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**OPTIMAL DISCOUNTING IN CONTROL PROBLEMS THAT
SPAN MULTIPLE GENERATIONS**

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SPAN MULTIPLE GENERATIONS**

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**OPTIMAL DISCOUNTING IN CONTROL PROBLEMS THAT
SPAN MULTIPLE GENERATIONS**

Frank Caliendo and Kenneth Lyon

ABSTRACT

The principal contribution of this paper is the linking together of separate control problems across multiple generations using the bequest motive, intergenerational altruism, rational expectations, and solution boundary conditions. We demonstrate that discounting at the market rate of interest is an endogenous characteristic of a general equilibrium, optimal control problem that spans multiple generations. Within the confines of our model, we prove that it is optimal to discount at the market rate of interest the social benefits to distant generations from immediate clean up at toxic waste sites if the current generation that bears the cleanup cost is perfectly altruistic towards future generations. Also, we show that this result holds for alternative assumptions regarding pure time preference. Moreover, the result holds regardless of whether selfish interim generations attempt to undo the provisions made for distant generations. In our distortion-free deterministic model, the evidence for intergenerational discounting at the market rate of interest is compelling.

OPTIMAL DISCOUNTING IN CONTROL PROBLEMS THAT SPAN MULTIPLE GENERATIONS

1. INTRODUCTION

A recent book entitled *Discounting and Intergenerational Equity*, published by Resources for the Future, represents an insightful compilation of the opinions and analysis of some of the foremost economists on the issue of discounting. The principle question being addressed is whether long-term environmental projects can be treated the same as short-term ones, as far as discounting is concerned. Some authors, Arrow (1999), Weitzman (1999), Bradford (1999), and Montgomery (1999) suggest that the market rate of interest is the appropriate discount rate even in projects that span multiple generations of time. Others, Cline (1999), Lind (1999), Schelling (1999), and Rothenberg (1999) hold that such projects cannot be viewed in the traditional cost-benefit sense.

This is a complex issue and both sets of opinions have merit. We do not presume to settle this debate, and we do not intend to give a detailed account of all the issues raised by these authors. Rather, our intention is to show that discounting at the market rate of interest is optimal in a distortion-free general equilibrium, environmental control problem that spans multiple generations of time, provided current generations are perfectly altruistic towards future generations. This important proof provides rigorous support for the market rate of interest as the appropriate discount rate in multigenerational public projects that deal with the environment.

Lyon (1996) proved that the discounting of future benefits from natural resources at the market rate of interest is a feature of a deterministic, general equilibrium optimal growth program for which distortions are absent. (See Weitzman 1994 and Weitzman 1998 for the effects of distortions and uncertainty on the social discount rate.) We extend Lyon's analysis to a multi-generational problem and show that under certain conditions his conclusions hold: the market rate of interest is the optimal discount rate, regardless of the time horizon.

We analyze a simple growth model where toxic waste sites affect the health and therefore the productivity of the work force, and waste can be removed in accordance with social cleanup expenditure. Waste removal generates good health and therefore additional income from increased productivity of workers. We consider separate, single generation continuous time

control problems with these features and we link these problems together (end-to-end) using the bequest motive, intergenerational altruism, rational expectations, and solution endpoint conditions. The result is an intergenerational control problem.

Section 2 analyzes two generations of time and shows that if the current generation is perfectly altruistic and has rational expectations of the behavior of the next generation, the separate control problems collapse to a single intergenerational control problem of twice the length in which future benefits are discounted at the market rate of interest. Section 3 shows that this result holds even for alternative assumptions regarding the rate of pure time preference. And, in Section 4 we consider three generations of time in which the middle generation is selfish and does not care about the level of toxic waste left for the last generation. Further, the middle generation may undo the cleanup efforts made by the first generation. Even so, the control model indicates that generation 1 will discount any income benefits accruing to generation 3 at the market rate of interest. Section 5 offers concluding remarks.

2. A MULTI-GENERATIONAL PROBLEM: TOXIC WASTE CLEANUP

The model economy produces income $y(t)$ according to the following concave, continuously differentiable production function

$$y(t) = f(k(t), h(w(t)))$$

where $k(t)$ is the stock of capital, and $h(w(t))$ is the level of human health which is a function of the amount of toxic waste at a disposal site, $w(t)$. We assume that toxic waste negatively affects human health, which in turn negatively affects the productivity of the workforce. All other factors used in the production of income are omitted for simplicity, and the same is true for other variables that affect health.

Society spends income on a consumption commodity, new capital, and waste removal: $c(t)$ is consumption spending, $z(t)$ is aggregate gross investment, and $g(t)$ is expenditure on waste removal. Thus,

$$y(t) = f(k(t), h(w(t))) = c(t) + z(t) + g(t)$$

We assume the unit price of each of these items is one (and there is no inflation in the model). Thus, $c(t)$, $z(t)$, and $g(t)$ also can be interpreted as quantities purchased at time t .

Capital and toxic waste are the state variables in this control problem. Capital accumulates according to

$$\frac{dk(t)}{dt} = z(t) - \delta k(t)$$

which can be rewritten as

$$(1) \quad \frac{dk(t)}{dt} = f(k(t), h(w(t))) - c(t) - g(t) - \delta k(t)$$

where δ is the rate of capital depreciation.

For simplicity the level of waste is reduced only according to cleanup expenditure (there is no biological depreciation of the waste – this assumption does not affect the results). Also, waste is no longer being produced. Imagine a toxic waste site that was once an area for active dumping, but that is now abandoned.

$$(2) \quad \frac{dw(t)}{dt} = -g(t)$$

Finally, society receives utility from consumption $U(c(t))$. This function is strictly concave and continuously differentiable. Of course, utility probably is also a function of health. The implications of this assumption are explored in a related working paper (Caliendo and Lyon 2003). To focus solely on the intergenerational dynamics we have chosen to model social utility as a function of consumption only.

Consider two periods of time: period 1 is $[0, T]$ and period 2 is $[T, 2T]$. There are two groups of people, those that exist on $[0, T]$, which we refer to as generation 1, and those that exist on $[T, 2T]$, which we refer to as generation 2. These generations do not overlap, they exist end-to-end. This assumption simplifies the mathematical model. Also, the generations do not need to be the same size, but for convenience they are.

Suppose a social planner initially operates the economy optimally on $[0, T]$, which along with other things means that $k(T)$ and $w(T)$ are chosen to maximize the lifetime utility of generation 1. Next the social planner maximizes the lifetime utility of generation 2, given $k(T)$ and $w(T)$ as initial conditions. Generation 2 is endowed with capital and waste according to the bequest of these stock variables by generation 1. Thus, $k(T)$ and $w(T)$ are the bequeathed stocks or solution endpoints from stage 1, and also the initial conditions for a new control problem starting at time T and finishing at $2T$.

The first stage of the multi-generational problem is

$$\max : \int_0^T \exp[-\rho t] U(c(t)) dt + \exp[-\rho T] B(k(T), w(T))$$

subject to equations (1) and (2) where $k(0)$ and $w(0)$ are given and $c(t)$ and $g(t)$ are the control variables. The exponential function is the subjective discount factor, which decreases annually at the rate of pure time preference, ρ . B is a strictly concave bequest function that measures the utility received by generation 1 from endowing generation 2 with $k(T)$ and $w(T)$. B is increasing in $k(T)$ and decreasing in $w(T)$.

The second stage is

$$\max : \int_T^{2T} \exp[-\rho(t-T)] U(c(t)) dt + \exp[-\rho(2T-T)] B(k(2T), w(2T))$$

subject to equations (1) and (2) where the initial conditions $k(T)$ and $w(T)$ are given. They are the solution endpoints to the first stage of the control problem. We assume the utility and bequest functions are the same for both generations, and that both generations are equally impatient about the time profile of their own utility (i.e., ρ is the same in both stages).

A number of authors (Cline 1999, Schelling 1999, Rothenberg 1999, and Toman 1999 to name a few) argue that pure time preference is an *intra*-generational phenomenon that should not be applied to *inter*-generational problems. One interpretation of this statement would be to *not* discount the bequest functions at the rate of time preference. We explore this alternative in the next section.

For notational clarity, and for reasons that will become apparent, the costate variables for stage 1 carry a “1” subscript and the costate variables for stage 2 carry a “2” subscript. The current value Hamiltonian function for stage 1 is

$$H(c(t), g(t), k(t), w(t), \lambda_1(t), \eta_1(t)) = U(c(t)) + \lambda_1(t) \{ f(k(t), h(w(t))) - c(t) - g(t) - \delta k(t) \} - \eta_1(t) g(t)$$

where $\lambda_1(t)$ and $\eta_1(t)$ are the current value costate variables in stage 1 associated with capital and waste, respectively.

The current value Hamiltonian function for stage 2 is

$$H(c(t), g(t), k(t), w(t), \lambda_2(t), \eta_2(t)) = U(c(t)) + \lambda_2(t) \{ f(k(t), h(w(t))) - c(t) - g(t) - \delta k(t) \} - \eta_2(t) g(t)$$

where $\lambda_2(t)$ and $\eta_2(t)$ are the current value costate variables in stage 2 associated with capital and waste, respectively.

Thus, $\lambda_i(t)$ is the value (at time t) to generation i of a unit of capital at time t , and $\eta_i(t)$ is the value (at time t) to generation i of a unit of waste at time t . (Detailed discussion of the costate variable can be found in Kamien and Schwartz 1991, Hoy et al. 1996, and Lyon 1999). $\eta_i(t)$ is negative because waste impairs worker productivity, and $-\eta_i(t)$ can be viewed as the value of a unit of waste removal at time t .

The following are first order necessary conditions in current value form that hold for both control problems (i.e., these conditions hold for generation 1 on $[0, T]$ and for generation 2 on $[T, 2T]$, only the initial conditions and transversality conditions differ from one problem to the next)

$$(3a) \quad \frac{\partial U(c^*(t))}{\partial c(t)} - \lambda_i^*(t) = 0 \quad i = 1, 2$$

$$(3b) \quad -\lambda_i^*(t) - \eta_i^*(t) = 0 \quad i = 1, 2$$

$$(3c) \quad \frac{d\lambda_i^*(t)}{dt} = \rho\lambda_i^*(t) - \left\{ \frac{\partial f(k^*(t), h(w^*(t)))}{\partial k(t)} - \delta \right\} \lambda_i^*(t) \quad i = 1, 2$$

$$(3d) \quad \frac{d\eta_i^*(t)}{dt} = \rho\eta_i^*(t) - \lambda_i^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \quad i = 1, 2$$

Asterisks denote optimal values. We assume that non-negativity constraints on the control and state variables are automatically satisfied. The transversality conditions for stage 1 are

$$\lambda_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial k(T)}$$

$$\eta_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)}$$

and for stage 2

$$\lambda_2^*(2T) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial k(2T)}$$

$$\eta_2^*(2T) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)}$$

The market rate of interest is defined as

$$r(t) \equiv \frac{\partial f(k^*(t), h(w^*(t)))}{\partial k(t)} - \delta$$

which follows naturally from Euler's theorem if the production function is homogeneous of degree one and if factor markets are competitive. Moreover, the implicit user cost of capital gives the same definition for the interest rate under certain conditions which can be expected to hold in this model. And it is worth mentioning that $r(t)$ is the own rate of interest of an additional unit of capital at time t , which is seen by differentiating equation (1) with respect to $k(t)$.

We define α as a parameter that measures the intergenerational altruism of generation 1 towards generation 2. α measures how much generation 1 cares for the welfare of generation 2 relative to how much generation 2 cares for their own welfare. Let $\alpha \in [0, 1]$. If $\alpha = 1$, then generation 1 cares exactly as much for generation 2 as generation 2 cares for themselves; if $\alpha = 0$, generation 1 cares nothing for the welfare of generation 2. Defining intergenerational altruism in this manner (which essentially is taken from Kennedy and Welling 1997) implies a bequest function for generation 1 with the following characteristics

$$\frac{\partial B(k(T), w(T))}{\partial k(T)} \equiv \alpha \lambda_2^*(T)$$

$$\frac{\partial B(k(T), w(T))}{\partial w(T)} \equiv \alpha \eta_2^*(T)$$

The second equation, for example, indicates that the marginal value to generation 1 of endowing generation 2 with an extra unit of waste is defined as α times the marginal value of that endowment to generation 2. This follows from the fact that at the beginning of stage 2, the solution costate variables for the second generation control problem are $\lambda_2^*(T)$ and $\eta_2^*(T)$, which represent the value that the second generation places on inheriting an extra unit of capital and waste, respectively, given $k(T)$ and $w(T)$. Implicit in this definition of the bequest function is the rational expectation on the part of generation 1 that generation 2 will dispose of the endowed waste in an optimal manner; this provides justification for evaluating the costate variables on the right hand side of the

above identities at their optimal values. Evaluated along the optimal path for generation 1 we have

$$\frac{\partial B(k^*(T), w^*(T))}{\partial k(T)} \equiv \alpha \lambda_2^*(T)$$

$$\frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \equiv \alpha \eta_2^*(T)$$

In what follows, we assume $\alpha = 1$, not only because this assumption is reasonable, but also because we are interested in analyzing optimal discounting in a multi-generational problem that is marked by perfect intergenerational altruism. This seems to be a popular theme – the current generation ought to plan for future generations as if they cared as much for future generations as future generations care for themselves. Thus, generation 1's bequest function has the characteristics

$$\frac{\partial B(k^*(T), w^*(T))}{\partial k(T)} \equiv \lambda_2^*(T)$$

$$\frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \equiv \eta_2^*(T)$$

This convenient definition of the partial derivatives of the bequest function allows us to write the costate variables in a form that spans multiple generations, and, therefore, provides for a discussion of intergenerational discounting.

Some readers may feel that perfect intergenerational altruism also implies that the bequest function should not be discounted at the rate of time preference. We do not necessarily disagree, but we show in Section 3 that the results of this paper are independent of whether or not the bequest function is discounted at the rate of time preference. This result, of course, strengthens the conclusions of this paper.

Equations (3c) and (3d) are linear differential equations (see, e.g., Simon and Blume 1994 for the general solutions to such differential equations). It is important to note that while the differential equations that govern the movement of the costate variables are the same in both generations, the *particular solutions* are not the same. Particular solutions, of course, depend on boundary conditions which may differ from one generation to the

next. The particular solutions to the boundary value problems from the two periods of time are (see Appendix A for derivations)

$$(4) \quad \lambda_1^*(t) = \frac{\partial B(k^*(T), w^*(T))}{\partial k(T)} \exp\left[\rho(t-T) + \int_t^T r(s)ds\right]$$

$$(5) \quad \lambda_2^*(t) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial k(2T)} \exp\left[\rho(t-2T) + \int_t^{2T} r(s)ds\right]$$

$$(6) \quad \eta_1^*(0) = \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \exp[-\rho T] + \int_0^T \lambda_1^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt$$

$$(7) \quad \eta_2^*(T) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} \exp[-\rho(2T-T)] + \int_T^{2T} \lambda_2^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t-T)] dt$$

Using our definition of perfect altruism, together with equation (7), we have

$$(8) \quad \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \equiv \eta_2^*(T) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} \exp[-\rho(2T-T)] + \int_T^{2T} \lambda_2^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t-T)] dt$$

Inserting equation (8) into equation (6) gives an intergenerational version of the costate variable

$$(9) \quad \eta_1^*(0) = \left\{ \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} \exp[-\rho(2T-T)] + \int_T^{2T} \lambda_2^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t-T)] dt \right\} \exp[-\rho T] + \int_0^T \lambda_1^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt$$

Equation (9) is the value to generation 1 of an additional unit of waste cleanup at time zero. We note that this value includes separate components for the additional income generated during the separate phases of the intergenerational problem.

It is useful to rewrite equations (4) and (5) by expressing the partial derivatives of the bequest function in terms of the initial values of the costate variables. Evaluate equation (4) at time zero, solve for $\partial B(k^*(T), w^*(T)) / \partial k(T)$, and insert back into equation (4) to obtain on $[0, T]$

$$(4') \quad \lambda_1^*(t) = \lambda_1^*(0) \exp \left[\rho t - \int_0^t r(s) ds \right]$$

Similarly, evaluate equation (5) at time T , solve for $\partial B(k^*(2T), w^*(2T)) / \partial k(2T)$, and insert back into equation (5) to obtain on $[T, 2T]$

$$(5') \quad \lambda_2^*(t) = \lambda_2^*(T) \exp \left[\rho(t - T) - \int_T^t r(s) ds \right]$$

Use the transversality condition from stage 1 along with the definition of perfect altruism to note that

$$\lambda_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial k(T)} \equiv \lambda_2^*(T)$$

Thus, equation (5') can be written as

$$(10) \quad \lambda_2^*(t) = \lambda_1^*(T) \exp \left[\rho(t - T) - \int_T^t r(s) ds \right]$$

We can evaluate equation (4') at time T and insert into equation (10) to obtain

$$\lambda_2^*(t) = \lambda_1^*(0) \exp \left[\rho T - \int_0^T r(s) ds \right] \exp \left[\rho(t - T) - \int_T^t r(s) ds \right]$$

which reduces to

$$(11) \quad \lambda_2^*(t) = \lambda_1^*(0) \exp \left[\rho t - \int_0^t r(s) ds \right]$$

Equations (4'), (9), and (11) jointly ensure that, at the optimum, future income benefits accruing to generations 1 and 2 that result from a unit of waste reduction at time zero will be discounted at the market rate of interest. This is seen by inserting equations (4') and (11) into equation (9) to obtain

$$\begin{aligned}
\eta_1^*(0) &= \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} \exp[-\rho(2T - T)] \exp[-\rho T] \\
&+ \int_T^{2T} \lambda_1^*(0) \exp\left[\rho t - \int_0^t r(s) ds\right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t - T)] dt \exp[-\rho T] \\
&+ \int_0^T \lambda_1^*(0) \exp\left[\rho t - \int_0^t r(s) ds\right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt
\end{aligned}$$

which reduces to

$$\begin{aligned}
(12) \quad \eta_1^*(0) &= \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} \exp[-\rho 2T] \\
&+ \int_T^{2T} \exp\left[-\int_0^t r(s) ds\right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt \\
&+ \int_0^T \exp\left[-\int_0^t r(s) ds\right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt
\end{aligned}$$

where $\lambda_1^*(0)$ has been normalized to one (we normalize the units of the utility function such that the marginal utility of consumption at time zero equals one, see equation 3a).

Equation (12) shows the intended result. Future income benefits that result from immediate removal of toxic waste are discounted at the market rate of interest on an optimal growth path, irrespective of the time at which the income benefits are received. Thus, whether the benefits accrue to current generations or to future generations, the optimal discount rate in this model is the market rate of interest.

Furthermore, the terms of equation (12) can be combined to obtain

$$\begin{aligned}
\eta_1^*(0) &= \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} \exp[-\rho 2T] \\
&+ \int_0^{2T} \exp\left[-\int_0^t r(s) ds\right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt
\end{aligned}$$

where it is clear that the value of a unit of a waste removal at time zero includes the discounted value of the future income stream across the entire multi-generational planning interval $[0, 2T]$. That is, if intergenerational altruism is *perfect*, then the multi-generational planning problem collapses to a single control problem with twice the length, and the market rate of interest is the optimal discount rate.

3. NON-DISCOUNTED BEQUEST FUNCTION

Some readers may feel that *not* discounting the bequest functions better represents perfect intergenerational altruism. This issue is examined below. We demonstrate that the conclusions in the previous section hold even if the bequest functions are not discounted at the rate of time preference in the objective functional.

In this case, all the first order necessary conditions remain the same, except for the transversality conditions, which become

$$\exp[-\rho T]\lambda_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial k(T)}$$

$$\exp[-\rho T]\eta_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)}$$

for stage 1, and for stage 2

$$\exp[-\rho(2T - T)]\lambda_2^*(2T) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial k(2T)}$$

$$\exp[-\rho(2T - T)]\eta_2^*(2T) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)}$$

The left sides of each of the transversality conditions are present value costate variables. It is standard that the present value costate variables, by definition, equal the partial derivatives of the bequest function when this function is not discounted at the rate of time preference.

These new transversality conditions alter the particular solutions. The analogue to equation (6) is

$$(13) \quad \eta_1^*(0) = \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} + \int_0^T \lambda_1^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt$$

The analogue to equation (8) is

$$(14) \quad \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \equiv \eta_2^*(T) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} + \int_T^{2T} \lambda_2^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t - T)] dt$$

Insert (14) into (13) to obtain

$$(15) \quad \eta_1^*(0) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} + \int_T^{2T} \lambda_2^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t-T)] dt \\ + \int_0^T \lambda_1^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt$$

The analogues to equations (4) and (5) are

$$(16) \quad \lambda_1^*(t) = \frac{\partial B(k^*(T), w^*(T))}{\partial k(T)} \exp\left[\rho t + \int_t^T r(s) ds\right]$$

$$(17) \quad \lambda_2^*(t) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial k(2T)} \exp\left[\rho(t-T) + \int_t^{2T} r(s) ds\right]$$

Following the method in the last section for transforming equations (4) and (5) into (4') and (5'), equations (16) and (17) also can be transformed identically into (4') and (5'). Thus, equations (4') and (5') hold whether the bequest functions are or are not discounted at the rate of time preference.

The transversality condition and the definition of perfect altruism imply

$$\exp[-\rho T] \lambda_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial k(T)} \equiv \lambda_2^*(T)$$

Using this together with equation (5')

$$\lambda_2^*(t) = \exp[-\rho T] \lambda_1^*(T) \exp\left[\rho(t-T) - \int_T^t r(s) ds\right]$$

which can be rewritten using equation (4') evaluated at time T as

$$\lambda_2^*(t) = \exp[-\rho T] \lambda_1^*(0) \exp\left[\rho T - \int_0^T r(s) ds\right] \exp\left[\rho(t-T) - \int_T^t r(s) ds\right]$$

This reduces to

$$(18) \quad \lambda_2^*(t) = \lambda_1^*(0) \exp\left[\rho(t-T) - \int_0^t r(s) ds\right]$$

Insert equations (18) and (4') into equation (15) to obtain

$$(19) \quad \eta_1^*(0) = \frac{\partial B(k^*(2T), w^*(2T))}{\partial w(2T)} + \int_T^{2T} \exp\left[-\int_0^t r(s) ds\right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt \\ + \int_0^T \exp\left[-\int_0^t r(s) ds\right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt$$

where $\lambda_1^*(0)$ has been normalized to one. Equation (19) shows that future income benefits from waste cleanup that accrue to future generations still are discounted at the

market rate of interest on the optimal path even if the bequest function is not discounted at the rate of time preference in the objective functional.

4. SELFISH INTERIM GENERATIONS

A common concern with discounting future benefits in projects that span multiple generations is the potential behavior of interim generations that could undo the provisions that are intended for the welfare of distant future generations. This section analyzes a related issue.

We expand the analysis of Section 2 to include a third generation. To achieve this we include the necessary conditions for the third generation, and by modifying equation (3b) as discussed below. Generation 3 exists on $[2T, 3T]$. We assume that each generation is perfectly altruistic towards the next generation in terms of the capital bequest. That is, generation i values the act of endowing generation $i + 1$ with capital exactly the same as generation $i + 1$ values receiving the capital endowment. However, we assume that only generations 1 and 3 are altruistic in terms of the bequest of toxic waste. Generation 2 is selfish and completely indifferent when it comes to preserving the environment for future generations. Generation 2 still may allocate funds to waste cleanup, but only to improve their own health.

Our motivation for considering this type of scenario comes from the idea that tastes for a clean environment may not be constant. Therefore, how should generation 1 discount the benefits from immediate waste cleanup that accrue to generation 3, provided that generation 2 is not concerned with reducing the amount of waste that generation 3 will inherit? Further, generation 2 may react to a smaller waste endowment from generation 1 by reducing their own cleanup expenditure on $[T, 2T]$, effectively undoing the cleanup efforts of generation 1. How does this affect intergenerational discounting? Importantly, we demonstrate that the answers to these interesting questions are consistent with the thrust of this paper: it is optimal for generation 1 to discount the future income benefits to that accrue to generation 3 at the market rate of interest, even if generation 2 is completely selfish.

Using the transversality conditions from the Section 2 together with the ideas mentioned in the above paragraphs, we have

$$\lambda_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial k(T)} \equiv \lambda_2^*(T)$$

$$\lambda_2^*(2T) = \frac{\partial B(k^*(2T))}{\partial k(2T)} \equiv \lambda_3^*(2T)$$

Thus, with respect to capital, generation 1 is altruistic towards generation 2, who in turn is altruistic towards generation 3. However, this link of altruism does not hold with respect to waste removal. Instead, we consider a case where generation 1 is perfectly altruistic towards both the remaining generations, and generation 2 cares only for themselves

$$\eta_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \equiv \eta_2^*(T) + \exp[-\rho(2T - T)]\eta_3^*(2T) \frac{\partial w^*(2T)}{\partial w^*(T)}$$

$$\eta_2^*(2T) = \frac{\partial B(k^*(2T))}{\partial w(2T)} \equiv 0$$

The marginal value (which is negative) that generation 1 places on the bequest of one extra unit of waste at time T is composed of the combined cost to generations 2 and 3. The partial derivative $\partial w^*(2T)/\partial w^*(T)$ captures the change in the amount of waste that generation 2 bequeaths to generation 3, as a result of an incremental change in generation 2's endowment of waste. We have also explored different possibilities for the manner in which perfect altruism is defined with respect to pure time preference. In Appendix B we extend the analysis of Section 3 where the bequest function is not discounted, as this may be more in line with "perfect" altruism. Importantly, the results of this section hold identically in either case.

The general solution to the differential equation for waste accumulation is

$$w^*(t) = C - \int_t^T g^*(s)ds$$

where C is a constant. Evaluate at time T , solve for C and substitute to obtain on $[T, 2T]$

$$w^*(t) = w^*(T) - \int_T^t g^*(s)ds$$

Evaluate at the end of generation 2, $t = 2T$, and differentiate

$$\frac{\partial w^*(2T)}{\partial w^*(T)} = 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt$$

If generation 1 endows generation 2 with an extra unit of waste, generation 2 will respond by endowing generation 3 with an extra unit, net of the change in their own waste removal spending on $[T, 2T]$.

Thus, we can rewrite the altruism condition as

(20)

$$\eta_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \equiv \eta_2^*(T) + \exp[-\rho(2T - T)]\eta_3^*(2T) \left\{ 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt \right\}$$

In the previous sections, we assumed that non-negativity constraints on the control and state variables were automatically satisfied. However, because generation 2 is selfish, we must explicitly account for a non-negativity constraint on waste removal in that generation. At time $2T$, the selfish second generation has the incentive to chose infinitely negative waste removal, which essentially means that income would be infinitely positive at time $2T$. Infinite income would be divided between consumption and gross investment, which implies that the value of the marginal capital endowment to generation 3 is zero. That is, generation 3 would be endowed with infinite capital, which consequently is worthless at the margin. Without a non-negative constraint on waste removal, this unrealistic scenario would be perfectly consistent with the first of necessary conditions of the problem.

Accounting for the non-negativity of $g(t)$ in generation 2 consists of altering equation (3b) to include the non-negative multiplier $\mu(t)$

$$-\lambda_2^*(t) - \eta_2^*(t) + \mu^*(t) = 0$$

and the complementary slackness condition

$$\mu^*(t)w^*(t) = 0$$

The non-negativity constraint will bind at time $2T$ and possibly before that time depending on the returns to health in the production of national income. That is, if the returns to good health are low relative to the cost of waste removal, the constraint will bind.

Generation 3 values an additional initial unit of waste according to

$$\begin{aligned}
(21) \quad \eta_3^*(2T) = & \frac{\partial B(k^*(3T), w^*(3T))}{\partial w(3T)} \exp[-\rho(3T - 2T)] \\
& + \int_{2T}^{3T} \lambda_3^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t - 2T)] dt
\end{aligned}$$

Thus, using (20), (21), and equation (7) with $\partial B(k^*(2T))/\partial w(2T) \equiv 0$, we can write the value of waste to generation 1 at time zero as (i.e., rewrite equation 6)

$$\begin{aligned}
(22) \quad \eta_1^*(0) = & \left\{ \frac{\partial B(k^*(3T), w^*(3T))}{\partial w(3T)} \exp[-\rho 3T] \right. \\
& + \left. \int_{2T}^{3T} \lambda_3^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt \right\} * \left\{ 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt \right\} \\
& + \int_T^{2T} \lambda_2^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt \\
& + \int_0^T \lambda_1^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt
\end{aligned}$$

The costate variable for physical capital during generation 3 is

$$\lambda_3^*(t) = \lambda_3^*(2T) \exp \left[\rho(t - 2T) - \int_{2T}^t r(s) ds \right]$$

which can be rewritten using the intergeneration altruism with respect to generation 2's capital bequest

$$(23) \quad \lambda_3^*(t) = \lambda_2^*(2T) \exp \left[\rho(t - 2T) - \int_{2T}^t r(s) ds \right]$$

Evaluate equation (11) at $t = 2T$ and insert into equation (23)

$$(24) \quad \lambda_3^*(t) = \lambda_1^*(0) \exp \left[\rho t - \int_0^t r(s) ds \right]$$

Insert equations (4'), (11), and (24) into equation (22) and normalize $\lambda_1^*(0) \equiv 1$ to obtain

$$\begin{aligned}
\eta_1^*(0) = & \frac{\partial B(k^*(3T), w^*(3T))}{\partial w(3T)} * \left\{ 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt \right\} \exp[-\rho 3T] \\
& + \int_{2T}^{3T} \exp \left[- \int_0^t r(s) ds \right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \left\{ 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt \right\} dt \\
& + \int_T^{2T} \exp \left[- \int_0^t r(s) ds \right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt \\
& + \int_0^T \exp \left[- \int_0^t r(s) ds \right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt
\end{aligned}$$

Thus, the value to generation 1 of an additional unit of waste removal at time zero includes the discounted stream of income to generation 3 from good health. Even though the gains to generation 3 may be small, due to a selfish interim generation that may respond to their own initial endowment at time T by reducing waste removal spending on $[T, 2T]$, these benefits are still discounted at the market rate of interest at the optimum. Again, see Appendix B for a derivation of the same conclusion under an alternative interpretation of perfect altruism (i.e., no intergenerational pure time preference).

5. CONCLUSION

This paper supports the discounting of income benefits at the market rate of interest in projects (like waste removal) that span multiple generations. To strengthen this conclusion, we have examined various scenarios involving pure time preference and selfish interim generations.

It is well documented that many economists are strongly against the use of positive pure time preference in intergenerational projects. Thus, we have considered the issue of waste removal under the cases where pure time preference does and does not exist across generations. We find that in either case, the market rate of interest is the optimal discount rate for future income benefits that result from immediate cleanup of toxic waste.

Finally, a common concern in intergenerational planning is the potential for selfish behavior of future generations that may undo environmental provisions made by the current generation, effectively reducing the benefits that are received by generations beyond the selfish generation. While we have not considered all the related possibilities, we do find that even if a selfish interim generation does not care about the amount of waste they leave behind, it is still optimal for an altruistic current generation to discount at the market rate of interest those benefits received by generations beyond the selfish generation. This result holds even if the selfish generation responds to a smaller initial endowment of waste by reducing their cleanup efforts.

APPENDIX A. DERIVATION OF PARTICULAR SOLUTIONS

Equation (3c) has the general solution

$$(A1) \quad \lambda_i^*(t) = C \exp \left[\rho t - \int_t^T r(s) ds \right]$$

where C is an arbitrary constant. Evaluate (A1) at $i = 1$ and $t = T$, solve for C , and insert back into (A1) to obtain

$$\lambda_1^*(t) = \lambda_1^*(T) \exp \left[\rho(t - T) + \int_t^T r(s) ds \right]$$

Use the transversality condition to find $\lambda_1^*(T)$, which gives equation (4).

To find equation (5), evaluate equation (A1) at $i = 2$ and $t = 2T$, solve for C , and insert back into (A1) to obtain

$$\lambda_2^*(t) = \lambda_2^*(2T) \exp \left[\rho(t - 2T) + \int_t^{2T} r(s) ds \right]$$

and use the transversality condition to find $\lambda_2^*(2T)$.

Equation (3d) has the general solution

$$(A2) \quad \eta_i^*(t) = \left\{ D - \int_t^T \lambda_i^*(s) \frac{\partial f(k^*(s), h(w^*(s)))}{\partial h(w(s))} \frac{\partial h(w^*(s))}{\partial w(s)} \exp[-\rho s] ds \right\} \exp[\rho t]$$

where D is a constant of integration. To find equation (6), evaluate equation (A2) at $i = 1$ and $t = T$, solve for D , insert back into equation (A2) to obtain

$$\eta_1^*(t) = \eta_1^*(T) \exp[-\rho(T - t)] + \int_t^T \lambda_1^*(s) \frac{\partial f(k^*(s), h(w^*(s)))}{\partial h(w(s))} \frac{\partial h(w^*(s))}{\partial w(s)} \exp[-\rho(s - t)] ds$$

and evaluate at $t = 0$ and use the transversality condition to obtain equation (6). To find equation (7), evaluate equation (A2) at $i = 2$ and $t = 2T$, solve for D , and insert into (A2) to obtain

$$\eta_2^*(t) = \eta_2^*(2T) \exp[-\rho(2T - t)] + \int_t^{2T} \lambda_2^*(s) \frac{\partial f(k^*(s), h(w^*(s)))}{\partial h(w(s))} \frac{\partial h(w^*(s))}{\partial w(s)} \exp[-\rho(s - t)] ds$$

Use the transversality condition and evaluate at $t = T$.

APPENDIX B. ALTERNATIVE TO SECTION 4

Here we rework Section 4, but with non-discounted bequest functions as in Section 3, and with the assumption that generation 1 cares for generations 2 and 3 exactly alike (without pure time preference). Thus,

$$\exp[-\rho T]\lambda_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial k(T)} \equiv \lambda_2^*(T)$$

$$\exp[-\rho(2T - T)]\lambda_2^*(2T) = \frac{\partial B(k^*(2T))}{\partial k(2T)} \equiv \lambda_3^*(2T)$$

$$\exp[-\rho T]\eta_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \equiv \eta_2^*(T) + \eta_3^*(2T) \frac{\partial w^*(2T)}{\partial w^*(T)}$$

$$\exp[-\rho(2T - T)]\eta_2^*(2T) = \frac{\partial B(k^*(2T))}{\partial w(2T)} \equiv 0$$

Note that generation 1 values their bequest of waste to the second and third generations exactly as much as those latter two generations collectively value a unit of waste, and there is no pure time preference in this relationship (that is, the value to generation 3 of a unit of waste removal at time T , $\eta_3^*(2T)\partial w^*(2T)/\partial w^*(T)$, is not discounted back to time T as in Section 4).

As in Section 4,

$$\frac{\partial w^*(2T)}{\partial w^*(T)} = 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt$$

Thus,

$$(B1) \quad \exp[-\rho T]\eta_1^*(T) = \frac{\partial B(k^*(T), w^*(T))}{\partial w(T)} \equiv \eta_2^*(T) + \eta_3^*(2T) \left\{ 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt \right\}$$

Generation 3 values an additional initial unit of waste according to

(B2)

$$\eta_3^*(2T) = \frac{\partial B(k^*(3T), w^*(3T))}{\partial w(3T)} + \int_{2T}^{3T} \lambda_3^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t - 2T)] dt$$

Thus, using (B1), (B2), and the right hand side of equation (14) with

$\partial B(k^*(2T))/\partial w(2T) = 0$, we can write the value of waste to generation 1 at time zero as (i.e., rewrite equation 13)

(B3)

$$\begin{aligned}
\eta_1^*(0) = & \left\{ \frac{\partial B(k^*(3T), w^*(3T))}{\partial w(3T)} \right. \\
& + \int_{2T}^{3T} \lambda_3^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t-2T)] dt \Big\} * \left\{ 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt \right\} \\
& + \int_T^{2T} \lambda_2^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho(t-T)] dt \\
& + \int_0^T \lambda_1^*(t) \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \exp[-\rho t] dt
\end{aligned}$$

As in previous sections

$$\lambda_3^*(t) = \lambda_3^*(2T) \exp \left[\rho(t-2T) - \int_{2T}^t r(s) ds \right]$$

which can be rewritten using the definition of intergeneration altruism with respect to generation 2's capital bequest

$$(B4) \quad \lambda_3^*(t) = \exp[-\rho(2T-T)] \lambda_2^*(2T) \exp \left[\rho(t-2T) - \int_{2T}^t r(s) ds \right]$$

Insert equation (18) evaluated at $t = 2T$ to obtain

$$(B5) \quad \lambda_3^*(t) = \lambda_1^*(0) \exp \left[\rho(t-2T) - \int_0^t r(s) ds \right]$$

Insert equations (4'), (18), and (B5) into equation (B3) and normalize $\lambda_1^*(0) \equiv 1$ to obtain

$$\begin{aligned}
\eta_1^*(0) = & \frac{\partial B(k^*(3T), w^*(3T))}{\partial w(3T)} * \left\{ 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt \right\} \\
& + \int_{2T}^{3T} \exp \left[- \int_0^t r(s) ds \right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} \left\{ 1 - \int_T^{2T} \frac{\partial g^*(t)}{\partial w^*(T)} dt \right\} dt \\
& + \int_T^{2T} \exp \left[- \int_0^t r(s) ds \right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt \\
& + \int_0^T \exp \left[- \int_0^t r(s) ds \right] \frac{\partial f(k^*(t), h(w^*(t)))}{\partial h(w(t))} \frac{\partial h(w^*(t))}{\partial w(t)} dt
\end{aligned}$$

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OPTIMAL DISCOUNTING IN CONTROL PROBLEMS THAT SPAN MULTIPLE GENERATIONS

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ABSTRACT

The principal contribution of this paper is the linking together of separate control problems across multiple generations using the bequest motive, intergenerational altruism, rational expectations, and solution boundary conditions. We demonstrate that discounting at the market rate of interest is an endogenous characteristic of a general equilibrium, optimal control problem that spans multiple generations. Within the confines of our model, we prove that it is optimal to discount at the market rate of interest the social benefits to distant generations from immediate clean up at toxic waste sites if the current generation that bears the cleanup cost is perfectly altruistic towards future generations. Also, we show that this result holds for alternative assumptions regarding pure time preference. Moreover, the result holds regardless of whether selfish interim generations attempt to undo the provisions made for distant generations. In our distortion-free deterministic model, the evidence for intergenerational discounting at the market rate of interest is compelling.

1. INTRODUCTION

A recent book entitled *Discounting and Intergenerational Equity*, published by Resources for the Future, represents an insightful compilation of the opinions and analysis of some of the foremost economists on the issue of discounting. The principle question being addressed is whether long-term environmental projects can be treated the same as short-term ones, as far as discounting is concerned. Some authors, Arrow (1999), Weitzman (1999), Bradford (1999), and Montgomery (1999) suggest that the market rate of interest is the appropriate discount rate even in projects that span multiple generations of time. Others, Cline (1999), Lind (1999), Schelling (1999), and Rothenberg (1999) hold that such projects cannot be viewed in the traditional cost-benefit sense.

This is a complex issue and both sets of opinions have merit. We do not presume to settle this debate, and we do not intend to give a detailed account of all the issues raised by these authors. Rather, our intention is to show that discounting at the market rate of interest is optimal in a distortion-free general equilibrium, environmental control problem that spans multiple generations of time, provided current generations are perfectly altruistic towards future generations. This important proof provides rigorous support for the market rate of interest as the appropriate discount rate in multigenerational public projects that deal with the environment.

Lyon (1996) proved that the discounting of future benefits from natural resources at the market rate of interest is a feature of a deterministic, general equilibrium optimal growth program for which distortions are absent. (See Weitzman 1994 and Weitzman 1998 for the effects of distortions and uncertainty on the social discount rate). We extend Lyon's analysis to a multi-generational problem and show that under certain conditions his conclusions hold: the market rate of interest is the optimal discount rate, regardless of the time horizon.

We analyze a simple growth model where toxic waste sites affect the health and therefore the productivity of the workforce, and waste can be removed in accordance with social cleanup expenditure. Waste removal generates good health and therefore additional income from increased productivity of workers. We consider separate, single generation continuous time control problems with these features and we link these

problems together (end-to-end) using the bequest motive, intergenerational altruism, rational expectations, and solution endpoint conditions. The result is an intergenerational control problem.