by a feasible portfolio strategy at arbitrage-free security prices, there exists a strategy that does so at the minimum cost. This minimum-cost hedging strategy can be obtained by solving a series of static hedging problems (2)-(3). Since in any event only a finite number of

securities are traded, each of these problems is a finite-dimensional convex program which minimizes the cost of a feasible portfolio in an event such that its one-period payoffs in immediate succeeding events exceed the minimum costs of hedging future liabilities. The minimum hedging cost is equal to the greatest present value of the future liabilities with respect to admissible stochastic discount factors.

The novelty of this result is that a solution to the static hedging problem in an event can be obtained without finding the solutions to the problems in other events. Thus optimal portfolios in different events can be solved for independently yet function together as a whole in hedging the (potentially infinite) liability stream at the minimum cost. In addition, equation (4) implies that if one's goal is merely to determine the minimum hedging cost, then one can accomplish the goal without actually solving for a minimum-cost hedging strategy.

5. Applications

To apply Theorem 4.1, one needs to use security market data q , R and Θ to identify admissible stochastic discount factors. Examples of admissible stochastic discount factors from $t + 1$ to t in an equilibrium model of security trading with sufficiently strong assumptions about preferences of consumers are intertemporal marginal rates of substitutions of the consumers who can purchase portfolios $\theta(s^t) \in \Theta(s^t)$ at prices $q(s^t)'\theta(s^t)$ in event s^t with payoffs $R(s^{t+1})'\theta(s^t)$ in events $s^{t+1} \in \{s_+^t\}$. In our present arbitrage pricing model with no explicit specification of preferences, the set of the systems of admissible stochastic discount factors A is completely characterized by (1). Since each $\Theta(s^t)^*$ is a polyhedral cone, A is determined by a system of linear inequalities.

In this section we derive the system of linear inequalities determining A for the four types of portfolio constraints introduced in Section 2. To help exposition we assume that in each event two securities are traded and both have positive prices,

and we denote by $r_i(s^t) = R_i(s^t)/q_i(s_-^t)$ the one-period rate of return on a security i that is traded in event s_-^t , for $s^t \neq s^0$, and for $i = 1, 2$.

- Linear inequalities determining A under short-sales constraint:

$$\sum_{s^{t+1} \in \{s_+^t\}} r_i(s^{t+1}) \left[\frac{a(s^{t+1})}{a(s^t)} \right] \leq 1, \quad a > 0, \quad i = 1, 2, \quad \forall s^t;$$

- Linear inequalities determining A with nonnegativity of portfolio net worth:

$$\sum_{s^{t+1} \in \{s_+^t\}} r_1(s^{t+1}) \left[\frac{a(s^{t+1})}{a(s^t)} \right] = \sum_{s^{t+1} \in \{s_+^t\}} r_2(s^{t+1}) \left[\frac{a(s^{t+1})}{a(s^t)} \right] \leq 1, \quad a > 0, \quad \forall s^t;$$

- Linear inequalities determining A under margin requirements:

$$\sum_{s^{t+1} \in \{s_+^t\}} \left\{ [1 + m_i(s^t)] r_j(s^{t+1}) - m_i(s^t) r_i(s^{t+1}) \right\} \left[\frac{a(s^{t+1})}{a(s^t)} \right] \leq 1, \\ a > 0, \quad i, j = 1, 2, \quad i \neq j, \quad \forall s^t;$$

- Linear inequalities determining A with target security proportions:

$$\sum_{s^{t+1} \in \{s_+^t\}} \left[\frac{t_{ij}(s^t) r_i(s^{t+1}) + r_j(s^{t+1})}{t_{ij}(s^t) + 1} \right] \left[\frac{a(s^{t+1})}{a(s^t)} \right] \leq 1, \\ a > 0, \quad i, j = 1, 2, \quad i \neq j, \quad \forall s^t.$$

In deriving the above systems of linear inequalities that determine A under the four portfolio constraints, first we use the definition of polar cone to obtain $\Theta(s^t)^*$ for each of the constraints, and then we apply relation (1).

6. Concluding Remarks

In this paper we study the problem of hedging a liability stream at minimum cost in infinite-horizon incomplete security markets with convex cone constraints on portfolio strategies. We prove the existence of a minimum-cost hedging strategy

under the condition of no feasible arbitrage, and we solve for the strategy by solving a series of independent one-period hedging problems. We show that the minimum hedging cost can be computed directly from security market data and information about the liability stream without finding an optimal hedging strategy.

An attractive feature of our results is that they are independent of preferences and of probability beliefs and they are applicable to an arbitrary liability stream. The results are useful in providing for solutions to other types of problems as well. Our results determine, for example, the highest price a market participant is willing to pay for a desired yet non-marketed payoff stream that it has to buy over-the-counter from an investment bank. For a corporation that needs to hedge a liability stream it has issued in financing its production plans, our results provide a profit-maximizing portfolio strategy for the corporation. Furthermore, the results here should be useful more generally in the characterization of budget sets and equilibria in a utility-maximization model of security trading.

Appendix

Proof of Theorem 3.1: Let $\bar{G}$ be the $k \times (m+1)$ matrix consisting of G amended by adding vector $-p$ as an additional column, i.e., $\bar{G} = [G \ -p]$. Since C is a polyhedral cone, there is a nonnegative integer n and a $k \times n$ matrix H such that, $x \in C$ if and only if $H'x \geq 0$. Suppose that (i) holds. Then, the system $\bar{G}'x \geq 0$ and $H'x \geq 0$ does not have a solution $x \in \mathbb{R}^k$ with $\bar{G}'x \neq 0$. By Tucker's Theorem of Alternatives, there exist $\bar{\alpha} \in \mathbb{R}_{++}^{m+1}$ and $\bar{\beta} \in \mathbb{R}_+^n$ such that $\bar{G}\bar{\alpha} + H\bar{\beta} = 0$. Denote by $\alpha \in \mathbb{R}_{++}^m$ the m -dimensional vector consisting of the first m elements of $\bar{\alpha}$, and denote by $\beta \in \mathbb{R}_+^n$ the n -dimensional vector $\bar{\beta}$, both normalized by the last element of $\bar{\alpha}$. It follows that $p - G\alpha = H\beta$. Since $(H\beta)'x = \beta'(H'x) \geq 0$ for all $x \in C$, we have $H\beta \in C^*$ by the definition of polar cone. Thus (ii) holds.

Suppose now that (ii) holds. Suppose, by contradiction, that there is $x \in C$

such that $G'x \geq 0$ and $p'x \leq 0$, with at least one strict inequality. Since $\alpha \in \mathbb{R}_{++}^m$, we have $0 \leq (p - G\alpha)'x = (p' - \alpha'G')x = p'x - \alpha'(G'x) < 0$, where we have again used the definition of polar cone. This is a contradiction. Thus (i) holds. $\square$

Proof of Theorem 4.1: The absence of feasible arbitrage implies that $A \neq \emptyset$ in light of the analysis in Section 3. Given that $z \geq 0$, it also implies that $q(s^t)' \theta(s^t) \geq 0$ for all s^t and for all feasible portfolio strategy θ with $z^\theta \geq z$. To prove this latter statement suppose, by contradiction, that $q(s^t)' \theta(s^t) < 0$ for some s^t . If $t = 0$, then θ is a feasible arbitrage which contradicts the assumption that there exists no feasible arbitrage. Suppose that $t > 0$, and consider a portfolio strategy $\tilde{\theta}$ that is equal to θ on $D(s^t)$ and is equal to zero security holdings elsewhere. Since zero security holdings are feasible with respect to $\Theta(s^t)$ for any s^t , $\tilde{\theta}$ is feasible with respect to Θ . Since $q(s^0)' \tilde{\theta}(s^0) = 0$, $z^{\tilde{\theta}}(s^t) = -q(s^t)' \theta(s^t) > 0$, $z^{\tilde{\theta}}(s^\tau) = z(s^\tau) \geq 0$ for all $s^\tau \in D(s^t) \setminus \{s^t\}$, and $z^{\tilde{\theta}}(s^\tau) = 0$ for every other s^τ , $\tilde{\theta}$ is a feasible arbitrage. A contradiction.

We can use the above result to show that

$$m(z) \geq \sup_{a \in A} \sum_{s^t \in D \setminus \{s^0\}} [a(s^t)/a(s^0)] z(s^t). \quad (5)$$

Let θ be a feasible portfolio strategy with $z^\theta \geq z$ and choose an arbitrary $a \in A$. By the definition of polar cone, the inner product of $\theta(s^t)$ and the left-hand side of (1) is nonnegative. Making use of this and $z^\theta \geq z$ repeatedly, we obtain

$$q(s^0)' \theta(s^0) \geq \sum_{t=1}^{\tau} \sum_{s^t \in N_t} \frac{a(s^t)}{a(s^0)} z(s^t) + \sum_{s^\tau \in N_\tau} \frac{a(s^\tau)}{a(s^0)} q(s^\tau)' \theta(s^\tau) \geq \sum_{t=1}^{\tau} \sum_{s^t \in N_t} \frac{a(s^t)}{a(s^0)} z(s^t)$$

for any $\tau \geq 1$, where the second inequality holds since $q(s^\tau)' \theta(s^\tau) \geq 0$ for all τ and all s^τ . Taking $\tau \rightarrow \infty$ on the right-hand side of this second inequality leads to $q(s^0)' \theta(s^0) \geq \sum_{s^t \in D \setminus \{s^0\}} [a(s^t)/a(s^0)] z(s^t)$. In the above inequality, taking the supremum on the right-hand side over all $a \in A$ and taking the infimum on the left-hand side over all feasible portfolio strategy θ with $z^\theta \geq z$ yield (5).

To prove the theorem, it remains to show that a portfolio strategy obtained by solving (2)-(3) at each s^t finances a payoff stream that is larger than or equal to z and has a date-0 price equal to the right-hand side of (5). To proceed, consider the following dual problem of (2)-(3):

$$\max_{\{\alpha(s^{t+1})\}} \sum_{s^{t+1} \in \{s_+^t\}} \alpha(s^{t+1}) \sup_{a \in A} \sum_{s^\tau \in D(s^{t+1})} [a(s^\tau)/a(s^{t+1})] z(s^\tau) \quad (6)$$

$$\begin{aligned} \text{s.t.} \quad & [q(s^t) - \sum_{s^{t+1} \in \{s_+^t\}} \alpha(s^{t+1}) R(s^{t+1})] \in \Theta(s^t)^*, \\ & \alpha(s^{t+1}) \geq 0, \quad s^{t+1} \in \{s_+^t\}. \end{aligned} \quad (7)$$

Since $A \neq \emptyset$, (7) has a feasible solution. The fact that (3) has a feasible solution can be demonstrated using an argument similar to that in the previous paragraph. It follows from the duality theorem of convex programming that both the primal problem and the dual problem have finite optimal solutions, and the values of their optimal objectives are equal. Since $A \neq \emptyset$, $\Theta(s^t)^*$ is a cone, and (6) is continuous in $\alpha(s^{t+1})$, the dual problem can be rewritten as

$$\begin{aligned} \sup_{\{\alpha(s^{t+1})\}} \sum_{s^{t+1} \in \{s_+^t\}} \alpha(s^{t+1}) \sup_{a \in A} \sum_{s^\tau \in D(s^{t+1})} [a(s^\tau)/a(s^{t+1})] z(s^\tau) \\ \text{s.t.} \quad & [q(s^t) - \sum_{s^{t+1} \in \{s_+^t\}} \alpha(s^{t+1}) R(s^{t+1})] \in \Theta(s^t)^*, \\ & \alpha(s^{t+1}) > 0, \quad s^{t+1} \in \{s_+^t\}. \end{aligned}$$

The value of the optimal objective of the above program is equal to

$$\sup \sum_{s^{t+1} \in \{s_+^t\}} [a(s^{t+1})/a(s^t)] \sup_{a \in A} \sum_{s^\tau \in D(s^{t+1})} [a(s^\tau)/a(s^{t+1})] z(s^\tau),$$

where the outer supremum is taken over all admissible stochastic discount factors $\{a(s^{t+1})/a(s^t)\}$ characterized by (1). By a dynamic programming argument, the value of this optimal objective is equal to $\sup_{a \in A} \sum_{s^\tau \in D(s^t) \setminus \{s^t\}} [a(s^\tau)/a(s^t)] z(s^\tau)$. This together with (3) imply that, a feasible portfolio strategy θ obtained by solving

(2)-(3) at each s^t has a date-0 price $q(s^0)' \theta(s^0)$ equal to the right-hand side of (5) and finances a payoff stream $z^\theta \geq z$. This coupled with (5) says that θ is a feasible portfolio strategy that hedges z at minimum cost, and that the minimum hedging cost is given by (4). $\square$

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On Infinite-Horizon Minimum-Cost Hedging under Cone Constraints*

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ABSTRACT. We prove there exists and analyze a strategy that minimizes the cost of hedging a liability stream in infinite-horizon incomplete security markets with a type of constraints that feasible portfolio strategies form a convex cone. We provide a theorem that extends Stiemke Lemma to over cone domains and we use the result to construct a series of primal-dual problems. Applying stochastic duality theory, dynamic programming technique and the theory of convex analysis to the dual formulation, we decompose the infinite-horizon dynamic hedging problem into one-period static hedging problems such that optimal portfolios in different events can be solved for independently.

Key Words: infinite horizon, minimum-cost hedging, cone constraints.

JEL Classification: C61, C63, G10, G20.

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1. Introduction

A market participant often needs to provide for a stream of payments stemming from contingent liability claims. Failure to meet such a claim may cause financial distress and insolvent liquidation. Two recent such tragedies are the bankruptcy of Barings Bank and of Orange County, both resulting from non-covered speculations in security markets.

What can the market participant do to reduce the default risk? The answer is, hedging. Hedging is a portfolio strategy that generates a payoff stream at least as large as the liability stream, so that it offsets the default risk. In general, there may exist multiple portfolio strategies that can serve to hedge the given liability stream. In such case the market participant may wish to find the least expensive such strategy, which is referred to as a minimum-cost hedging strategy.

Cost minimization is often adopted in the literature as an optimality criterion. The main advantage of this criterion is that the optimal solutions are independent of preferences and of probability beliefs of market participants. Edirisinghe, Naik and Uppal (1993) and Naik and Uppal (1994) provide extensive discussions about other favorable attributes of the cost-minimization criterion and its relation to the utility-maximization approach.

In finite-horizon complete frictionless markets, a simple strategy of replicating the underlying liability stream provides the minimum-cost hedging at any security prices, as long as there are no arbitrage opportunities. Black and Scholes (1973), Merton (1973) and Cox, Ross and Rubinstein (1979) pioneer this approach in their classic work on hedging and valuation of call and put options.

Recent research work has relaxed the assumptions that markets are complete and frictionless. In such generalized environment a liability stream desired to be hedged may be not marketed and, even if it is marketed, exact replication may no longer provide the least expensive hedging. Aliprantis, Brown and Werner (2000)