The Schwarzschild Solution and Timelike Geodesics

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Abstract

Using the techniques of Riemannian Geometry and General Relativity this work will derive the Schwarzschild metric tensor field by solving Einstein's Equations in a vacuum from a general spherically symmetric metric tensor field. The derived Schwarzschild metric tensor field will be used to calculate the motion of massive particles orbiting around and falling into a static, spherical black hole by solving the geodesic equation. The radial in fall of a particle shows the proper time for these massive particles to fall into and orbit around a black hole. Looking at circular orbits, it is shown that there is a limit for stable circular orbits for massive particles. This limit occurs at a radius of 3 times the Schwarzschild radius (R_s) where there is an unstable orbit possible at $3R_s$. There are no stable orbits inside of that.

Derivation

Starting with the general metric tensor field:

$$g_{\mu\nu} = \begin{pmatrix} -f^2(r) & 0 & 0 & 0 \\ 0 & h^2(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix}$$

The following steps are made to solve the Einstein equation in a vacuum [1]:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$$

$$R^{\alpha}_{\ \beta\mu\nu} = \Gamma^{\alpha}_{\ \beta\nu,\mu} - \Gamma^{\alpha}_{\ \beta\mu\nu} + \Gamma^{\alpha}_{\ \mu\rho}\Gamma^{\rho}_{\ \beta\nu} - \Gamma^{\alpha}_{\ \nu\rho}\Gamma^{\rho}_{\ \beta\mu}$$

$$R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

By taking the trace of the Einstein Tensor, it can be shown that $R = 0$ and thus the Einstein Equation can be solved by setting $R_{\mu\nu} = 0$. Solving these four differential equations yields: $f^2(r) = \left(1 - \frac{1}{Kr}\right)$ and $h^2(r) = \left(1 - \frac{1}{Kr}\right)^{-1}$

where K is from the integration constants.

By plugging these functions into the geodesic equation for large values of r, the functions $f^2(r)$ and $h^2(r)$ should look like Newton's laws of gravity. Doing this shows that:

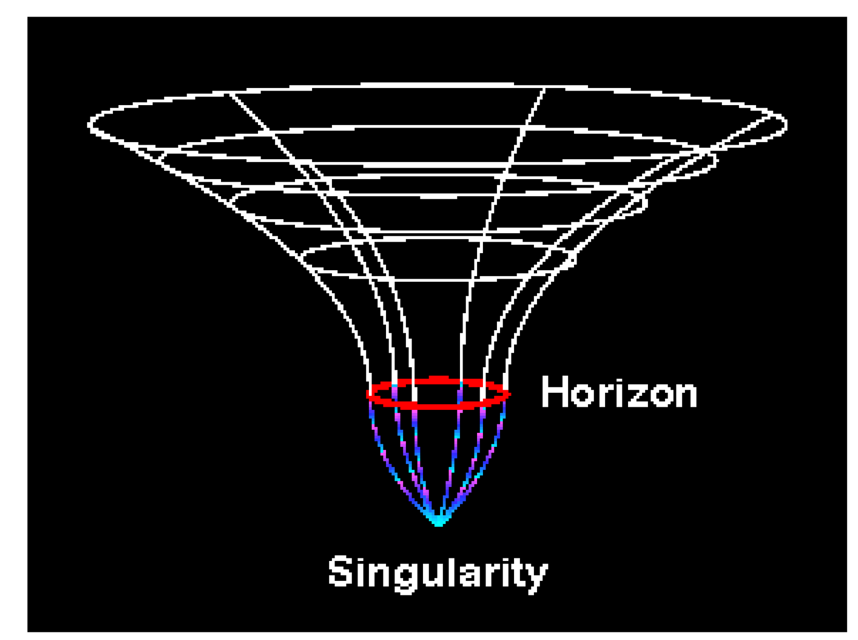
$$K = \frac{c^2}{2GM}$$

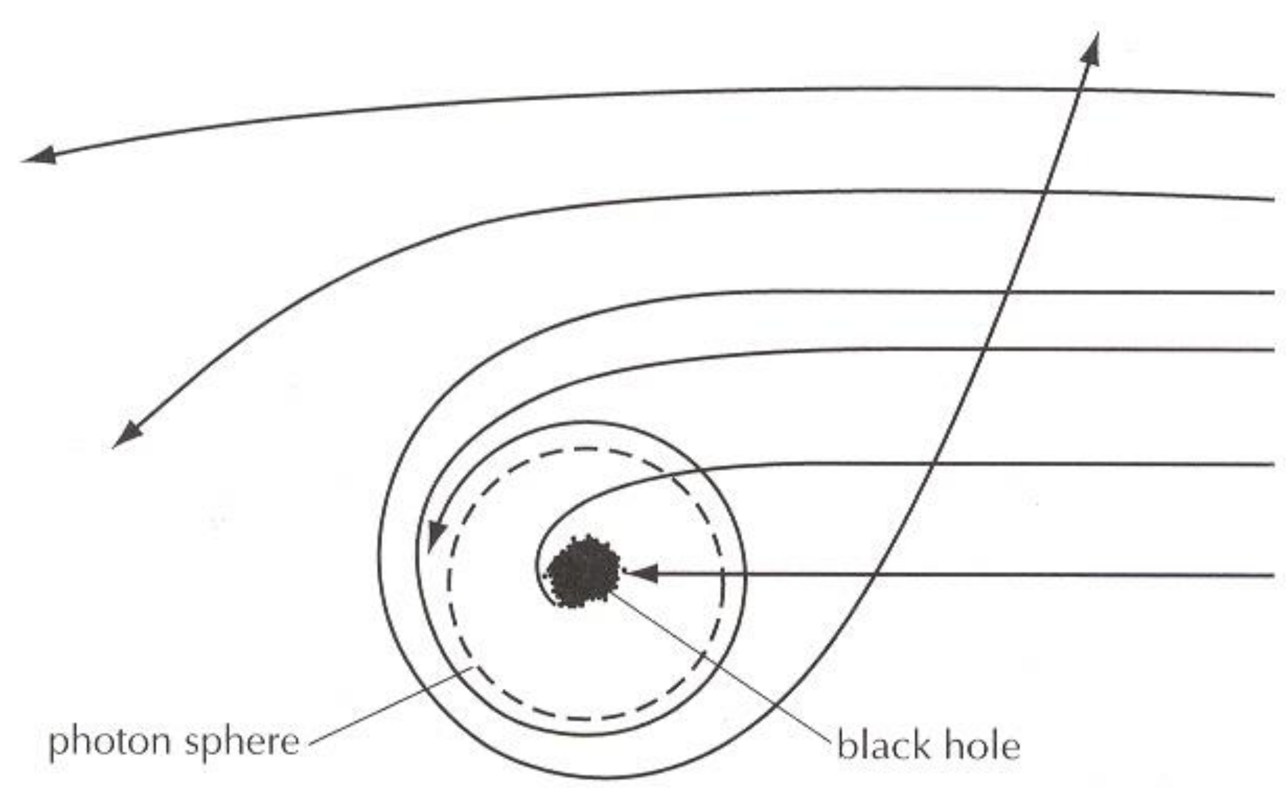
which is the inverse of the Schwarzschild radius. Plugging this value back into our original equations, the Schwarzschild metric tensor field can be shown to be:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{rc^2}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{rc^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix}$$

Background

The equations used are commonly found in Riemannian geometry and are closely related. Finding components of each symbol and tensor allows for further calculations. Once a metric tensor field is found with its corresponding Christoffel symbols, equations of motion can be calculated by using initial conditions and the geodesic equation.





Future Plans

I would like to continue similar geodesic calculations with photons and find null geodesics. With these null geodesics, it would be interesting to find if it is possible to have a photon or a beam of light travel around a black hole for several complete orbits and then leave the orbit. This is similar to what spacecraft do for longer trips into the cosmos. It would also be interesting to investigate the nature of geodesics around exotic forms of mass such as genus 1 and genus 2 black holes.

Calculation

The geodesic equation is:

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^{\alpha}_{\ \beta\mu} u^\beta u^\mu$$

Where x^α is the four-position vector and u^α is the four-velocity. Derivatives are taken with respect to proper-time where the line element is:

$$-d\tau^2 = ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2$$

The units are chosen with $G = c = 1$, so $M = \frac{GM}{c^2}$.

The first calculation is radial infall with no initial velocity. The initial conditions are as follows:

$$x_0^\alpha = \left(0, r_0, \frac{\pi}{2}, 0\right) \qquad u_0^\alpha = \left(u_0^0, 0, 0, 0\right)$$

The geodesic equations are integrated with these initial conditions. The θ and ϕ components are found to remain constant, $\frac{d\theta}{d\tau} = \frac{d\varphi}{d\tau} = 0$.

Integrating the time and radial components yield the 4-velocity:

$$u^0 = \frac{1}{u_0^0}\left(1 - \frac{2M}{r}\right)^{-1} \qquad u^r = \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r} - \frac{2M}{r_0}}$$

By integrating the radial component we find the proper time that elapses for a particle falling from an initial position r_0 to the Schwarzschild radius [2]. The result, putting back in units for G and c is:

$$\tau = \frac{r_0}{c}\left(\sqrt{\frac{r_0c^2}{2GM}}\left(\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{2GM}{r_0c^2}}\right) + \sqrt{1 - \frac{2GM}{r_0c^2}}\right)$$

For a particle which starts far from the horizon, $r_0 \gg \frac{2MG}{c^2}$, this is approximately

$$\tau = \sqrt{\frac{\pi^2 r_0^3}{8GM}}$$

For a particle which starts in the neighborhood of a black hole, say at $r_0 = \frac{8GM}{c^2}$,

$$\tau = \frac{r_0}{c}\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) \approx \frac{3r_0}{c}$$

The next calculation is for a circular orbit, so the particle has a velocity in the ϕ-direction, with the radius held fixed. The initial conditions for a circular orbit are:

$$x_0^\alpha = \left(0, r_0, \frac{\pi}{2}, 0\right) \qquad u_0^\alpha = \left(u_0^0, 0, 0, u_0^3\right)$$

Integrating the geodesic equation with these initial conditions yield the 4-velocity:

$$u^\alpha = \frac{1}{\sqrt{r_0 - 3M}}\left(\sqrt{r_0}, 0, 0, \sqrt{M}\right)$$

This shows that orbits at radius 3M are unstable. Circular orbits inside of that do not exist.

References

[1]Schutz, B. (2009). A First Course in General Relativity. Cambridge, UK, Cambridge University Press.

[2]Wheeler, J. (2006) Schwarzschild Geometry. Radial Geodesics in Schwarzschild Spacetime 1-4

[3]http://hubblesite.org/explore_astronomy/black_holes/encyc_mod3_q13.html

[4]http://casa.colorado.edu/~ajsh/schwp.html

[5]http://www.patana.ac.th/secondary/science/anrophysics/relativity_option/images/photon_sphere.png