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THE COSTATE VARIABLE IN A STOCHASTIC RENEWABLE RESOURCE MODEL

Kenneth S. Lyon and Saket Pande

ABSTRACT

In this paper we discuss the costate variable in a stochastic optimal control model of a renewable natural resource, which we call a fishery. The role of the costate variable in deterministic control models has been discussed extensively in the literature. See, for example, Lyon (1999), Clark (1990, pp. 102-107), and Arrow and Kurz (1970, pp. 35-37); however, there is little discussion of this variable for stochastic models, even though the costate variable has similar roles in the two models. In both models the costate variable is a shadow value of the associated state variable, and as such has the role of rationing the use of the state variable. In addition, as has been shown in Lyon (1999), in natural resource problems the costate variable can be partitioned into a scarcity effect and a cost effect. We show that this same partitioning can be done in the stochastic renewable resource problem. We discuss and contrast the similarities and differences in these concepts for deterministic and stochastic models. In addition, we present a numerical example to help solidity the results.

Key words: costate variable, stochastic dynamic programming, deterministic dynamic programming, renewable resource model, simulation
THE COSTATE VARIABLE IN A STOCHASTIC RENEWABLE RESOURCE MODEL

Introduction

In this paper we discuss the costate variable in a stochastic optimal control model of a renewable natural resource, which we call a fishery. The role of the costate variable in deterministic control models has been discussed extensively in the literature. See for example Lyon (1999), Clark (1990, pp.102-107), and Arrow and Kurz (1970, pp. 35-37); however, there is little discussion of this variable for stochastic models, even though the costate variable has similar roles in the two models. In both models the costate variable is a shadow value of the associated state variable, and as such has the role of rationing the use of the state variable. In addition, as has been shown in Lyon (1999), in natural resource problems the costate variable can be partitioned into a scarcity effect and a cost effect. We show that this same partitioning can be done in the stochastic renewable resource problem. We discuss and contrast the similarities and differences in these concepts for deterministic and stochastic models. In addition, we present a numerical example help solidify the results.

Below we first identify discrete time deterministic and stochastic optimal control models and manipulate them to identify the desired concepts. This includes the identification of the costate variable. We use discrete time rather than continuous time, because the stochastic elements require less overhead in discrete time. We then discuss the costate variable. After this we present a numerical stochastic fishery model to illustrate the concepts. The final section contains the summary and conclusions.

The Discrete Time Models

The models are of a renewable resource which we will refer to as a fishery. In the deterministic model the objective is to maximize the present value of the resource and in the stochastic model it is to maximize the expected value of the resource. In both models we assume that current values of the variables are known with certainty; however, in the stochastic model the law of motion for the fish stock will have a random component.

The real interest rate, \( r \), is assumed to be constant to simplify the exposition, and we define \( \beta = 1/(1+r) \), the discount factor. To achieve the objective we maximize the present value of the net surplus stream, \( s(h_t) \), and the expected value of the net surplus stream, \( E[s(h_t)] \), in the two models respectively, where \( h_t \) is the harvest on the time horizon \( t = 1, 2, ..., T \). We let \( x_t \) give the time path, and \( E[x_t] \) give the expected time path of the resource stock in the two models respectively on the same time horizon. In both the initial condition for \( x \) is \( x_0 = x^0 \), and in both models the harvest cost function is written \( c(h_t, x_t) \) with the following characteristics. Marginal harvest costs are positive and increase with the rate of harvest, \( c_h(h, x) > 0 \), \( c_{hh}(h, x) \geq 0 \). In addition, harvest costs decrease as the resource stock increases, \( c_x(h, x) \leq 0 \), which is the usual relationship. This is the result of greater abundance lowering harvest costs. The demand function in inverse form is written as \( D(h) \) with the usual negative slope, \( D'(h) < 0 \). Net surplus can be written

\[
s(h_t, x_t) = \int_0^{h_t} D(v)dv - c(h_t, x_t).\]

For the deterministic model the objective functional can be written
(1) \[ W = \sum_{t=0}^{T} \beta^t s(h_t, x_t) + \beta^{T+1} S(x_{T+1}), \]

which is maximized subject to

(2) \[ x_{t+1} = x_t + g(x_t) - h_t, \]

(3) \[ x_0 = x^0, \text{ given} \]

(4) \[ h_t, x_t \geq 0 \quad \text{for} \quad t = 0, 1, ..., T \text{ with } T \text{ given}, \]

where, \( g(x) \) is the growth, recruitment, or reproduction function for the resource stock, fish, \( S \) is the terminal value function, and \( T \) is the end of the current time horizon. The graph of the growth function, \( g(x) \), is assumed to have an inverted "U" shape such as that related to the logistic curve, and is assumed to be differentiable; however, in the numerical example we use an inverted "V" shape. Subtracting from this growth the harvest, \( h_t \), yields the net growth, \( x_{t+1} - x_t \). The terminal value function, \( S \), can be thought of as a bequest function for the fish stock that is left to some distant generation. The end of the current time horizon, \( T \), is assumed to be sufficiently distant that the system will have evolved to the stationary state, the state where \( x_{t+1} - x_t = 0 \).

The necessary conditions can be found using Bellman’s principal of optimality or using the Lagrange multiplier method. We present both to show the relationship between the two. Define

\[
V_t(x_t) = \max_{h_t} \sum_{t=0}^{T} \beta^{t-t'} s(h_t, x_t) + \beta^{T+1} S(x_{T+1}) \]

subject to Equations (2) and (4) with \( x_t \) given. Using the Bellman equation (principal of optimality) we can write

\[
V_t(x_t) = \max_h \{s(h_t, x_t) + \beta V_{t+1}(x_{t+1})\} \]

subject to Equations (2) and (4) with \( x_t \) given. Substituting Equation (4) into this last equation yields:

(5) \[ V_t(x_t) = \max_h \{s(h_t, x_t) + \beta V_{t+1}(x_{t+1} + g(x_t) - h_t)\} \]

The necessary conditions for an internal solution (i.e., where Equation (4) is satisfied) are:

(5a) \[ \frac{\partial s(h'_t, x'_t)}{\partial h_t} - \beta \frac{\partial V_{t+1}(x'_{t+1})}{\partial x_{t+1}} = 0, \quad t = 0, 1, ..., T - 1 \]

(5b) \[ \frac{\partial s(h'_t, x'_t)}{\partial h_t} - \beta S'(x'_{t+1}) = 0 \]
(5c) \[ x_{t+1}^* = x_t^* + g(x_t^*) - h_t^*, \quad x_0^* = x^0, \quad t = 0,1,...,T \]

where the asterisk indicates optimum value. In addition, by the Envelope Theorem we have

(5d) \[ \frac{\partial V_t(x_t^*)}{\partial x_t^*} = \frac{\partial s(h_t^*, x_t^*)}{\partial x_t^*} + \beta \frac{\partial V_{t+1}(x_{t+1}^*)}{\partial x_{t+1}^*} [1 + g'(x_t^*)]. \]

Equations (5a) - (5d) will be put into a correspondence with equations from the Lagrange multiplier method, which we now present.

In the Lagrange multiplier method (Chow, 1997) we define:

(6) \[ L = \sum_{t=0}^{T} \beta^t s(h_t, x_t) + \beta^{T+1} \lambda_{T+1} (x_T + g(x_T) - h_T - x_{T+1}) + \beta^{T+1} S(x_{T+1}) \]

where the \( \lambda \)'s are the Lagrangian multipliers. We ignore the non-negativity constraints and examine an internal solution where the Lagrangian multipliers associated with these constraints are all zero. The necessary conditions for an internal solution of the problem in Equation (6) are:

(6a) \[ \frac{\partial s(h_t^*, x_t^*)}{\partial h_t} - \beta \lambda_{t+1}^* = 0, \quad t = 0,1,...,T \]

(6b) \[ x_{t+1}^* = x_t^* + g(x_t^*) - h_t^*, \quad x_0^* = x^0, \quad t = 0,1,...,T \]

(6c) \[ \lambda_t^* = \frac{\partial s(h_t^*, x_t^*)}{\partial x_t} + \beta \lambda_{t+1}^* [1 + g'(x_t^*)], \quad t = 0,1,...,T, \]

(6d) \[ \lambda_{T+1}^* = S'(x_{T+1}^*). \]

A comparison of Equation (5a) - (5d) with (6a) - (6b), yields the conclusion that

\[ \lambda_{t+1}^* = \frac{\partial V_{t+1}(x_{t+1}^*)}{\partial x_{t+1}^*} \]

because they are necessary conditions for the same maximization problem. The \( \lambda \)'s are costate variables and are as usual the shadow value of the resource stock.

We proceed now with the identification of the cost and scarcity effects in the costate variable. To achieve this we use \( \lambda_0^* \) to simplify the writing of the subscripts; however, we could use any \( t \) from 0 to \( T-1 \). We use Equation (6c) recursively as follows:
\[ \lambda_0 = \frac{\partial s(h_0^*, x_0^*)}{\partial x_0} + \beta \lambda_1^* [1 + g'(x_0^*)], \]

\[ \lambda_0 = \frac{\partial s(h_0^*, x_0^*)}{\partial x_0} + \beta \left( \frac{\partial s(h_1^*, x_1^*)}{\partial x_1} + \beta \lambda_2^* [1 + g'(x_1^*)] \right) [1 + g'(x_0^*)], \]

\[ \lambda_0 = \frac{\partial s(h_0^*, x_0^*)}{\partial x_0} + \beta \left( \frac{\partial s(h_1^*, x_1^*)}{\partial x_1} + \beta \left( \frac{\partial s(h_2^*, x_2^*)}{\partial x_2} + \beta \lambda_3^* [1 + g'(x_2^*)] \right) [1 + g'(x_1^*)] \right) [1 + g'(x_0^*)]. \]

We now shorthand \( \partial s(h_0^*, x_0^*) / \partial x_0 = \partial s_1^* / \partial x_0 \) and \( 1 + g'(x_0^*) = 1 + g_{1,0}^* \), and continue

\[ \lambda_0^* = \frac{\partial s_0^*}{\partial x_0} + \beta \frac{\partial s_1^*}{\partial x_1} (1 + g_{0,0}^*) + \beta^2 \frac{\partial s_2^*}{\partial x_2} (1 + g_{0,0}^*) (1 + g_{1,1}^*) + \beta^3 \frac{\partial s_3^*}{\partial x_3} (1 + g_{0,0}^*) (1 + g_{1,1}^*) (1 + g_{2,2}^*) + \ldots \]

\[ + \beta^{r+1} (1 + g_{0,0}^*) (1 + g_{1,1}^*) \ldots (1 + g_{r,r}^*) \lambda_{r+1}^* \]

(7) \[ \lambda_0^* = \frac{\partial s_0^*}{\partial x_0} + \sum_{r=1}^{T} \beta^r \frac{\partial s_r^*}{\partial x_r} \left( \prod_{k=0}^{r-1} (1 + g_{k,k}^*) \right) + \beta^{T+1} \left( \prod_{k=0}^{T} (1 + g_{k,k}^*) \right) \lambda_{T+1}^* \]

Note that \( \partial s(h_i^*, x_i^*) / \partial x_i = \partial s_i^* / \partial x_i \) where the sub-x indicates partial derivative. Also note that \( \prod_{k=0}^{T} (1 + g_{k,k}^*) \) can be viewed as giving the time path of an additional unit of \( x \) originating at time zero, \( t = 0 \), which we call \( \xi_{t,0} \). Note that \( g'(x) \) has been called the own biological rate of interest by Quirk and Smith (1970). It is the discrete annual growth rate or physical return to one unit of the resource. For the more general case where the new unit originates in time \( \tau \) we have

\[ \xi_{t,\tau} = \prod_{k=0}^{t-1} (1 + g_{k,k}^*) . \]

Using these definitions we can rewrite Equation (7) as

(8) \[ \lambda_0^* = -\sum_{r=0}^{T} \beta^r \xi_{t,0} \xi_{t,\tau} (h_i^*, x_i^*) + \beta^{T+1} \xi_{T+1,0} \lambda_{T+1}^* \]

For the more general case we have

(8') \[ \lambda_0^* = -\sum_{r=\tau}^{T} \beta^r \xi_{t,\tau} c_x (h_i^*, x_i^*) + \beta^{T+1} \xi_{T+1,\tau} \lambda_{T+1}^* \]

In Equation (8')

(9a) \[ \beta^{T+1} \xi_{T+1,\tau} \lambda_{T+1}^* \]
is the Scarcity Effect given that \( c_x = 0 \), and

\[
(9b) \quad - \sum_{i=t}^{T} \beta^{i-t} \xi_{t,i} c_x(h_x^*, x_x^*)
\]

is the Cost Effect given that \( T \) is sufficiently large that \( \beta^{T+1} \) is nil. These results parallel those given in Lyon (1999) for the continuous time case, and these effects have exactly the same interpretation as in the continuous time case. When \( c_x(h_x, x_t) = 0 \) for all \( x_t \) (i.e., \( c_x = 0 \)) it is easy to see that the Cost Effect disappears; thus the value of the additional unit of the resource stock at time \( r \) is the present value of \( \xi_{T+1, r} \) times the value of one unit at \( T+1 \). The additional unit of resource at \( r \) grows or shrinks to \( \xi_{T+1, r} \) units in time \( T+1 \). When the Cost Effect is non-null we can select \( T \) to be arbitrarily large so that Equation (9a) is nil. Since \( c_x \) is negative, the Cost Effect is positive, and identifies the cost saving associated with the additional resource stock. The corresponding terms will be identified below for the stochastic costate variable.

We now develop the parallel results for the stochastic model. We consider the class of "admissible" control laws (policies) that consist of a finite sequence of functions \( \pi = \{ \mu_0, \mu_1, ..., \mu_T \} \), where \( \mu_t : S_t \rightarrow C_t \) and such that \( \mu_t(x_t) \in U_t(x_t) \forall x_t \in S_t \). Here \( C_t, S_t \) are spaces of elements \( h_t \) and \( x \), respectively. The control \( h_t \) is constrained to take values from a given non-empty subset \( U_t(x_t) \) of \( C_t \), which depends on the current value of the state variable. It is assumed that for some admissible policy vector, the optimal expected value could be achieved. The problem of maximizing the expected value of the net surplus stream then has the objective functional

\[
E_0[W] = E_{\xi_t=0,T} \left[ \sum_{t=0}^{T} \beta^t s(\mu_t(x_t), x_t) + \beta^{T+1} S(x_{T+1}) \right]
\]

which is to be maximized subject to

\[
(11) \quad x_{t+1} = x_t + g(x_t) - \mu_t(x_t) + \xi_t
\]

\[
(12) \quad x_0 = x^0, \text{ given}
\]

\[
(13) \quad \mu_t(x_t) \in U_t(x_t) \subseteq C_t, x_t \in S_t \geq 0 \text{ for } t = 0, 1, ..., T \text{ with } T \text{ given,}
\]

where \( \xi_t \in D_t \) (\( D_t \) is space of elements \( \xi_t \)) is a random variable that is independently and identically distributed through time with mean zero and density function \( P_t(\xi) \). We assume that at time \( t \) there is certainty about all variables except \( \xi_t \), which becomes known by time \( t+1 \). The term \( g(x_t) + \xi_t \) is the stochastic growth function over the time period \( t \) to \( t+1 \). The term \( g(x_t) \) gives the mean growth since the mean value of \( \xi_t \) is zero.

However, the above formulation is equivalent to (Bertsekas, 1987, p. 14)
which is to be maximized subject to (12) and

\[ x_{t+1} = x_t + g(x_t) - h_t + \varepsilon_t \]

\[ h_t \in U_t(x_t) \subseteq C_t, x_t \in S, \geq 0 \text{ for } t = 0, 1, ..., T \text{ with } T \text{ given.} \]

Since \( \varepsilon_t \sim P_t(\varepsilon) \). From (11)

\[ \varepsilon_t \sim P_t(x_{t+1} - x_t - g(x_t) + \mu_t(x_t)) = P_t(x_{t+1} | x_t). \]

If the expectation operator is defined as in (10), then

\[ E[\cdot] = \int P_t(\varepsilon) \mathrm{d}\varepsilon = \int P(x_{t+1} | x_t) \mathrm{d}x_{t+1} = E_t[\cdot | x_t] \]

As in the deterministic problem the necessary conditions can be found using the Bellman equation or using the Lagrange multiplier method. As above we present both to show the relationship between the two methods. Define for any \( t' \) with \( 0 \leq t' \leq T \)

\[ v_{t'}(x_{t'}) = \max_{t \in {0, T}} E_t \left[ \sum_{t'=t}^{T} \beta^{t'-t} s(h_{t'}, x_{t'}) + \beta^{T+1} S(x_{T+1}) | x_{t'} \right] \]

subject to Equations (11) and (13) with \( x_{t'} \) given. Using the Bellman equation for \( t \in [0, T] \) we can write

\[ v_{t'}(x_{t'}) = \max_{h_t} E_t \left[ \{s(h_{t'}, x_{t'}) + \beta v_{t+1}(x_{t+1}) \} | x_{t'} \right] \]

subject to Equations (11) and (13) with \( x_t \) given. We define \( v_{T+1}(x_{T+1}) = S(x_{T+1}) \). Substituting Equation (11) into this last equation yields:

\[ v_{t'}(x_{t'}) = \max_{h_t} E_t \left[ \{s(h_{t'}, x_{t'}) + \beta v_{t+1}(x_{t+1}) + g(x_{t'}) - h_t + \varepsilon_t \} | x_{t'} \right] \]

or

\[ v_{t'}(x_{t'}) = \max_{h_t} \{s(h_{t'}, x_{t'}) + \beta E_t \left[ v_{t+1}(x_{t+1}) + g(x_{t'}) - h_t + \varepsilon_t \} | x_{t'} \} \]

The necessary conditions for an internal solution are:

\[ \frac{\partial s(h^*_{t'}, x_{t'})}{\partial h_t} - \beta E_t \left[ \frac{\partial v_{t+1}(x_{t+1})}{\partial x_{t+1}} \right] | x_{t'} = 0, \ t = 0, 1, ..., T - 1 \]
Applying the Envelope Theorem to Equation (14) yields

\[
\frac{\partial s(h^*_t, x_t)}{\partial h_t} - \beta E_T \left[ S'(x_{T+1}) | x_T \right] = 0
\]

(14c) \[ x_{t+1} = x_t + g(x_t) - h^*_t, \quad x_0 = x^0, \quad t = 0, 1, \ldots, T \]

We now identify the costate variables by comparing these conclusions to those from the Lagrange multiplier method. We define:

\[
\mathcal{L} = E_0 \left[ \sum_{t=0}^{T} \beta^t s(h_t, x_t) + \beta^{t+1} \Lambda_{t+1}(x_t + g(x_t) - h_t + \varepsilon_t) + \beta^{T+1} S(x_{T+1}) | x_0 \right]
\]

where the \( \Lambda \)'s are Lagrangian multipliers. The necessary conditions for an internal solution of the problem in Equation (15) are:

(15a) \[ \frac{\partial s(h^*_t, x_t)}{\partial h_t} - \beta E_t \left[ \Lambda^*_t(x_{t+1}) | x_t \right] = 0, \quad t = 0, 1, \ldots, T \]

(15b) \[ x_{t+1} = x_t + g(x_t) - h^*_t, \quad x_0 = x^0, \quad t = 0, 1, \ldots, T \]

(15c) \[ \Lambda^*_t(x_t) = \frac{\partial s(h^*_t, x_t)}{\partial x_t} + \beta E_t \left[ \Lambda^*_t(x_{t+1}) | x_t \right] 1 + g'(x_t) \], \quad t = 0, 1, \ldots, T

(15d) \[ \Lambda^*_{T+1}(x_{T+1}) = E_T \left[ S'(x_{T+1}) | x_T \right]. \]

Equations (14a)-(14d) and (15a)-(15d) are for the same maximization problem; hence we can conclude from Equations (14d) and (15c) that

\[
\Lambda^*_t(x_t) = \frac{\partial v_t(x_t)}{\partial x_t}.
\]

The \( \Lambda \)'s are costate variables and are shadow values of the resource stock. The stochastic nature of the resource stock, however, does add some new information.
We proceed now with the identification of the cost and scarcity effects in the costate variable. To achieve this we use $\Lambda_0^*(x_0)$ to simplify the writing of the subscripts; however, we could use any $t$ from 0 to $T-1$. We use Equation (15c) recursively as follows:

\[
\Lambda_0^*(x_0) = \frac{\partial s(h_0^*, x_0)}{\partial x_0} + \beta E_0 \left[ \Lambda_1^*(x_1) \mid x_0 \right] [1 + g'(x_0)] \]

\[
\Lambda_0^*(x_0) = \frac{\partial s(h_0^*, x_0)}{\partial x_0} + \beta \left[ \frac{\partial s(h_1^*, x_1)}{\partial x_1} + \beta E_1 \left[ \Lambda_2^*(x_2) \mid x_1 \right] [1 + g'(x_1)] \right] \mid x_0 \] [1 + g'(x_0)] \]

\[
\Lambda_0^*(x_0) = \frac{\partial s(h_0^*, x_0)}{\partial x_0} + \beta E_1 \left[ \frac{\partial s(h_1^*, x_1)}{\partial x_1} + \beta E_2 \left[ \Lambda_3^*(x_2) \mid x_1 \right] [1 + g'(x_1)] \right] \mid x_0 \] [1 + g'(x_0)] \]

\[
(16) \quad \Lambda_0^*(x_0) = \frac{\partial s(h_0^*, x_0)}{\partial x_0} + \beta [1 + g'(x_0)] E_0 \left[ \frac{\partial s(h_1^*, x_1)}{\partial x_1} \mid x_0 \right] + \beta^2 [1 + g'(x_0)] E_0 \left[ \frac{\partial s(h_2^*, x_2)}{\partial x_2} \mid x_1 \right] \mid x_0 \]

\[
+ \beta^2 [1 + g'(x_0)] E_0 \left[ \frac{\partial s(h_3^*, x_3)}{\partial x_2} \mid x_2 \right] \mid x_1 \] [1 + g'(x_1)] \mid x_0 \]

The growth of one unit of additional stock originating at the beginning of time period $t$, over the period would be $[1 + g'(x_0)]$. However, if the system has only reached the beginning of time period $t-1$ and not $t$, one's expected growth of one unit of additional stock injected at the beginning of time period $t-1$ would be (by the end of time period $t$)

\[
[1 + g'(x_{t-1})] E_{t-1} \left[ [1 + g'(x_t)] \right] x_{t-1} \].

Thus by backward induction, the expected growth (by the end of time period $t$) of one unit of additional stock introduced at the beginning of time period zero would be

\[
E_0 \left[ \xi_{0,0} \mid x_0 \right] = [1 + g'(x_0)] E_0 \left[ [1 + g'(x_1)] E_1 \left[ [1 + g'(x_2)] E_2 \left[ \frac{\partial s(h_3^*, x_3)}{\partial x_2} \mid x_2 \right] \mid x_1 \right] \right] [1 + g'(x_0)] \mid x_0 \]

\[
t \in [1, T].
\]

Thus terms involving $(1 + g'(\cdot))$ give the expected time path of an additional unit of the resource stock at originating at time zero, $E_0 \left[ \xi_{0,0} \mid x_0 \right]$.

We have $\frac{\partial s(h_i^*, x_i)}{\partial x_i} = -c_i(h_i^*, x_i)$ where the sub-x indicates partial derivative. Similar to the treatment given above, one additional unit of the stock injected at the beginning of time
period \( t \) would yield a return of \([1 + g'(x_i)]E,[c_x(h^*_t, x_{r+1})]x_i \] from expected cost savings in the next time period. By iteration, we therefore define

\[
(17a) \quad \mathcal{E}_{t,0} = [1 + g'(x_0)]E_0[1 + g'(x_1)]E_1[\ldots E_{t-1}[1 + g'(x_i)]E_i[c_x(h^*_t, x_{r+1})]x_i] x_{t-1} \ldots | x_0]
\]

\( t \in [1, T - 1] \) and

\[
(17b) \quad \mathcal{E}_{T,0} = [1 + g'(x_0)]E_0[1 + g'(x_1)]E_1[\ldots E_{T-1}[1 + g'(x_i)]E_T[c_x(h^*_T, x_{T+1})]x_T] x_{T-1} \ldots | x_0]
\]

\( \mathcal{E}_{t,0} \) defines the expected return on a unit of additional stock injected at the beginning of time period 0 due to cost saving in time period \( t+1 \).

For the more general case where the new unit originates in time \( \tau \) we further define

\[
(18a) \quad \mathcal{E}_{\tau,0} = [1 + g'(x_0)]E_{\tau}[1 + g'(x_1)]E_{\tau+1}[\ldots [1 + g'(x_i)]E_i[c_x(h^*_\tau, x_{r+1})]x_\tau] x_{\tau-1} \ldots | x_0]
\]

\( t \in [\tau + 1, T - 1] \) and

\[
\mathcal{E}_{\tau,0} = [1 + g'(x_0)]E_{\tau}[1 + g'(x_1)]E_{\tau+1}[\ldots [1 + g'(x_\tau)]E_{\tau}[c_x(h^*_\tau, x_{r+1})]x_\tau] x_{\tau-1} \ldots | x_0]
\]

By Law of iterated expectation (See Appendix A), equations (17a, b) could be rewritten as

\[
(18a) \quad \mathcal{E}_{\tau,0} = [1 + g'(x_0)]E_{\tau}[1 + g'(x_1)]E_{\tau+1}[\ldots [1 + g'(x_i)]E_i[c_x(h^*_\tau, x_{r+1})]x_0]
\]

\[
(18b) \quad \mathcal{E}_{\tau,0} = [1 + g'(x_0)]E_{\tau}[1 + g'(x_1)]E_{\tau+1}[\ldots [1 + g'(x_\tau)]E_{\tau}[c_x(h^*_\tau, x_{r+1})]x_\tau]
\]

For the more general case

\[
(19) \quad \Lambda^*_0(x_0) = -\sum_{t=0}^{T} \beta^t E_{\{x_t, \ldots, x_{t+1}\}}[\xi_{t,0} c_x(h^*_t, x_t)] x_0 + \beta^{T+1} E_{\{x_T, \ldots, x_{T+1}\}}[\xi_{T+1,0} c_x(h^*_T, x_T)] x_0
\]
For the more general case we have

\[(20) \Lambda^*_T(x_r) = \sum_{t=1}^{T} \beta^t E_{(x_t,...,x_{t+1})} \left[ \xi_{t,0} c_x(h_t, x_t) \right] | x_r \] + \[ E_{(x_t,...,x_{T+1})} \left[ \beta^{T-t+1} \xi_{T+1,0} \Lambda^*_T(x_{T+1}) \right] | x_r \]

In Equation (20)

\[(21a) \ E_{(x_t,...,x_{t+1})} \left[ \beta^{T-t} \xi_{T+1,0} \Lambda^*_T(x_{T+1}) \right] | x_r \]

is the expected value of the Scarcity Effect given that \(c_x = 0\), and

\[(21b) \ \sum_{t=1}^{T} \beta^t E_{(x_t,...,x_{t+1})} \left[ \xi_{t,0} c_x(h_t, x_t) \right] | x_r \]

is the expected value of the Cost Effect. As in the deterministic model the Scarcity Effect exists only when the Cost Effect is null. The difference is that the term is expected value. The time path of the additional unit is given by a Markov process, and since \(x_{T+1}\) is a random variable so is the terminal costate variable. In the Cost Effect, the expected value of the summation is likewise considered. These have similar explanations. To emphasize the differences between the deterministic and stochastic Cost Effects we present them for time 0 in an expanded form:

\[9b’] \ -c_x(h_0, x_0) - \beta c_x(h_1, x_1)(1 + g'(x_0)) - \beta^2 c_x(h_2, x_2)(1 + g'(x_0))(1 + g'(x_1)) - \beta^3 c_x(h_0, x_0)(1 + g'(x_0)(1 + g'(x_1))(1 + g'(x_2)) + ... \]

\[21b’ \]

\[-c_x(h_0, x_0) - E_{(x_0)} \left[ \beta[1 + g'(x_0)]c_x(h_1, x_1) | x_0 \right] - E_{(x_1, x_2)} \left[ \beta^2[1 + g'(x_0)][1 + g'(x_1)]c_x(h_2, x_2) | x_0 \right] - E_{(x_1, x_2, x_3)} \left[ \beta^3[1 + g'(x_0)][1 + g'(x_1)][1 + g'(x_2)]c_x(h_3, x_3) | x_0 \right] + ... \]

\[21b”] \ -c_x(h_0, x_0) - \beta \mathcal{E}_{0,0} - \beta^2 \mathcal{E}_{2,0} - \beta^3 \mathcal{E}_{3,0} + ... \]

This form emphasizes the fact that starting with \(t = 0\) the resource stock is a stochastic variable, and that all terms that include \(x_t\) for \(t > 0\) are given as expected values, by law of iterated expectations.

We next present a fishery numerical example to illustrate the stochastic nature of these concepts.

**Fishery Numerical Example**

We now present a linear-quadratic stochastic fishery numerical example and its solution. The growth function, \(g(x_t)\), is linear and the net surplus function, \(s(h_t, x_t)\), is quadratic, which gives us a problem that is solvable. This problem can be thought of as the problem of interest or
as a simplification of the problem of interest. Few stochastic dynamic programming, SDP, problems have a closed form solution, and most numerical SDP problems require some form of linearization to be solvable.

The problem is to find a time independent decision rule (policy function), \( \mu(x) \), that solves Equation (10) subject to Equations (11)-(13), where

\[
D(h) = a_0 - b_0 h
\]

\[
c(h, x) = a_1 h + .5b_1 h^2 - a_2 x + .5b_2 x^2
\]

so

\[
(19) \quad s(h, x) = (a_0 - a_1) h - .5(b_0 + b_1) h^2 + a_2 x - .5b_2 x^2,
\]

and

\[
(20) \quad g(x) = \begin{cases} \eta x & \text{for } x \leq x_a \\ 2\eta x_a - \eta x & \text{for } x_a < x \leq 2x_a. \end{cases}
\]

The growth function, \( g(x) \), has an inverted V shape with a peak at \( x_a \) and is equal to zero at \( x=0 \) and \( x=2x_a \).

The value function, \( V_0(x) \), takes the form

\[
V_0(x) = p_1 x + p_2 x^2 + p_3 + d.
\]

Standard techniques are used to solve for the \( p_i \)'s. This is detailed in Appendix A, and the parameter values that we use are stated there also. For the parameter values we use the long run solution lies on the right branch of the growth function, and we restrict all of our analysis to that branch, \( x_a < x < 2x_a \). The resulting policy function is

\[
\mu(x) = h^*(x) = 0.7119 + 0.0049x.
\]

In addition,

\[
x_{t+1}^* = 4.95 - \mu(x_t) + \varepsilon_t \quad \text{with } \quad x_0 = 41,
\]

and

\[
\lambda^*_t = \frac{\partial V(x_t)}{\partial x_t} = p_1 + 2p_2 x_t = 19.0461 - 0.0714x_t.
\]

We use the uniform density function on the interval \([-1, 1]\) which yields an expected value of \( \varepsilon \) of 0, \( E[\varepsilon] = 0 \), and a variance of \( \varepsilon \) of 1/3, \( E[\varepsilon^2] = 1/3 \). We next present the simulation results.
Figure 1 Optimal Time Paths of the Fish Stock

Figure 1 shows the time path of the fish stock for 100 simulations and the time path for the deterministic model, which is given by the white line down the middle of the simulation paths. The white line can also be thought of as the time path of the expected value of the stock. We start at a stock level that is less than the long run optimal level; hence the fish stock grows through time. This is achieved, of course, through optimal harvests that allow the growth. We
note that the simulation time paths are clustered around the deterministic or expected time path.

Figure 2 Optimal Time Paths of the Harvest
Figure 2 shows the harvest time paths for the simulations and the deterministic model, and

Figure 3 Optimal Time Paths of the Costate Variable

Figure 3 shows the costate variable time path for the same computer runs. The three figures show a lot of similarity, which follows from the fact that the optimal harvest and the optimal costate variable are both linear functions of the stock. This diagram illustrates that the costate variable is stochastic. An alternative way to depict this variability is to view the variance of the simulated values at each $t$ and the expected or theoretical variance at each $t$. These concepts are developed
in Appendix A, and are presented in Figure 4.

Figure 4 Time Path of the Variance of the Costate Variable

The solid line is the variance of the simulated values at each $t$, and the dashed line is the theoretical or expected variance. Note that during the first few years the variance increases and then stabilizes around a long run value. This same feature is also noted in Figure 3 where the band of simulated values gradually increases and then stabilizes.
The Costate Variable in a Stochastic Renewable Resource Model

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Abstract

In this paper we discuss the costate variable in a stochastic optimal control model of a renewable natural resource, which we call a fishery. The role of the costate variable in deterministic control models has been discussed extensively in the literature. See for example Lyon (1999), Clark (1990, pp.102-107), and Arrow and Kurz (1970, pp. 35-37); however, there is little discussion of this variable for stochastic models, even though the costate variable has similar roles in the two models. In both models the costate variable is a shadow value of the associated state variable, and as such has the role of rationing the use of the state variable. In addition, as has been shown in Lyon (1999), in natural resource problems the costate variable can be partitioned into a scarcity effect and a cost effect. We show that this same partitioning can be done in the stochastic renewable resource problem. We discuss and contrast the similarities and differences in these concepts for deterministic and stochastic models. In addition, we present a numerical example to help solidify the results.

Keywords: Costate variable, stochastic dynamic programming, deterministic dynamic programming, renewable resource model, simulation