Simple Stability criteria for Uncertain Continuous-Time Linear Systems

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Abstract- **This paper mainly deals with the stability problem of continuous-time linear systems having uncertainties. Instead of using the tradition types of Lyapunov functions, this paper provides a very different method to investigate the stability of such systems. Hence, it reduces the conservativeness of having structured uncertainties belonging to convex sets. Based on a famous theorem that specifies regions containing all the eigenvalues of a complex square matrix, sufficient criteria are proposed to guarantee the asymptotic stability of linear systems. The main merit of this method is in analyzing linear systems having uncertainties. Moreover, the proposed criteria can also be used to investigate the stability of linear time-variant systems. In this case, a new stability criterion has been introduced. Finally, a rectangular region has been obtained which illustrates the region including spectra of the state matrix of any linear system even those having uncertainties. The usefulness of the suggested stability criteria and the proposed rectangular regions have been illustrated by several examples.**

Index Terms- **Asymptotic stability, LTI systems, Uncertainty, Gerschgorin theorem, Eigenvalue estimation.**

I. INTRODUCTION

Stability analysis of LTI systems has been investigated via various methods. Famous methods such as Routh-Hurwitz stability criterion, Lyapunov functions, and calculating the eigenvalues of the state matrices are some of the standard approaches for checking the stability of such systems. But when having systems with high dimension state matrices accompanied by uncertainties or time-variant parameters, using such methods seem to be abortive or time-consuming.

The stability problem of uncertain linear systems has received considerable attention and numerous criteria have been proposed for this issue during the last three decades [1-12]. It is worth noting that most of these stability conditions have been based on Lyapunov method [1-10]. One of the traditional methods for the stability investigation of such systems is using a single quadratic Lyapunov function. However, using this method for the aim of robust stability investigation seems to be conservative. Then, in order to reduce the conservatism, other methods such as parameter-dependent Lyapunov functions and piecewise Lyapunov functions were introduced. Based on using parameter-dependent Lyapunov functions, many criteria have been proposed to check the robust stability of systems having time-varying uncertain parameters which are less conservative than quadratic stability [4-8,10]. Let's review some of the previous works in this area.

Based on Lyapunov function and by approximating the uncertain region with a convex hyper-polyhedron, Gu et al [1] provided a necessary and sufficient condition to guarantee the quadratic stability of uncertain linear systems. Interestingly, it has also been shown that their method can be used for checking the stability of other types of systems like uncertain Takagi-Sugeno fuzzy models [13]. Fang and Loparo [3] used a Lyapunov equation for the nominal system and proposed an approach to check the robust stability of linear systems having structured uncertainty. Though improving the obtained bounds for the structured uncertainty comparing to the previous works, it requires satisfying one of the three defined inequalities which seems to be time-consuming. Romas and Peres [4] introduced a sufficient condition for the robust stability of continuous-time uncertain linear systems with convex bounded uncertainties. Based on linear matrix inequalities, they constructed a parameter dependent Lyapunov function to guarantee the stability of any matrix inside the defined uncertainty domain. Montanger et al [5] proposed a sufficient condition for the robust stability of linear systems with time-varying uncertainty. They guaranteed the stability of means of a parameter-dependent Lyapunov function and imposing bounds on the time derivatives of the uncertain parameters. Their method was based on two assumptions: the uncertain parameters belong to a polytope and the time derivatives of them are defined in certain bounds.

In terms of linear matrix inequalities and based on parameter-dependent Lyapunov function, Zhai et al [7] investigated the robust stability of linear systems having real parametric uncertainty. However, the method requires checking the existence of some symmetric matrices and a skew matrix which seems not to be an easy task. Yang and Dong [8] derived stability criteria for the existence of a parameter-dependent Lyapunov function in order to guarantee the robust stability of linear systems having polytopic uncertainty. Amao et al [9] considered the robust stability problem for linear uncertain systems subjected to parametric time-varying uncertainties. In this case, they made use of polyhedral Lyapunov functions and obtained less conservative results for such systems compared to the classical quadratic stability method.

It can be seen that many of those aforementioned studies on this issue have been based on the parameter dependent Lyapunov functions. However, the introduced stability conditions seem to be rather conservative and some impose very restrictive assumptions. Moreover, most of the obtained criteria for the stability problem are based on the fact that the uncertainties are structured and are confined to convex sets. These restrictions have led the researchers to think of introducing approaches which are not dependent on Lyapunov functions [11,12]. Gong and Thompson [11] considered the stability problem of systems having unstructured parameter perturbation. Their stability condition was derived from the polar decomposition of the nominal system matrix. The polar decomposition of a matrix A is a matrix decomposition of the form A=UP where U is a unitary matrix and P is a positive semi-definite Hermitian matrix. Ren et al [12] studied linear systems having constant parameter uncertainty. By applying a Guardian map, they derived sufficient criteria to guarantee the robust stability of such systems. Though proposing a different and novel approach, their method is firstly limited to a certain type of uncertainties. Secondly, checking the stability via this method requires satisfying number of inequalities which will be increased when having higher-dimensional state matrices.

As have been stated before, most of the previous studies for the stability problem of linear systems having uncertainties are based on Lyapunov function with the assumption of having structured uncertainties belonging to convex sets. In this paper, the authors seek to propose a different method that has not the restrictions and conservativeness of the famous Lyapunov function. In this case, we will investigate the stability via estimating the eigenvalues of the perturbed system directly. The idea is inspired from [14-18,20,21], where the authors studied the estimation methods for the eigenvalues of any arbitrary matrix.

The estimation and location of eigenvalues have been always one of the important topics in matrix theory. In this area, the famous Gerschgorin disk theorem estimates all the eigenvalues of any arbitrary complex matrix in the union of defined disks [15-17]. C. K. Li and R. C. Li in [18] improved the previous results in estimating the eigenvalues of Hermitian matrices. They estimated the eigenvalues of such matrices from the eigenvalues of their corresponding diagonal matrices. In other words, the sub-diagonal entries were considered to be perturbations on the diagonal matrix which was defined as the nominal matrix. Aside from just estimating the spectra, this method can also be used for the stability investigation of linear systems. In this case, Shekaramiz and Sheikholeslam took advantage of such a method to analyze the stability of continuous-time linear systems [19]. Zou and Jiang [20] estimated the eigenvalues and the smallest singular value of matrices. They showed that all the eigenvalues of an arbitrary complex matrix can be found in one closed disk. It was a better estimation in comparison to the previous works but was not an easy task to compute. Xingdong et al [21] dealt with the eigenvalue estimation in order to propose a solution to the perturbed matrix Lyapunov equation. It is a novel work applicable to control theory and linear system stability. However, the method is again based on Lyapunov method and the obtained inequality seems not to be easily fulfilled.

In this paper, firstly, Gerschgorin circle theorem will be considered. Gerschgorin circle theorem is a method that specifies regions in which the spectra of any complex square matrix do exist. By using the aforementioned theorem, three sufficient criteria will then be proposed in order to check the asymptotic stability of LTI systems. Moreover, the stability problem of linear systems having uncertainties will be considered. In this case, a sufficient criterion is proposed to guarantee the asymptotic stability of such systems. Then, by taking advantage of the proposed criteria, a theorem will be represented which specifies a rectangular region in which spectra of the state matrix for any LTI system exist. Since the method is based on the Gerschgorin Circle theorem, it is called "*Gerschgorin Rectangle Theorem*". Finally, the obtained rectangular region will be extended in order to estimate the spectra of systems having uncertainties.

This paper is organized as follows: The Gerschgorin theorem will be presented in section II. In section III, three sufficient criteria will be introduced to seek the asymptotic stability of LTI systems. Section IV will provide sufficient stability conditions for continuous-time linear systems having uncertainties. Furthermore, numerical examples will be presented to demonstrate the effectiveness of the proposed methods. Then in section V, we will propose new regions in which the spectra of both LTI systems and those having uncertainties can be found. Finally, the concluding remarks are given in section VI.

II. PRELIMINARIES

As has been states before, Gerschgorin circle theorem seeks the location of eigenvalues for any arbitrary square matrix. In other words, it specified circular regions in which the entire eigenvalues of a square matrix can be found. The theorem is described below.

Gerschgorin Circle Theorem 1 [14,15]: Consider an arbitrary complex square matrix $A_{n\times n}$ as follows.

$$A_{n\times n} = \lfloor a_{ij} \rfloor, i, j = 1,2,..., n. \tag{1}$$

Then, any eigenvalue λ of the matrix A is located in at least one of the closed disks of the complex plane centered at a_{ii} and having the defined radius R_i^R below. These disks are called Gerschgorin disks.

$$R_i^R = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \tag{2}$$

In other words,

$$\forall \lambda \in \delta(A), \exists i \quad \text{such that} \quad |\lambda - a_{ii}| \le R_i^R \tag{3}$$

where $\delta(A)$ is the spectrum of matrix A such that

$\delta(A) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$.

Corollary 1 [16]: Since the above result also holds for the transpose of matrix A, we can formulate a version of the Gerschgorin circle theorem based on columns sums instead of rows sums and reach to the following results.

$$\forall \lambda \in \delta(A), \exists j \quad \text{such that} \quad |\lambda - a_{jj}| \le R_j^C \tag{4}$$

where,

$$R_j^C = \sum_{\substack{i=1 \\ i \ne j}}^{n} |a_{ij}| \tag{5}$$

The result is obtained from the fact that the eigenvalues of any arbitrary matrix are equal to the eigenvalues of its transpose.

III. STABILITY ANALYSIS OF CONTINUOUS-TIME LTI SYSTEMS

In this section, based on Gerschgorin circle theorem, sufficient criteria will be proposed to investigate the stability of LTI systems. Please note that we are dealing with continuous-time linear systems in which all the entries of their relevant state matrices are assumed to have real values. Hence according to Gerschgorin circle theorem, all of the Gerschgorin disks for such systems will be centered on the real-axis of the complex plane. Now, consider the following definitions.

$$\overline{H} = \max_i \{a_{ii} + \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|\} \tag{6}$$

$$\underline{H} = \min_i \{a_{ii} - \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|\} \tag{7}$$

In the above definitions, $\overline{H}$ and $\underline{H}$ denote the maximum and minimum values that the real-part of eigenvalues of square matrix A may have possess, respectively. For better understanding, ponder the following example.

Example 1: Consider matrix A_1 as follows.

$$A_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

In the figure below the related Gerschgorin disks and the values of $\overline{H}$ and $\underline{H}$ have been illustrated.

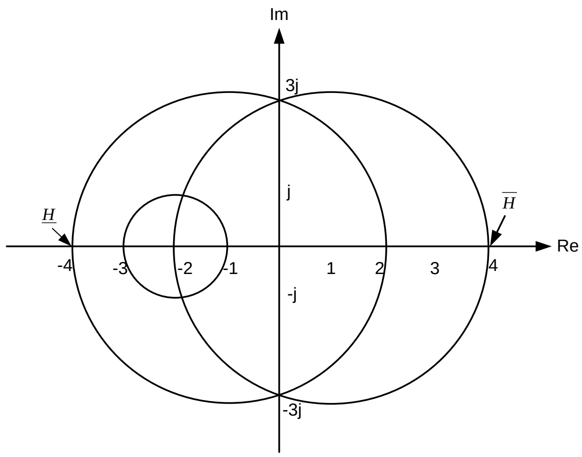

Figure 1: Gerschgorin disks for matrix A_1 and values of $\overline{H}$ and $\underline{H}$ obtained from Eq. (6) and Eq. (7), respectively.

Since all the entries of matrix A_1 have real values, the centers of all Gerschgorin disks are placed on the real-axis of the complex plane. This also can be seen in Fig. (1).

In the theorem below, a sufficient criterion will be proposed in order to investigate the asymptotic stability of linear time-invariant systems.

Theorem 2: Consider the unforced LTI system $\dot{X}(t) = A_{n\times n}\, X(t)$. Where,

$A_{n\times n} = \left[a_{ij}\right], i, j = 1,2,..., n.$

Then, the system is asymptotically stable if it has a negative value for the following definition.

$$\overline{H} = \max_{i} \{a_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|\} \tag{8}$$

Proof: Assume that one of the Gerschgorin disks of matrix A has been drawn as follows.

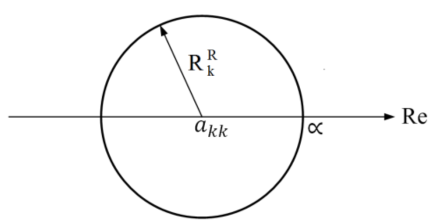

Figure 2: One of the Gerschgorin disks of matrix A in the complex plane.

Please note that all of the entries of matrix A have been assumed to have real values; $a_{kk} \in \mathrm{R}$. Then, the real-part of any arbitrary point being in or on the boundary of the above closed disk would have the value equal or less than α. Now, consider Eq. (2) and Eq. (3) defined in the Gerschgorin circle theorem.

$$\forall\, \lambda \in \delta(A)\ , \exists i \ \text{ such that } \ \left|\lambda - a_{ii}\right| \le \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|$$

By having real values for the center of Gerschgorin disks, we can then rewrite the above inequality as follows.

$$a_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right| \le \mathrm{Re}(\lambda) \le a_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|$$

Now define $\overline{H}_i$ as

$$\overline{H}_i = a_{ii} + \sum_{\substack{j=1 \\ i\neq j}}^{n} \left|a_{ij}\right| \tag{9}$$

The term $\overline{H}_i$ denotes the maximum permissible value for λ_k related to the k^th Gerschgorin disk. Then define

$$\overline{H} = \max_{i} (\overline{H}_i) \tag{10}$$

Now, having a negative value for $\overline{H}$ concludes that all of the eigenvalues of matrix A have negative real-part values. In this case, the system is asymptotically stable.

□

Example 2: Consider an LTI system having the below state matrix.

$$A_2 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -4 & 2 \\ 2 & -2 & -5 \end{bmatrix}$$

By applying definition (8) to this matrix, we obtain $\overline{H}(A_2) = -1 \langle 0$. Therefore, according to Theorem 2, the system is asymptotically stable. For better understanding the Gerschgorin disks are shown below.

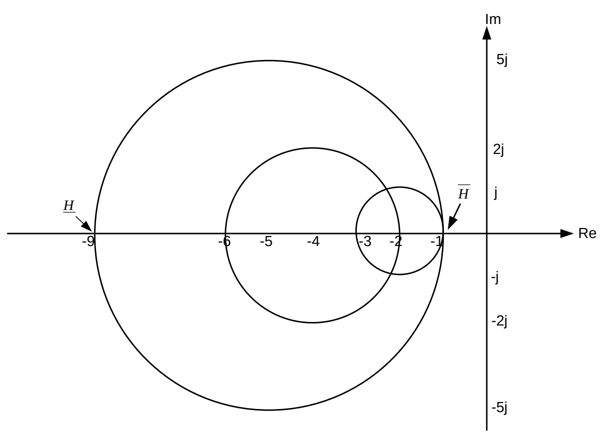

Figure 3: The Gerschgorin disks for matrix A_2 obtained from the Gerschgorin circle Theorem.

Finally, in order to verify the stability conclusion, the eigenvalues of matrix A_2 have been calculated below.

$$\delta(A_2) = -4.787 \pm 2.105\,j, -1.426$$

So far, a sufficient criterion has been proposed in order to guarantee the asymptotic stability of LTI systems. But, there exist many systems that have non-negative values for $\overline{H}$ while the systems are asymptotically stable. Let's have a look at the following example.

Example 3: Consider an LTI system with the below state matrix.

$$A_3 = \begin{bmatrix} -3 & 1 & -6 & 2.5 \\ 1.2 & -9 & 3.6 & -1 \\ 1 & 5 & -12 & 1.8 \\ -0.6 & 2 & 2 & -6 \end{bmatrix}$$

In this example, applying theorem 2 concludes no results about the stability of the system; $\overline{H}(A_3) = 6.5 \rangle 0$. But, calculating the eigenvalues of matrix A_3 guarantees the asymptotic stability of the system.

$$\delta(A_3) = -4.1461, -5.4051 \pm 1.6328\,j, -15.0436$$

Hence, our proposed theorem 2 still seems to be conservative. Now, let's introduce another stability criterion for LTI systems.

Corollary 2: Consider the LTI system $\dot{X}(t) = AX(t)$. The system is asymptotically stable if the below definition has negative value.

$$\overline{\overline{H}} = \max_{j} \{a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|\} \tag{11}$$

Proof: The proof can be easily obtained from Gerschgorin circle theorem 1, corollary 1, and theorem 2 and is omitted.

□

Example 4: Consider the system described in Ex. (3). By applying definition (11) to the system, we reach to $\overline{\overline{H}}(A_3) = -0.2 \langle 0$. Therefore according to corollary 2, the system is asymptotically stable.

Please note that the proposed criterion in corollary 2 also seems conservative. This is because of the fact that there exist many stable systems in which the stability criterion of corollary 2 is not fulfilled. Now, by taking advantage of what were introduced in theorem 2 and corollary 2, we can define the less conservative theorem below.

Theorem 3: Consider an unforced LTI system $\dot{X}(t) = A_{n\times n}\, X(t)$.

Where,

$$A_{n\times n} = [a_{ij}],\ i, j = 1,2,\dots, n.$$

Then, the system is asymptotically stable if the following inequality is satisfied.

$$\overset{\Delta}{H} = \min(\overline{H}, \overline{\overline{H}}) \langle 0 \tag{12}$$

Where,

$$\overline{H} = \max_i \{a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|\} \qquad \text{Revisited (8)}$$

$$\overline{\overline{H}} = \max_j \{a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|\} \qquad \text{Revisited (11)}$$

Proof: The proof can be easily obtained from theorem 2 and corollary 2 and is omitted.

□

Example 5: Consider a continuous-time LTI system having the same state matrix as Ex. 3.

By applying the definitions of theorem 3, we reach to the following results.

$$\overline{H}(A_3) = 6.5, \overline{\overline{H}}(A_3) = -0.2 \quad \Rightarrow \overset{\Delta}{H}(A_3) = -0.2 \langle 0$$

Then, the system is asymptotically stable.

Now, reconsider the state matrix A_3 represented in Ex. 5. It seems that stability investigation for systems having rather higher dimension state matrices by methods such as Routh-Hurwitz criterion, Lyapunov methods, or directly calculating the eigenvalues are much more time-consuming when compared with our proposed stability criteria.

IV. STABILITY ANALYSIS OF SYSTEMS HAVING UNCERTAINTIES

So far, some sufficient criteria have been introduced in order to guarantee the asymptotic stability of LTI systems. In this section, the stability problem of systems having uncertainties will be considered. In this case, based on the Gerschgorin circle theorem and the obtained stability criteria of the previous section, a new stability condition for such systems will be proposed.

Theorem 4: Consider the following unforced continuous-time linear system.

$$\dot{X}(t) = (A_0 + \Delta A(t))\, X(t) \tag{13}$$

Where,

$$A_0 = [a_{ij}],\ \Delta A(t) = [\Delta a_{ij}(t)],\ |\Delta a_{ij}(t)| \le M_{ij}, \quad i, j = 1,2,..., n. \tag{14}$$

Matrices A_0 and $\Delta A(t)$ denote the nominal state matrix and the uncertainty, respectively. Assume that the system with its nominal state matrix A_0 be asymptotically stable. Then, the system (13) is still asymptotically stable if the following criterion is satisfied.

$$\overset{\Delta}{H}_{A_0+\Delta A(t)} = \min(\overline{H}_{A_0+\Delta A(t)}, \overline{\overline{H}}_{A_0+\Delta A(t)}) \langle 0 \tag{15}$$

Where,

$$R^{R}_{i,A_0+\Delta A(t)} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(|a_{ij}| + M_{ij} \right), i = 1,2,,\ldots,n. \quad (16)$$

$$R^{C}_{j,A_0+\Delta A(t)} = \sum_{\substack{i=1 \\ i\neq j}}^{n} \left(|a_{ij}| + M_{ij} \right), j = 1,2,,\ldots,n. \quad (17)$$

$$C_i = a_{ii} + M_{ii} \ , i = 1,2,,\ldots, n. \quad (18)$$

$$\overline{H}_{A_0+\Delta A(t)} = \max_i (C_i + R^{R}_{i,A_0+\Delta A(t)}) \ , i = 1,2,,\ldots,n. \quad (19)$$

$$\overline{\overline{H}}_{A_0+\Delta A(t)} = \max_j (C_j + R^{C}_{j,A_0+\Delta A(t)}) \ , j = 1,2,,\ldots,n. \quad (20)$$

The above notations represent the following definitions.

$R^{R}_{i,A_0+\Delta A(t)}$: Maximum permissible value for the radius of i[th] Gerschgorin disk corresponding to i[th] row of $A_0 + \Delta A(t)$.

$R^{C}_{j,A_0+\Delta A(t)}$: Maximum permissible value for the radius of jth Gerschgorin disk corresponding to jth column of $A_0 + \Delta A(t)$.

C_i : Maximum permissible value for the center of i[th] Gerschgorin disk corresponding to i[th] row of $A_0 + \Delta A(t)$.

$\overline{H}_{A_0+\Delta A(t)}$: Maximum permissible value for the real-part of spectra of $A_0 + \Delta A(t)$ based on rows sums.

$\overline{\overline{H}}_{A_0+\Delta A(t)}$: Maximum permissible value for the real-part of spectra of $A_0 + \Delta A(t)$ based on columns sums

Proof: Having maximum values for the radii and centers of Gerschgorin disks, one can obtain the maximum permissible value for the $\overset{\Delta}{H}_{A_0+\Delta A(t)}$ defined in Eq. (15). This value represents the maximum permissible value for the real-part of spectra of matrix $A_0 + \Delta A(t)$. According to theorem 3, having negative value for $\overset{\Delta}{H}_{A_0+\Delta A(t)}$ guarantees the asymptotic stability of the system. The approach for the proof is almost the same as those described in theorem 2 and theorem 3 and is omitted.

□

Example 6: Consider the uncertain linear system defined in Eq. (13) and Eq. (14). Assume that the state matrix of the system be as follows.

$$A_0 = \begin{bmatrix} -4 & 2 & -5 & 2 \\ 0.8 & -9 & 3 & -1 \\ 1 & 3 & -12 & 1 \\ -0.5 & 1 & 2 & -6 \end{bmatrix} \text{ and } \Delta A(t) = \begin{bmatrix} a(t) & b(t) & c(t) & e(t) \\ b(t) & c(t) & d(t) & b(t) \\ a(t) & e(t) & b(t) & c(t) \\ d(t) & c(t) & b(t) & a(t) \end{bmatrix}$$

Where,

$$|a(t)| \le 0.1, \ |b(t)| \le 0.3, \ |c(t)| \le 0.5, \ |d(t)| \le 0.8, \ |e(t)| \le 1.0$$

Now, by applying theorem 4, the following results are obtained.

Table 1: Stability investigation of Ex.6 by applying theorem 4.

Defined terms	***Applied equation***	***Obtained values form the applied equation***	***Stability conclusion of the nominal system***	***Stability conclusion of the system having uncertainties***
$\overline{H}_{A_0}$	Eq. (8)	5	No results	-
$\overline{\overline{H}}_{A_0}$	Eq. (11)	-1.7	Asymptotically stable	-
$\overset{\Delta}{H}_{A_0}$	Eq. (12)	-1.7	Asymptotically stable	-
$\overline{H}_{A_0+\Delta A(t)}$	Eq. (19)	Not useful	-	No results
$\overline{\overline{H}}_{A_0+\Delta A(t)}$	Eq. (20)	-0.1	-	Asymptotically stable
$\overset{\Delta}{H}_{A_0+\Delta A(t)}$	Eq. (15)	-0.1	-	Asymptotically stable

Therefore by having the results obtained in Table 1, one can conclude that for any permissible values of uncertainties being in their relevant bounds, the system is asymptotically stable. In the figure below, the spectra of matrix $A_0 + \Delta A(t)$ for different values of uncertainties have been illustrated. It can also be seen in the figure that the spectra of such system are placed on the left-hand side of the imaginary axis in the complex plane.

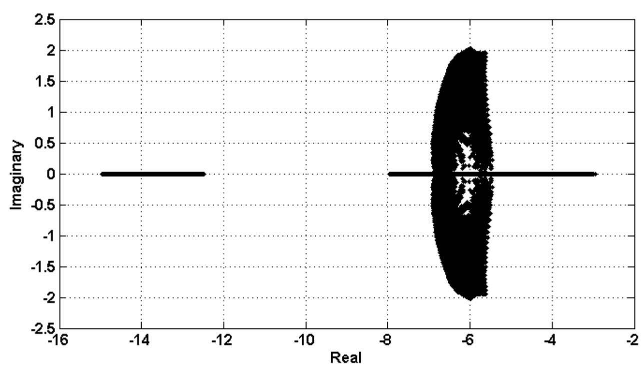

Figure 4: The spectra of matrix $A_0 + \Delta A(t)$ for all permissible values of uncertainties.

V. INVESTIGATING A REGION INCLUDING THE SPECTRA OF LINEAR SYSTEMS

In this section, a new rectangular region specifying all the eigenvalues of the state matrix of any arbitrary continuous-time linear system will be introduced. The following theorem, seeks the rectangular region in which the spectra of the state matrix for LTI systems can be found.

Lemma 1 "Gerschgorin Rectangle Theorem": For any LTI system $\dot{X} = AX(t)$ and more generally for any arbitrary square matrix having real entries, the spectra of matrix A are located in the following closed rectangular region in the complex plane.

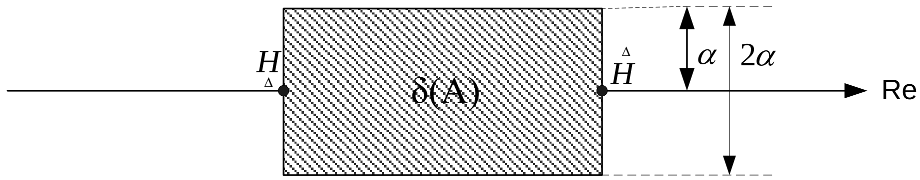

Figure 5: The closed rectangular region contains the spectra of any real square matrix.

Where,

$$\overset{\Delta}{H} = \min(\overline{H}, \overline{\overline{H}}) \qquad \text{Revisited (12)}$$

$$\underset{\Delta}{H} = \max(\underline{H}, \underline{\underline{H}}) \qquad (21)$$

$$\alpha = \min\left(R^R_{Max}, R^C_{Max}\right) \qquad (22)$$

$$\overline{H} = \max_i \{ a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \} \qquad \text{Revisited (8)}$$

$$\overline{\overline{H}} = \max_j \{ a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \} \qquad \text{Revisited (11)}$$

$$\underline{H} = \min_i \{ a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \} \qquad \text{Revisited (7)}$$

$$\underline{\underline{H}} = \min_j \{ a_{jj} - \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \} \qquad (23)$$

$$R^R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad \text{Revisited (2)}$$

$$R^R_{Max} = \max_i \{ R^R_i \} \qquad (24)$$

$$R^C_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \qquad \text{Revisited (5)}$$

$$R^C_{Max} = \max_j \{ R^C_j \} \qquad (25)$$

Proof: According to theorem 3, $\overset{\Delta}{H}$, as defined in Eq. (12), is the maximum permissible value that the real-part of spectra of matrix A may possess in the complex plane. Having this value, it can be concluded that the spectra of such matrix are located in the shaded region below in the complex plane.

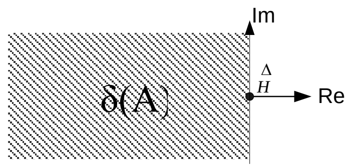

Figure 6: The obtained region for the spectra of matrix A in the complex plane by having the value of $\overset{\Delta}{H}$.

As has been stated before, $\underline{H}$, defined in Eq. (7), is the minimum permissible value that the real-parts of the spectra of matrix A matrix may possess. Similarly, the term $\underline{\underline{H}}$ also denotes the minimum permissible value for the real-parts of the spectra of matrix A. This term is defined by considering the Gerschgorin disks with respect to columns sums. Hence, one can claim that the maximum of $\underline{H}$ and $\underline{\underline{H}}$ can also demonstrate the minimum permissible value for the real-parts of the spectra of such matrix: $\underset{\Delta}{H} = \max(\underline{H}, \underline{\underline{H}})$. The shaded region in the following figure illustrates the obtained region for the spectra of matrix A by applying the value of $\underset{\Delta}{H}$ to Fig. (6).

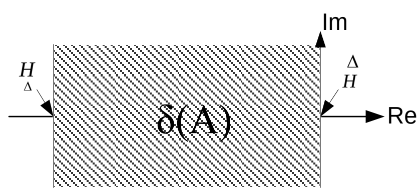

Figure 7: The obtained region for the spectra of matrix A in the complex plane by having the values of $\overset{\Delta}{H}$ and $\underset{\Delta}{H}$.

Now, we want to obtain bounds for the imaginary-parts of the spectra of matrix A as well. According to theorem 1, the Gerschgorin disks have centers of a_{ii} and radii of R_i^R as defined in Eq. (2). Please note that for LTI systems, the entries of the state matrix A are all real values. In the other words, all Gerschgorin disks are centered on the real-axis of the complex plane. Now, consider the disk with the greatest radii among all of the Gerschgorin disks. It is obvious that the maximum permissible imaginary-part for the spectra of matrix A is equal to the radius of such disk. Therefore for finding the aforementioned maximum permissible value, it suffices to calculate the radius of the greatest Gerschgorin disk as follows.

$$R_{Max}^R = \max_i \{R_i^R\},\ R_i^R = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|,\ i = 1,2,\ldots,n. \qquad \text{Revisited (24)}$$

Similarly, the same approach can be applied to corollary 1 where the Gerschgorin disks are obtained from columns sums. In this case, one can conclude that the maximum permissible value for the imaginary-part of the spectra of the matrix A is as follows.

$$R_{Max}^C = \max_j \{R_j^C\},\ R_j^C = \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|,\ j = 1,2,\ldots,n. \qquad \text{Revisited (25)}$$

Having both the values of R_{Max}^{R} and R_{Max}^{C} as the maximum permissible values for the imaginary-parts of the spectra of matrix A, it is obvious that the term $\alpha = \min\left(R_{Max}^{R}, R_{Max}^{C}\right)$ can also demonstrate the maximum permissible values for the imaginary-parts of the spectra of such matrix. Moreover, all Gerschgorin disks are centered on the real-axis of the complex plane: the Gerschgorin disks are symmetric with respect to the real-axis of the complex plane. This means that $-\alpha$ is the minimum permissible value of the imaginary-parts of the spectra of matrix A. Finally, the shaded region in Fig. (5) illustrates the estimated region for the spectra of matrix A.

Example 7: Consider an LTI system having the below state matrix. □

$$A_4 = \begin{bmatrix} -3 & 1 & -6 & 2.5 \\ 1.2 & -9 & 3.6 & -1 \\ 1 & 5 & -12 & 1.8 \\ -0.6 & 2 & 2 & -6 \end{bmatrix}$$

In the figure below, both the Gerschgorin disks of theorem 1 and the obtained shaded rectangular region form lemma 1 have been illustrated.

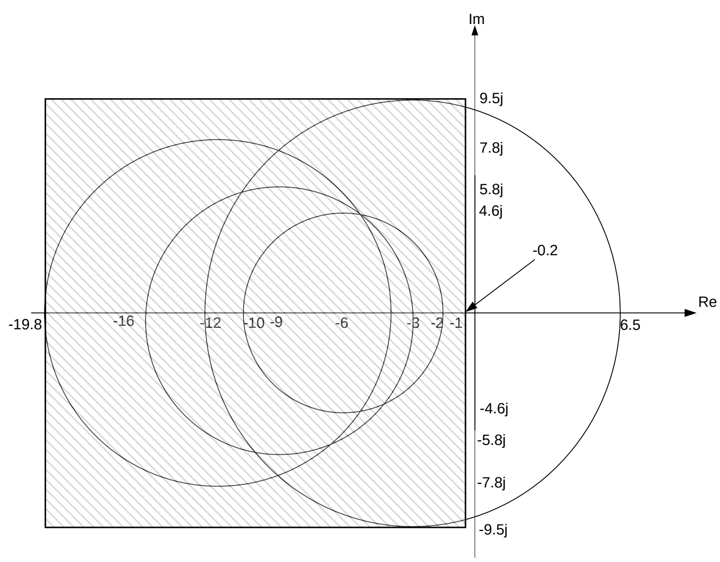

Figure 8: Comparison of Gerschgorin disks obtained from theorem 1 with rectangular region of lemma 1.

Similarly, the following figure illustrates both the Gerschgorin disks and the shaded rectangular region where obtained from corollary 1 and lemma 1, respectively.

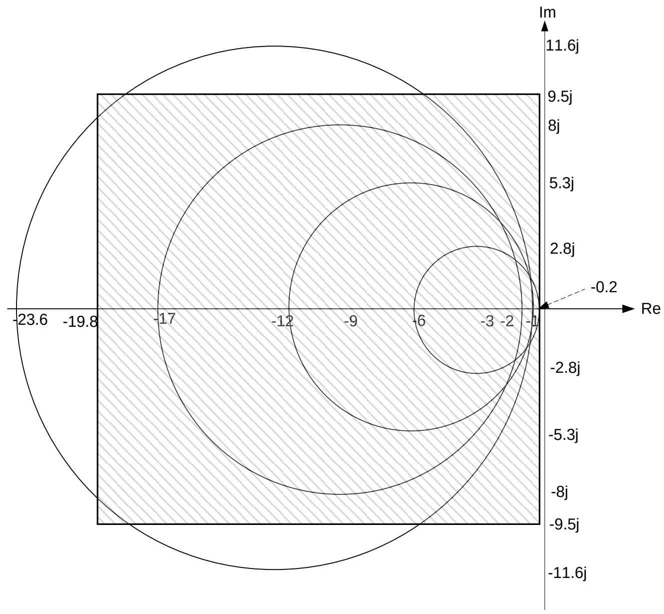

Figure 9: Comparison of Gerschgorin disks obtained from Corollary 1 with rectangular region of lemma 1.

In summary, it seems that the rectangular region provides tighter region for estimating the spectra of matrix A_4 when compared with the results of theorem 1 and corollary 1. Finally, for the purpose of comparison, the spectra of matrix A_4 have been calculated directly as follows.

$$\delta(A_4) = -4.1461, -5.4051 \pm 1.6328j, -15.0436$$

Now, in the following theorem provides a region in which the spectra of the state matrix for any linear systems having uncertainties will be investigated.

Theorem 5: Consider the following unforced continuous-time linear system.

$$\dot{X}(t) = (A_0 + \Delta A(t))\, X(t) \qquad \text{Revisited (13)}$$

Where,

$$A_0 = [a_{ij}],\ \Delta A(t) = [\Delta a_{ij}(t)],\ |\Delta a_{ij}(t)| \le M_{ij},\quad i, j = 1,2,\ldots, n. \qquad \text{Revisited (14)}$$

Matrices A_0 and $\Delta A(t)$ denote the nominal state matrix and the uncertainty, respectively. Assume that the system with its nominal state matrix A_0 be asymptotically stable. Then, the spectra of the uncertain system for any arbitrary values of bounded uncertainties are placed in the following rectangular region.

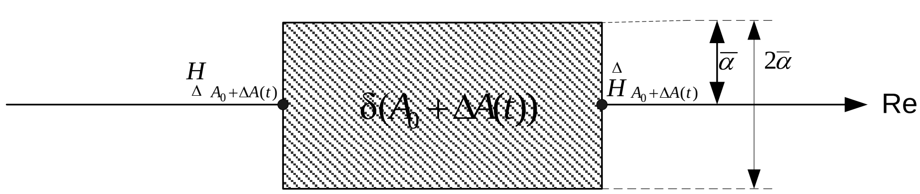

Figure 10: The closed rectangular region contains the spectra of the state matrix $A_0 + \Delta A(t)$.

Where,

$$\overset{\Delta}{H}_{A_0+\Delta A(t)} = \min\left(\overline{H}_{A_0+\Delta A(t)}, \overline{\overline{H}}_{A_0+\Delta A(t)}\right) \qquad \text{Revisited (15)}$$

$$\underset{\Delta}{H}_{A_0+\Delta A(t)} = \max\left(\underline{H}_{A_0+\Delta A(t)}, \underline{\underline{H}}_{A_0+\Delta A(t)}\right) \qquad (26)$$

$$\overline{\alpha} = \min\left(R^R_{Max,\Delta}, R^C_{Max,\Delta}\right) \qquad (27)$$

$$C_i = a_{ii} + M_{ii}, \; i = 1,2,,...,n. \qquad \text{Revisited (18)}$$

$$\overline{H}_{A_0+\Delta A(t)} = \max_i \left(C_i + R^R_{i,A_0+\Delta A(t)}\right), \; i = 1,2,,\ldots,n. \qquad \text{Revisited (19)}$$

$$\overline{\overline{H}}_{A_0+\Delta A(t)} = \max_j \left(C_j + R^C_{j,A_0+\Delta A(t)}\right), \; j = 1,2,,\ldots,n. \qquad \text{Revisited (20)}$$

$$\overline{C}_i = a_{ii} - M_{ii}, \; i = 1,2,,...,n. \qquad (28)$$

$$\underline{H}_{A_0+\Delta A(t)} = \min_i \{\overline{C}_i - R^R_{i,A_0+\Delta A(t)}\} \qquad (29)$$

$$\underline{\underline{H}}_{A_0+\Delta A(t)} = \min_j \{\overline{C}_j - R^C_{j,A_0+\Delta A(t)}\} \qquad (30)$$

$$R^R_{i,A_0+\Delta A(t)} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(|a_{ij}| + M_{ij}\right), \; i = 1,2,,\ldots,n. \qquad \text{Revisited (16)}$$

$$R^R_{Max,\Delta} = \max_i \{R^R_{i,A_0+\Delta A(t)}\} \qquad (31)$$

$$R^C_{j,A_0+\Delta A(t)} = \sum_{\substack{i=1 \\ i\neq j}}^{n} \left(|a_{ij}| + M_{ij}\right), \; j = 1,2,,\ldots,n. \qquad \text{Revisited (17)}$$

$$R^C_{Max,\Delta} = \max_j \{R^C_{j,A_0+\Delta A(t)}\} \qquad (32)$$

Proof: The proof is straight forward and is omitted. □

Example 8: Consider the LTI system described in Ex. 6. By applying theorem 5, we reach to the following region. The region contains all the spectra of the state matrix for any arbitrary values of uncertainties being the defined bounds.

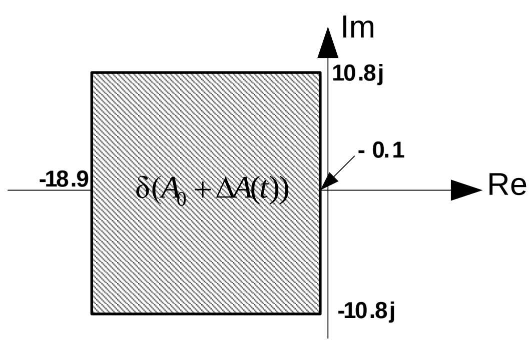

Figure 11: The closed rectangular region contains the spectra of the state matrix $A_0 + \Delta A(t)$.

It can be easily seen that the illustrated spectra of the state matrix of Fig. 5 are placed in the obtained rectangular region of Fig. 11.

VI. CONCLUSION

The stability problem of continuous-time LTI systems has been considered. Sufficient criteria have been proposed to guarantee the asymptotic stability of LTI systems. The merit of this approach is mainly in linear systems having uncertainties in their state matrices or in systems having time-variant state matrices. In this case, a theorem has been proposed in order to investigate the asymptotic stability of such systems. Unlike most of the previous works in this area, our method is independent of any types of Lyapunov function. Therefore, it is not confined to having special types of uncertainties. Then, by taking advantage from Gerschgorin theorem in estimating the spectra of any arbitrary matrix, a rectangular region has been obtained which demonstrates the region of existence for the spectra of the state matrix of any LTI system. Finally, the obtained region has been extended to specify the spectra's region for linear systems having uncertainties. The result can be used for both estimating the spectra and investigating the stability of linear systems.

FUTURE WORK

As a future work, the authors are now studying different methods of estimating the spectra of matrices. As has been stated in the introduction part, different methods have been introduced to solve this problem. Though showing to be useful for the stability problem of uncertain linear systems, Gerschgorin theorem has still the defect and that is the estimation is directly related to the main-diagonal entries of the state matrix. In other words, having positive values for those entries, our criteria fail to check the stability. Therefore, the authors seek for the estimations methods which are not directly related to the main-diagonal of matrices.

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