Simulation of Steady and Unsteady Flows in Channels and Rivers

Roland W. Jeppson

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This report deals with work that was done to provide prediction capability of the hydraulics of flow to an aquatic model. The aquatic model is being developed to simulate the production and standing crop of fish and other aquatic organisms in a stream or river, with particular emphasis toward what minimum stream flows are necessary for the maintenance of viable habitats for trout. Since the hydraulics of streams and rivers, including depth of flow, velocity of flow, and flow rates are necessary input to the aquatic model, this hydraulic model was developed and programmed. The hydraulic model has wide application on its own merits and, therefore, is described in this separate report.
The unsteady, one-dimensional Saint-Venant equations are solved by an implicit finite difference scheme to handle general channel and river flows. The initial conditions for the unsteady flow are provided by solving the steady varied flow equation for the specified boundary conditions. The solution for the unsteady flow allows any of eight separate boundary conditions to be specified which are composed of combinations of specifying the depth or discharge as functions of time at either the upstream or downstream ends, with the stage-discharge relation or constant depth and flow rate specified at the other end. Typical solutions showing the spatial and time dependency of such flow characteristics as flow rate, depth and velocity are given for example problems, which include lateral inflow, and channels whose geometry, slope, and Manning's $n$ vary with respective to distance along the channel.
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This report describes a computer program which is based on the one-dimensional open-channel flow principles widely used in engineering practice (Chow, 1959 or Henderson, 1966). The model predicts the steady state or transient flow characteristics from information giving the channel geometry and a measure of the flow resistance through values of the Gauckler-Manning $^{1/}n$. Using hydraulic terminology, the flow conditions are determined by solving the appropriate equations for steady and unsteady free surface flow allowing for lateral inflow or outflow if accretions or diversion occur in the river. The geometric and hydraulic properties of the channel are allowed to vary with the position along the channel. If steady state flow occurs the ordinary differential equation for varied flow is solved, and if the flow is unsteady the Saint-Venant equations are solved.

Many well known principles of open channel flow are included herein since this report is intended for mathematically trained individual and not just hydraulic engineers.

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$^{1/}$ The name Gauckler-Manning is recommended by Williams 1970, instead of just Manning.
Readers with backgrounds in open channel flow will find it to their advantage to skip, or at most scan those sections dealing with theory of open channel flow, development of the gradually varied flow equation and the Saint-Venant equations.

The computer program has been written under the assumption that at selected sections along the channel or river the geometric and hydraulic properties will be given. Consequently as input, the program requires the upstream or downstream flow rate, and for unsteady flow the depth as a function of time at one of these boundaries, as well as the following at each of several designated sections: (1) the geometry, (2) the slope of the channel bottom, (3) values for Gauckler-Manning n and (4) accretions or losses between these sections. The variables at sections along the channel will be denoted by a subscript \( i = 1, 2, \ldots, n \). Two options are available to specify the geometry at each section. The first assumes a trapezoidal shape (of which rectangular and triangular are special case), and the second allows for any arbitrary section. Use of the trapezoidal shape is generally easier requiring only that the following be given at each section as defined in Fig. 1: (a) the bottom width \( b_i \) and, (b) the slope of the channel side \( m_i \). If the option of the arbitrary section is used it is necessary that each of the following be given at each section for a number of specified depths, denoted by a \( j \) subscript, at that section (See Fig. 2): (a) the cross-sectional area \( A_{ij} \), (b) the wetted perimeter \( P_{ij} \), (c) the top width, \( T_{ij} \).
Fig. 1. Trapezoidal channel section.
Area -- $A_i = (b_i + m_i) y_i$;
Top width -- $T_i = b_i + 2m_i y_i$;
Wetted perimeter -- $P_i = b_i + 2y_i/m_i^2 + 1$

Fig. 2. Arbitrary channel section.

Additional input specifies the total length of channel, and how many sections this length should be divided into at which the depth and other computed values will be given. When these latter sections do not coincide with the sections at which the input is given, which would generally be the situation, then data of each three consecutive input sections are fit by a second degree polynomial by means of Lagrange formula and intermediated values interpolated, or extrapolated at the ends if necessary. If unsteady (or transient) flow is to be simulated then the time dependent depth, or flow rate, at either the upstream or downstream end of the channel must be specified.

The computer solution provides the following at each output section, some of which are computed by interpolation of the input data, and some of which are computed by numerically solving the differential equations describing open channel flow: (1) the distance or $x$-coordinate.
(2) the discharge, (3) the geometry of the channel (If a trapezoidal channel is specified, this data includes, the bottom width, the slope of the channel side and the slope of the channel bottom. If a arbitrary section
is specified, this data includes the area, the wetted perimeter and the
top width for several depth increments), (4). Values of the Gauckler-
Manning coefficient, (5) the slope of the channel bottom, (6) the critical
depth, (7) the critical slope, (8) the normal depth, (9) the varied flow
depth from the specified boundary condition, (10) the area corresponding
to the depth of #9, (11) the wetted perimeter corresponding to #9, (12)
the top width corresponding to #9, (13) the depth for each of the time
steps specified if a transient situation is called for, as well as
#10 thru #13 corresponding to each of these depths. Each of these items
will be discussed fully in the following sections.

Fundamentals of Open Channel Flow

Definitions

Before describing the solution method, some terminology used in
connection with open channel flow will be defined. Some of these terms
were used in the introduction without defining them.

1. Steady flow exists when none of the variables describing the flow
such as the depth $y$, the velocity $V$ or the flow rate $Q$ are functions of
time. Steady flow is expressed mathematically as $\frac{\partial y}{\partial t} = 0$, $\frac{\partial V}{\partial t} = 0$, $\frac{\partial Q}{\partial t} = 0$, etc.

2. Unsteady or transient flow occurs if flow conditions at any section
along the channel change with time. Mathematically unsteady flow exists
if $\frac{\partial y}{\partial t} \neq 0$, $\frac{\partial V}{\partial t} \neq 0$ or $\frac{\partial Q}{\partial t} \neq 0$, etc.

3. Uniform flow exists when none of the variables describing the flow
vary with position along the channel. If $x$ is the coordinate along the
channel, uniform flow is described mathematically as $\frac{\partial y}{\partial x} = 0$, $\frac{\partial V}{\partial x} = 0$, $\frac{\partial Q}{\partial x} = 0$, etc.
Gradually varied flow occurs if conditions do change with position along the channel, but these changes are small enough that the one-dimensional equations of open channel flow are valid for practical applications. Flow over dam spillways, weir, etc., are rapidly varied. For such problems the flows must be considered two (or even three) dimensional, i.e. the dependent variables of the flow are functions of x and y (or even x, y and z) as well as possibly time. Mathematically, varied flow exists if \( \partial y / \partial x \neq 0, \partial V / \partial x \neq 0 \), but the flow rate Q is constant with x.

Spatially varied flow is a varied flow for which lateral inflow or outflow occurs. Mathematically \( \partial Q / \partial x = q \neq 0 \)

Combinations of the above flows: such as steady-uniform, unsteady-varied are used to completely define a flow in open channels.

Laminar or turbulent flow are distinguished on the basis of a dimensionless parameter called Reynolds number, representing the ratio of inertia to viscous forces acting within the flow. The Reynolds Number is

\[
R_e = \frac{V(A/P)}{\nu} \ldots \ldots \ldots \ldots \ldots (1)
\]

in which \( V \) is the average velocity, \( A \) is the cross sectional area, \( P \) is the wetted perimeter, and \( \nu \) is the kinematic viscosity of the fluid. When \( R_e \) is less than 500 the flow is laminar, otherwise the flow is turbulent. Laminar flows are rare in open channels, existing only as sheet flow over highways or land surfaces, where depths and velocities are small.

Subcritical, Critical or Supercritical flow is an additional classification depending respectively upon whether the average velocity of the flow is less than, equal to, or greater than the propagation speed.
and
\[
\frac{V}{g} \frac{\partial V}{\partial x} + \frac{\partial y}{\partial x} - S_o + S_f + F_q = -\frac{1}{g} \frac{\partial V}{\partial t} \quad \text{...... (15)}
\]

(motion)
in which \( q = \frac{dQ}{dx} \) (steady) is the lateral inflow (positive) and should not be confused with \( \partial Q/\partial x \) in subsequent equations. \( F_q \) accounts for the momentum flux per unit mass for lateral inflow or outflow. Reasonable values to give \( F_q \) are:

\( F_q = 0 \) (for bulk lateral outflow since each pound of such outflow carries with it the same momentum as each pound remaining in the flow.)

\( F_q = \frac{Vq}{2gA} \) (for seepage outflow since seepage outflow removes water from the channel bottom with zero velocity.)

\( F_q = \frac{V-Uq}{gA} q + \frac{Z}{A} \frac{\partial A}{\partial x} \quad y,t \) (for lateral inflow in which \( U \) is the velocity component of the inflow in the direction of the channel and \( Z \) is the depth from the water surface to the centroid of the area.)

The second form of the Saint-Venant equations considers the depth \( y \) and the flow rate \( Q \), instead of the velocity \( V \) as the primary dependent variables. These equations can be obtained from Eqs. 14 and 15 by noting that \( Q = VA \) and are:

\[
\frac{\partial Q}{\partial x} - q + \frac{\partial A}{\partial t} = 0 \quad \text{...... (16)}
\]

(continuity)

and

\[
\frac{2Q}{gA^2} \frac{\partial Q}{\partial x} + (1 - F_q) \frac{\partial^2 y}{\partial x^2} - \frac{Q^2}{gA^3} \frac{\partial A}{\partial x} \quad y,t \quad + S_f - S_o + F_q - \frac{Qq}{gA^2} + \frac{1}{gA} \frac{\partial Q}{\partial t} = 0 \quad \text{...... (17)}
\]
in which all terms are as defined previously. (In obtaining Eq. 17 the \( \partial A/\partial t \) has been eliminated by substituting from Eq. 16.)

The second form, Eqs. 16 and 17, of the Saint-Venant equations has been selected for use herein, primarily because these equations reduce
more directly to the equations most frequently used to solve the problem of steady-spatially varied flow.

The Saint-Venant Eqs. (14 and 15) or (16 and 17) describe unsteady-spatially varied flow in a channel whose hydraulic and geometric properties vary with x. These equations simplify for less general applications. If the channel's geometry does not depend on x then \( \frac{\partial A}{\partial x} \) becomes zero, and if no lateral outflow occurs, both \( q \) the \( F_q \) become zero. These simplifications might be considered special cases of the more general problem in which these terms are simply equated to zero, but the solution uses the same technique as for the general problem. However, if the flow is steady, the continuity equation simplifies to an algebraic equation and the equation of motion simplifies to an ordinary differential equation. To accomplish this simplification note that for unsteady flows \( Q \) and \( y \) are functions of \( x \) and \( t \), but for steady flows these dependent variables are only functions of \( x \). Consequently all derivatives with respect to time \( t \) are identically zero (this is the definition of steady flow). The partial derivatives of \( x \) become total derivatives and therefore for steady flow Eq. 16 becomes,

\[
Q = Q_o + qx 
\]

in which \( Q_o \) is the flow rate in the channel where \( x = 0 \).

The equation of motion, Eq. 17, for steady flow (i.e. when \( \partial Q/\partial t = 0 \)) becomes.

\[
S_f = S_o - (1 - F_r^2) \frac{dy}{dx} + \frac{Q^2}{gA^3} \frac{\partial A}{\partial x} y - \frac{Qq}{gA^2} - F_q 
\]
The arrows accompanied by the descriptions below Eq. 19 show how by deleting terms the single equation defines different types of steady open channel flows.

Since the Froude number squared, \( F_r^2 = \frac{Q^2T}{(gA^3)} \) and since \( q = \frac{\partial Q}{\partial x} \) for steady flow, Eq. 19 is identical to Eq. 12 which defines the friction slope as the negative of the slope of the energy line. The only exception is the term \( F \) in Eq. 19, which accounts for the possibility that the lateral inflow may possess more (or less) energy per pound (or per Newton) in the \( x \)-direction than the fluid in the main channel. However, in developing Eq. 12 it was assumed that all fluid contained equal energy per pound (or per Newton).

Simplification of Eq. 19 for special cases is accomplished by dropping terms. If no lateral inflow occurs the last two terms containing \( q \) and \( Fq \) vanish. If in addition, the cross-section of the channel is unvarying with \( x \), the third from the last term \( \frac{Q^2}{(gA^3)} \left( \frac{\partial A}{\partial x} \right) \big|_y \) becomes zero. Finally if the flow is uniform, \( S_f = S_o \).

For convenience in solution, Eq. 19 is rewritten so that \( \frac{dy}{dx} \) stands by itself on the left of the equal sign, or

\[
\frac{dy}{dx} = \frac{S_o - S_f + \frac{Q^2}{gA^3} \frac{\partial A}{\partial x} \big|_y - \frac{Qgq}{gA} - Fq}{1 - F_r^2} \quad \cdots \quad (20)
\]

For a trapezoidal channel \( \frac{\partial A}{\partial x} \big|_y \) can be evaluated as,

\[
\frac{\partial A}{\partial x} \big|_y = y \frac{db}{dx} + y^2 \frac{dm}{dx}
\]

and for a general shaped channel must be evaluated by determining the change in cross-sectional area at adjacent sections with the depth constant.
Written in the form of Eq. 20, x is assumed to be the independent variable and y the dependent variable. In this form the depths at specified intervals of x are desired. If the positions (i.e. x's) are desired where specified depths will occur, then x becomes the dependent variable and y the independent variable. For such applications the reciprocal of Eq. 20 is the appropriate differential equation. This latter form of the differential equation is more readily solved, particularly if the channel is prismatic, because the right side of the equation depends only on y. Upon separating variables the solution can be obtained by a simple integration, albeit numerical for the general problem. Even for non-prismatic channels the latter form is better adapted for numerical solution, under most circumstances, since the magnitude of the right side of the equation is more heavily influenced by y than x. Despite these advantages in considering y the independent variable, the requirements of this project dictate that y be considered the independent variable.

Since Eq. 20 is a first order ordinary differential equation with the flow rate in it defined by the algebraic Eq. 18, instead of a pair of simultaneous partial differential equations, as is the case with the general Saint-Venant equations, solutions to steady flow are much easier to obtain than solutions to unsteady flows. However, since A is a non linear function of y in general, and Q, q and $S_o$ may be arbitrary functions of x, no closed form solution to even Eq. 20 can be obtained. Its solution must therefore be obtained by numerical methods such as described below. Obviously, the general Saint-Venant equations must also be solved by numerical methods. The method used to solve the steady spatially varied flow Eq. 20 will be discussed in the next section. Thereafter the method of solution of the general Saint-Venant equations will be
Text books dealing with open channel hydraulics generally present tabular techniques, designed for hand computations, for solving gradually varied flow problems. These techniques consist of relatively crude numerical solutions of the ordinary differential equation. While a computer solution could easily use these techniques, a better alternative is to take advantage of the considerable work by numerical analysts that has gone into numerically solving general ordinary differential equations. Use of this alternative allows the computer program designed to solve a problem of steady varied flow to simply call upon general purpose algorithms that are available on most computing systems such as the IBM scientific package or the UNIVAC Math-Stat pack. Initially such subroutines from the UNIVAC Math-Stat pack were used. In order to make the computer program self contained, and capable of execution on any system, as well as to increase the computation efficiency that can be achieved with a special purpose algorithm over the general purpose algorithm, the numerical solution algorithm was incorporated into the computer program. Two versions of the subroutine to carry out this numerical solution were developed. The first uses the Euler Method to begin the solution and the Hamming Method (see for example Carnaham, Luther and Wilkes, 1969) to continue the solution. The other version uses the Euler Method to continue as well as begin the solution, and results in a shorter computer program. Even though the Euler method provides a lower order approximation of the derivative its use to continue the solution is justified considering the accuracy with which the geometry, bottom slope, roughness parameter, and etc. are generally determined.
For the sake of completeness, brief descriptions of the Euler and Hamming methods are given.

**Euler Method**

The Euler method is a self-starting predictor-corrector technique. The first prediction (the first approximation at step $\Delta x$ beyond where the dependent variable $y$ is known) to $y_{i+1}$ is given by,

$$y_{i+1}^{(0)} = y_i + \Delta x \frac{dy}{dx}$$

Subsequent predictions may be based on a second order difference equation,

$$y_{i+1}^{(0)} = y_{i-1} + 2 \Delta x \frac{dy}{dx}$$

After the prediction is completed, the value $y_{i+1}$ is corrected by the trapezoidal formula,

$$y_{i+1}^{(n+1)} = y_i + \frac{\Delta x}{2} \left[ \frac{dy}{dx}^{(n)}_{i+1} + \frac{dy}{dx}_i \right]$$

Equation 23, referred to as the Euler corrector, is iteratively applied until the change between consecutive iterations becomes less than a selected small quantity. In Eqs. 21 thru 23 the value of $dy/dx$ is determined from Eq. 20 with $x$ and $y$ evaluated at the section indicated by the subscripts.

**Hamming Method**

The Hamming Method is a stable form of Milne's predictor-corrector method. It consists of first applying a predictor, then a modifier, before applying a corrector, which is customarily applied only once at each interval, but which might be iteratively applied, and following the corrector by a final value equation. These equations are:

**Predictor:**

$$y_{i+1}^{(0)} = y_{i-3} + \frac{4 \Delta x}{3} \left[ 2 \frac{dy}{dx}_{i} - \frac{dy}{dx}_{i-1} + 2 \frac{dy}{dx}_{i-2} \right]$$
Modifier:

\[ y_{i+1}^{(1)} = y_{i+1}^{(0)} - \frac{12}{121} \left[ y_1^{(0)} - y_1^{(2)} \right] \]  \hspace{1cm} (25)

Corrector:

\[ y_{i+1}^{(j)} = \frac{1}{8} \left[ 9 y_i^{(j-1)} - y_{i-2}^{(j-1)} + 3 \Delta x \left( \frac{dy}{dx} \right)_{i+1}^{(j-1)} + 2 \left( \frac{dy}{dx} \right)_{i}^{(j-1)} - \left( \frac{dy}{dx} \right)_{i-1}^{(j-1)} \right] \]  \hspace{1cm} (26)

\( j = 2, 3 \ldots n \), but generally \( j \) equal only 2

Final value:

\[ y_{i+1} = y_i + \frac{9}{121} \left( y_{i+1}^{(0)} - y_{i+1}^{(n)} \right) \]

To obtain the first value of the modifier, i.e. \( y_{4}^{(1)} \), the Hamming Method estimates

\[ y_3^{(0)} - y_3^{(n)} = \frac{242}{27} \left\{ y_3 - y_0 - \frac{3 \Delta x}{8} \left[ \frac{dy}{dx} \right]_3 + 3 \frac{dy}{dx} \right\}_2 + 3 \frac{dy}{dx} \right\}_1 + \frac{dy}{dx} \right\}_0 \} \]  \hspace{1cm} (28)

The method changes the step size according to: If \( \left| y_{i}^{(0)} - y_{i}^{(n)} \right| < \alpha_1 \)

The interval size is doubled before proceeding to the next steps, or if \( \left| y_{i}^{(0)} - y_{i}^{(n)} \right| > \alpha_2 \), the interval size is halved, and \( y_i \) is recomputed.

Because the output in the program has been specified at given intervals, no allowance has been built into the algorithm for changing step sizes.

Characteristics of solution

It is necessary to apply the numerical method described above judiciously based on an understanding of water surface profiles that can exist, or the solution will bear little or no resemblance to the actual flow depths and velocities. For instance, should a solution start at a free overfall and proceed upstream with the boundary condition for \( y \) specifying a value less than the critical depth the solution would indicate a decreasing depth as one moved upstream instead of the increasing depth which does occur. The solution would be attempting to define the
so-called M₃ water surface profile instead of the M₂ water surface profile. Actually for all situations except when the slope of the channel bottom is just right to produce critical depth, three possible water surface profiles exist and some preliminary analysis of the total flow situation is needed to determine which of the three possibilities will in fact occur under given conditions. Furthermore, Eq. 20 becomes singular at critical depth. At critical depth the Froude number equals 1 and the denominator of Eq. 20 becomes zero. Obviously no numerical technique can adequately cope with an infinite derivative even if the computer, someway, could perform a division by zero to produce infinity. Actually as the water surface approaches critical depth, the change in the depth of flow becomes too rapid for the one-dimensional flow equation to be valid. Consequently, Eq. 20 is only valid for flow depths a small amount above and below critical depth. Also the water surface, according to Eq. 20, only asymptotically approaches the normal depth since the numerator of Eq. 20 becomes zero. Therefore, boundary conditions on y cannot be equal to the normal depths but must be slightly greater or less than the normal depth.

Gradually varied flow problem

A description of the subject of water surface profiles is given to provide additional understanding of the general nature of valid solutions for known flow conditions. For purposes of this description the three last terms in the numerator of Eq. 20 will be deleted giving the differential equation for gradually varied flow in a prismatic channel without lateral inflow

\[ \frac{dy}{dx} = \frac{S_o - S_f}{1 - \frac{F}{r^2}} \]  

(29)
The general conclusions regarding shapes of water surface profiles obtained from Eq. 29 will be valid only in-as-far as $S_o - S_f$ dominates the numerator of Eq. 20, but none-the-less are instructive regarding open channel flows.

Whether a gradually varied flow increases or decreases in depth in the downstream direction depends upon whether the numerator and denominator of Eq. 29 have like or opposite signs. Like signs result in increasing depths and opposite signs in decreasing depth in the downstream direction. A letter with a subscript is used to identify each possible type of gradually varied profile. The letter will denote whether the channel will cause supercritical, critical, or subcritical flow under uniform flow conditions, according to:

S (for steep) will produce supercritical uniform flow.
C (for critical) will produce critical uniform flow.
M (for mild) will produce subcritical uniform flow.
H (for horizontal) the channel bottom slope equal zero.
A (for adverse) the channel bottom slope is negative, or upward in the direction of flow.

The subscript will be 1, 2, or 3 depending respectively upon whether the actual depth is above both the normal and critical depths, between the normal and critical depths, or below both the normal and critical depths. With this notation a water surface above the normal depth in a mild channel is called an $M_1$ - profile, and a water surface in a steep channel below the critical depth but above the normal depth is called an $S_2$ - profile.

The signs of the numerator and denominator of Eq. 29 are determined by observing that: (1) whenever the depth is greater than the normal depth the numerator is positive since the friction slope $S_f$ is less than the slope of the channel bottom $S_o$, and whenever the depth is less than the
normal depth the numerator is negative, and (2) whenever the depth is
greater than the critical depth the denominator is positive because the
Froude Number is less than 1, and whenever the depth is less than the
critical depth the denominator is negative because the Froude Number is
greater than 1. Take an $M_1$ profile for example. Both numerator and
denominator are positive, and therefore the water surface increases.
Figure 3 shows the generally shape of all the water surface profiles.

The flow can change from an $M_3$ to an $M_1$ (or an $S_2$ to an $S_1$) profiles
trough a hydraulic jump. A hydraulic jump will occur under appropriate
conditions provided the depths from the $M_3$ and $M_1$ (or $S_2$ and $S_1$)
profiles equals the "conjugate depths" $y_1$ and $y_2$ in the hydraulic jump
equation,

$$\frac{Q_1^2}{gA_1} + A_1 \overline{Z}_1 = \frac{Q_2^2}{gA_2} + A_2 \overline{Z}_2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (30)$$

in which $\overline{Z}$ is the depth from the water surface to the centroid of the
cross-section.

The water surface profiles given by Eq. 20 may deviate from those on
Figure 3 depending upon the magnitude of the terms dropped in obtaining
Eq. 29. For spatially varied flow, or flow in a non-prismatic channel both
the normal and critical depths vary with $x$. A water surface can only go
from an $M_3$ (or $S_2$) profile which is below critical depth to an $M_1$ (or $S_1$)
profile through a hydraulic jump. From this discussion it should be clear
to the reader that the solution to Eq. 20 requires that the general type of
flow conditions be specified. Fortunately with only rare exceptions,
is flow in natural streams and river supercritical. Only in man made
channels with linings do supercritical flows occur. Consequently for natural
streams and channels only $M_1$ and $M_2$ profiles need be considered, and
possibly $M_3$ profiles in short reaches below man made structures.
<table>
<thead>
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<th>Profile Designation</th>
<th>Sign Associated with Eq. 36.2</th>
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<tr>
<td><strong>Mild</strong></td>
<td>$M_1$</td>
<td>$\frac{dy}{dx} = + = +$</td>
</tr>
<tr>
<td></td>
<td>$M_2$</td>
<td>$\frac{dy}{dx} = - = -$</td>
</tr>
<tr>
<td></td>
<td>$M_3$</td>
<td>$\frac{dy}{dx} = - = +$</td>
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<tr>
<td><strong>Steep</strong></td>
<td>$S_1$</td>
<td>$\frac{dy}{dx} = + = -$</td>
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<td></td>
<td>$S_2$</td>
<td>$\frac{dy}{dx} = - = +$</td>
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<tr>
<td></td>
<td>$S_3$</td>
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</tr>
<tr>
<td><strong>Horizontal</strong></td>
<td>$H_2$</td>
<td>$\frac{dy}{dx} = - = -$</td>
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<tr>
<td></td>
<td>$H_3$</td>
<td>$\frac{dy}{dx} = + = +$</td>
</tr>
<tr>
<td><strong>Critical</strong></td>
<td>$C_1$</td>
<td>$\frac{dy}{dx} = + = -$</td>
</tr>
<tr>
<td></td>
<td>$C_3$</td>
<td>$\frac{dy}{dx} = - = +$</td>
</tr>
<tr>
<td><strong>Adverse (or negative slope)</strong></td>
<td>$A_2$</td>
<td>$\frac{dy}{dx} = + = -$</td>
</tr>
<tr>
<td></td>
<td>$A_3$</td>
<td>$\frac{dy}{dx} = - = +$</td>
</tr>
</tbody>
</table>

Fig. 3. Gradually varied flow profiles.
SOLUTION OF THE SAINT-VENANT EQUATIONS

Methods for Solution

There are obviously many more difficulties, and considerations in numerically solving the Saint-Venant equations, than solving the steady varied flow equation. The Saint-Venant equations can have discontinuous solutions, even when initial and boundary conditions are continuous and smooth. Only the integral form of these equations, which is not given here, will provide solutions to such discontinuities. In the differential form these discontinuities must be allowed for by the hydraulic jump equation providing connective values. In real channel flows these discontinuities are spontaneous formations of such phenomena as hydraulic bores, or standing waves. An extreme amount of computer logic would be required to adequately test and allow for all of the possibilities even though much is know about the subject, and consequently all these possibilities could be incorporated as logic into routines for handling the many possibilities. This has not been done, however, in the present program which does not allow for any discontinuities in the water surface. Consequently, the program will only handle unsteady situations in which the water surface at a boundary is falling, or rising slowly enough so that spontaneous formation of hydraulic bores or standing waves do not occur. This limitation does little in restricting the use of the program to real streams and channels, however, since the formation of such discontinuities occurs very infrequently. Exceptions will be observed when the stream or channel discharges directly in the ocean or an estuary subjected to tidal action.

The Saint-Venant equations are hyperbolic, which means they have real and
distinct characteristics. This places them in the category of the wave
equation. For mathematically well posed hyperbolic equations, initial
conditions on both the magnitude and the derivative of the dependent
variable, or variables are needed as well as possibly boundary conditions
depending upon whether the problem is considered finite or infinite in
length. Numerical solutions to the Saint-Venant equations, as discussed by
Strelkoff, 1970, generally fall into one of the following categories:
(1) Utilization of the characteristics to change the equations to ordinary
differential equations along the characteristic lines (2) Explicit finite-
differencing of characteristic equations on a rectangular network in the
x-t plane. (3) Direct, explicit finite-differencing of the Saint-Venant
equations of continuity and motion in a rectangular network, and (4) Direct,
implicit finite-differencing of the equations in a rectangular network.

As more and more solutions to the Saint-Venant equations appear in
the literature it will be easier to determine which of the above categories
provides the best suited approach to solve a specific application. It
is the writer's opinion that category 1 or 2 are generally best, but
herein utilization of characteristics has a distinct disadvantage, since
the characteristics are not straight lines, they do not provide values
directly at the designated stations at a given time. Use of explicit
finite differencing of category 3 are restricted to small time steps
($\Delta t \leq \Delta x (|V| + c)$) on the basis of stability considerations. Generally
this severely restricts the size of the time step. In consideration of these
limitations, the implicit method listed as (4) above has been selected to
solve the Saint-Venant equations. Its implementation is based on the
stability criteria described by Strelkoff, 1970.
Methods of Differencing

The implicit method of solving the Saint-Venant equations will be explained in reference to the rectangular grid network in the x-t plane shown in Figure 4. The vertical grid lines, spaced at intervals on x, represent the sections along the channel where the, depth, velocity, flow rate, etc. are to be given for each time step. The horizontal lines, spaced at intervals of Δt, represent the different times for which the solution results are to be given. The finite difference solution discretizes the continuous variables Q, V, y, etc. of the problem to values at the points of intersection of the horizontal and vertical grid lines. The word implicit implies that to advance the difference solution through a time step it is necessary to solve implicit equations (in this case a system of linear equations) simultaneously. These equations are obtained by replacing the derivatives in the Saint-Venant equations by differences. The space derivatives are replaced by second order central differences, centered at the grid point, i.e. on the appropriate vertical line and on the time line \( t^{j+1} \). The time derivative is based on a backward first order difference. Furthermore to obtain a system of linear algebraic equations, the coefficients of the derivative are evaluated on the time line \( t^j \). If these coefficients were evaluated on the time line \( t^{j+1} \) the resulting difference equations would be nonlinear, and it would then be necessary to solve this nonlinear system by some iterative technique like the Newton Method. Evaluating the coefficients on the \( t^j \) time line does reduce the accuracy of the solution, but if does not make the method unstable for larger time steps \( Δt \) as occurs in explicit methods. However, as Strelkoff, 1970, points out stability consideration dictate that the friction slope \( S_f \) be taken on the \( t^{j+1} \) time line.
Fig. 4. Finite difference grid network in the x-t plane.
If \( K \) is defined by

\[
\frac{1.49}{n^{5/3}} \frac{A^{5/3}}{\rho^{2/3}}
\]

then Eq. 13 can be written

\[
S_f = \frac{Q_i}{K^2}
\]  

The first terms of a Taylor series of Eq. 31 gives,

\[
(S_f)_{j+1} \approx (S_f)_j + \frac{\partial S_f}{\partial Q} \left( Q_{i+1}^j - Q_i^j \right) + \frac{\partial S_f}{\partial K} \frac{\partial K}{\partial y} \left( y_{i+1}^j - y_i^j \right)
\]

in which the derivatives in Eq. 32 are:

\[
\frac{\partial S_f}{\partial Q} = \frac{2S_f}{Q}
\]

\[
\frac{\partial S_f}{\partial K} = -\frac{2S_f}{K^2}
\]

\[
\frac{\partial K}{\partial y} = \frac{K}{A} \left( \frac{5T}{3} - \frac{2A}{3P} \frac{\partial P}{\partial y} \right)
\]

Use of the above described scheme to difference the flow rate form of

the Saint-Venant Eqs. 16 and 17 gives the following equations after some

algebraic manipulation:

\[
-0.5Q_{i+1}^j + \frac{\Delta x}{\Delta t} T_{i}^j y_{i+1}^j + 0.5 Q_{i+1}^j = \frac{\Delta x}{\Delta t} T_{i}^j y_{i}^j + q_{i+1}^j
\]

and

\[
- \frac{T(c^2 - v^2)}{2 \Delta x}^j \left( Q_{i-1}^j - V_{i}^j Q_{i-1}^j \right) - 2g \left[ S_f \frac{5T}{3} - \frac{2A}{3P} \frac{\partial P}{\partial y} \right] y_i^j
\]

\[
+ \left[ \frac{1}{\Delta t} + 2g \frac{\partial S_f}{\partial Q} \right] y_i^j + \left[ T \frac{(c^2 - v^2)}{2 \Delta x} \right] y_{i+1}^j + \frac{V_{i}^j}{\Delta x} Q_{i+1}^j
\]

\[
= Q_{i}^j + \frac{V^2}{\Delta x} y_{i}^j \right] + g \left[ A \left( (S_0 - F_{i,j+1}) + S_f \right) - 2S_f \frac{5T}{3} - \frac{2A}{3P} \frac{\partial P}{\partial y} \right] y_i^j
\]

\[
+ (Vq)^j
\]

in which \( F_{i,j+1} \) is computed according to the equations below Eq. 15 with

\( V \) evaluated on the \( j \)-th time line, and \( q \) and \( U_q \) at the \( j+1 \) time line.
Boundary Conditions

At the boundaries \( i = 1 \) and \( i = n \), difference equations must be obtained from boundary conditions which appropriately define the actual flow conditions at these ends. For instance, consider the problem in which the depth at the downstream boundary is varied as a function of time by raising or lowering a gate, and that we are concerned with solutions only up to the time when the depth first begins to drop at the upstream end.

Then \( y_1 \) and \( Q_1 \) do not vary with time (i.e. the boundary \( i = 1 \) has a Dirichlet condition \( y(0,t) = y(0,0) \) and \( Q(0,t) = Q(0,0) \). At the downstream boundary \( i = n \), \( y(L,t) = y_b(t) \) (a known function of time), and the condition for \( Q(L,t) \) must satisfy the continuity equation.

\[
\frac{\partial Q}{\partial x} = q - T \frac{\partial y}{\partial t}
\]

Using second order differences to evaluate \( \frac{\partial y}{\partial x} \) leads to,

\[
0.5Q_{n-2} - 2Q_{n-1} + 1.5Q_n = \Delta x(q - T \frac{\partial y}{\partial t}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)
\]

as the boundary difference operator for \( Q \) at \( i = n \).

Many other boundary conditions are possible. However, only the above conditions will be used to illustrate how a solution is obtained. Seven other combination of boundary conditions are incorporated into the computer program or described later, however.

Solving difference equations

When Eqs. 36 and 37 are written simultaneously for all grid points on any time line \( t^{j+1} \) including the boundary points, if the variable is unknown on the boundary, a system of simultaneous equations results equal in number to the number of unknowns \( Q_{1}^{j+1} \) and \( y_{1}^{j+1} \) (i.e. generally twice the number of grid points along any time line). A solution of this system advances the solution to the problem through one time step. After obtaining that solution, the \( j + 1 \) time line becomes the \( j \)-th time line.
and the process is repeated. The initial conditions, $Q(x,0)$ and $y(x,0)$, are used to start the solution for the first time line above the axis (i.e. $j = 2$). These initial values for $Q(x,0)$ and $y(x,0)$ have been taken as the solution to the steady state problem as defined by Eq. 17.

Using matrix notation the system of equations for any time line can be written as,

$$AZ = B$$  \hspace{2cm} (39)

in which $\hat{Z}$ is the vector of unknowns $y_{i}^{j+1}$ and $Q_{i}^{j+1}$, $\hat{B}$ is the vector of knowns on the right of the equal signs of Eqs. 36, 37, and 38, and $A$ is the matrix of the coefficients of $y_{i}^{j+1}$ and $Q_{i}^{j+1}$ on the left of the equal sign in Eqs. 36, 37 and 38.

It would be possible to utilize a standard linear algebra algorithm which reside on most computing system to solve the system represented by Eq. 39. To do this would result in very inefficient use of computer storage, and require many more computations than are actually necessary, because of the special character of the coefficient matrix $A$. In Figure 5, Eq. 39 is shown containing its individual elements. The system of equations represented in Figure 5 has been obtained by writing Eq. 37 first and then Eq. 36 at each grid point from $i = 2, 3 \ldots n-1$. The individual elements of the coefficient matrix are given by a subscript to denote whether they are a coefficient of $y$ or $Q$, and a superscript according to the equation number to denote they are different numerical values. All non-zero elements of the coefficient matrix are on the diagonal, two position in front of the diagonal and a maximum of three positions beyond the diagonal. The only exception to this is the final row which has three non-zero elements in front of the diagonal. By Gaussian elimination the third element of this last row can be made equal to zero. The solution to the
Fig. 5. Elements of matrix equation 39.
system is thereafter readily accomplished by two passes through each row to eliminate first the second and then the first elements before the diagonal followed by back substitution in obtaining the values of the unknowns \( y_i \) and \( Q_i \). Also storage requirements for matrix A are only \((2n-3) \times 6\) instead of \((2n-3) \times (2n-3)\) if standard linear algebra subroutine were to be used.

To illustrate the algebra required for a solution, the elements of the coefficient matrix A will be denoted by a double subscript and the superscript denoting the row. The first subscript \( \lambda \) will denote the row and the second subscript the element along that row so that the diagonal element on each row has a value of 3 for the second subscript. The elements of the \( \vec{Z} \) and \( \vec{B} \) vectors will be distinguished by a single subscript for the row. Then the matrix A and vectors \( \vec{Z} \) and \( \vec{B} \) are given by,

\[
a_{\lambda j}, \ \lambda = 1, 2, 3 \ldots 2n-3, \text{ and } j = 1, 2 \ldots 6
\]

and

\[
z_{\lambda}, \ \lambda = 1, 2 \ldots 2n-3
\]

\[
b_{\lambda}, \ \lambda = 1, 2 \ldots 2n-3 . . . . . . . . . . . . . . . \ldots (40)
\]

After elimination of the third non-zero element in front of the diagonal in the final row as described above the solution proceeds using the following algorithm to reduce the matrix to upper triangular.

\[
\text{for } k = 1 \text{ and } 2
\]

\[
c_{\lambda k} = \frac{a_{\lambda k}}{a_{\lambda-1, r}}
\]

\[
a_{\lambda j} = a_{\lambda j} - c_{\lambda k} a_{\lambda-1, j+1}
\]

\[
b_{\lambda} = b_{\lambda} - c_{\lambda k} b_{\lambda-1}
\]

for \( \lambda = 4 - k, 4 - k + 1, \ldots, 2n - 3 \) and \( j = k + 1, k + 2, \ldots \)

(4, if \( \lambda \) is odd or 5 if \( \lambda \) is even)
Thereafter, the solution is obtained by back substitution, or

\[ z_{2n-3} = \frac{b_{2n-3}}{a_{2n-3}} \]

\[ z_{2n-4} = \left(\frac{b_{2n-4} - z_{2n-3} a_{2n-4,4}}{a_{2n-4,4}}\right) / a_{2n-4,4} \]

for

\[ m = 2n-5, 2n-7, 2n-9, \ldots, 2 \]

\[ z_m = \left(\frac{b_m - z_{m+1} a_{m+4} - z_{m+2} a_{m+5,4}}{a_{m+5,4}}\right) / a_{m+5,4} \]

\[ z_{m-1} = \left(\frac{b_{m-1} - z_m a_{m-1,4} - z_{m+1} a_{m-1,5} - z_{m+2} a_{m-1,6}}{a_{m-1,6}}\right) / a_{m-1,6} \ldots \]

The elements of the solution vector \( z_m \) are the values of the flow rate \( Q \) whenever the subscript \( m \) is even (with the exception that \( z_{2n-3} = Q_n \)), and whenever this subscript is odd, that element represents the depth \( y \).

The details of the solution procedure, as described above, applies only for the boundary conditions given, i.e. both \( Q \) and \( y \) at the upstream end invariant with time and \( y \) specified as a function of time at the downstream end.

Combination of boundary conditions accommodated

Other boundary conditions may consist of specifying the discharge at either the upstream or downstream ends of the reach of channel being considered, or the depth at the upstream end of the channel. When either the discharge or the depth is specified at either the upstream or downstream end of the channel the other variable must be determined to satisfy the conditions of the problem. Furthermore, conditions must be applied to the other end of the channel. These conditions should be formulated to define mathematically what is most likely to occur in the real stream. To build all such possible boundary conditions into a
single computer program would be prohibitive. However, it is possible to build those conditions into a program which describes quite adequately the more common situations. Two separate versions of the subroutines which carries out the computation for solving unsteady flow have been written; each of which allows for 8 possible boundary conditions. The one version allows for the following eight combination of boundary conditions:

1. The depth at the downstream end is specified as a function of time but the flowrate at this end is unknown. The depth and flowrate at the upstream end do not change with time.

2. The depth at the downstream end is specified as a function of time but the flowrate at this end is unknown. The stage-discharge relationship is specified by input data at the upstream end.

3. The depth at the upstream end is specified as a function of time but the flowrate at this end is unknown. The depth and flowrate at the downstream end do not change with time.

4. The depth at the upstream end is specified as a function of time but the flowrate at this end is unknown. The stage-discharge relationship is specified by input data at the downstream end.

5. The flowrate at the downstream end is specified as a function of time, and the depth at this end is unknown. The depth and flowrate at the upstream end do not change with time.

6. The flowrate at the downstream end is specified as a function of time, and the depth at this end is unknown. The stage-discharge relationship is specified by input data at the upstream end.

7. The flowrate at the upstream end is specified as a function of time, and the depth at this end is unknown. The depth and flowrate at the downstream
end do not change with time.

8. The flowrate at the upstream end is specified as a function of time, and the depth at this end is unknown. The stage-discharge relationship is specified by input data at the downstream end.

These 8 possible boundary conditions are illustrated in Figure 6.

The other version allows for essentially the same eight combinations of boundary conditions with the exception that the stage-discharge relationship in no's. 2, 4, 6, and 8 are replaced by a normal derivative of the depth with respect to x, or,

$$\frac{\partial y}{\partial x} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (43)$$

This normal derivative describes those flow situations relatively well that occur if the channel configuration, etc., is such that the depth at the boundary is constant with respect to distance, for the unsteady flows to be included in the solution.

In the later version of the computer program subroutine for unsteady flow, the continuity equation or the equation of motion is used as the basis to develop the finite difference operator for the unknown variables to replace Eqs. 46 or 47 (given later); and the stage-discharge relationship. In operating this latter program a tendency for the depth to gradually increase or decrease was noted for those situations in which the boundary does not exist at a point of constant depth under normal flow conditions. For this reason and also not to unduly expand the size of this report only details for implementing the eight possible boundary conditions for the former version of the program with the stage-discharge relationships is described herein.

In describing the methods used for including the boundary conditions, several ideas will be discussed which apply regardless of which of the 8
(a) Case #1

\[ y(0, t) = y_0 \text{ (Const.)} \]
\[ Q(0, t) = Q_0 \text{ (Const.)} \]

y(o, t) \text{ determined to satisfy Eq. 47.}

(b) Case #2

\[ y(0, t) = y_0(t) \]
\[ Q(0, t) \text{ determined to satisfy Eq. 47.} \]

(c) Case #3

\[ y(0, t) = y_0(t) \]
\[ Q(0, t) \text{ determined to satisfy Eq. 47.} \]

(d) Case #4

\[ y(0, t) = y_0 \text{ (Const.)} \]
\[ Q(0, t) = Q_0 \text{ (Const.)} \]

(e) Case #5

\[ Q(0, t) = Q_0 \text{ (Const.)} \]
\[ y(0, t) = y_0 \text{ (Const.)} \]

y(\lambda, t) = y_\lambda(t)
Q(\lambda, t) \text{ determined to satisfy Eq. 46.}

y(\lambda, t) = y_\lambda(t)
Q(\lambda, t) \text{ determined to satisfy Eq. 46.}

y(\lambda, t) = y_\lambda(t)
Q(\lambda, t) \text{ determined to satisfy Eq. 46.}

y(\lambda, t) = y_\lambda(t)
Q(\lambda, t) \text{ determined to satisfy Eq. 46.}
$Q_1 = f(y_1)$ - to satisfy stage-discharge relation.

Case 1

$Q(0, t) = Q_b(t)$
$y(0, t)$ determined to satisfy Eq. 47.

Case 2

$y(\ell, t) = y_o \text{ (Const.)}$
$Q(\ell, t) = Q_0 \text{ (Const.)}$

(g) Case 7

$Q(0, t) = Q_b(t)$
$y(0, t)$ determined to satisfy Eq. 47.

(h) Case 8

$Q_n = f(y_n)$ to satisfy stage-discharge relation.

Fig. 6. Problem cases depending upon boundary conditions selected.
possible boundary conditions is being considered. For each unsteady problem solved four boundary conditions, or more specifically the finite difference operators, are needed to supply the four values \( y_j^{j+1}, Q_1^{j+1}, y_n^{j+1}, Q_n^{j+1} \) (see Figure 4 for notation), for each new time step of the solution. For those boundary conditions which require no change in the depth and discharge (upstream for case 1, downstream for case 3, upstream for case 5 or downstream for case 7) no finite difference operators are needed, since these values are known. For those boundaries for which either the depth or discharge is specified as a function of time a finite difference operator is needed to supply an equation involving the other unknown value of either discharge or depth. This equation becomes part of the system of equations along with the equations at the interior grid points whose solution provides values for the unknowns described earlier.

The approach used to obtain these additional finite difference equations is first to combine the Saint-Venant Equations linearly so that the combination of partial derivatives can be interpreted as a total derivative with respect to time, along the characteristics curves. If \( c \) is multiplied by Eq. 16 and added to Eq. 17, the following equation is obtained:

\[
\left( \frac{\partial Q}{\partial t} + (V+c) \frac{\partial Q}{\partial x} \right) - T(V - c) \left( \frac{\partial y}{\partial t} + T(V + c) \frac{\partial y}{\partial x} \right) = gA \left( S_0 - F_q - S_f \right) + v^2 \frac{\partial A}{\partial x} + q \ c
\]  

(44)

If \( c \) times Eq. 16 is subtracted from Eq. 17. The equation

\[
\left( \frac{\partial Q}{\partial t} + (V - c) \frac{\partial Q}{\partial x} \right) - T(V+c) \left( \frac{\partial y}{\partial t} + T(V-c) \frac{\partial y}{\partial x} \right) = gA \left( S_0 - F_q - S_f \right) + v^2 \frac{\partial A}{\partial x} - q \ c
\]

(45)

is produced.
Upon differencing Eq. 44, the following equation is produced for the downstream boundary,

\[ \frac{\Delta x}{\Delta t} \left[ \frac{(V - c)^j}{n} y_{n-1}^j + \frac{(V+c)^j}{n} Q_{n-1}^{j+1} + \left( \begin{array}{l} 2g S_{f}^{j} \left( \frac{2}{3} R \frac{\partial P}{\partial y} - \frac{5}{3} T \right) n^{j} - \frac{(V-c)^j}{n} \Delta t \end{array} \right) \right] = \frac{(V-c)^j}{\Delta t} y_{n}^{j+1} + gA_{n}^{j} \left( S_{o} - F_{q} + S_{f} \right) y_{n}^{j+1} + gA_{n}^{j} \left( S_{o} - F_{q} + S_{f} \right) y_{n}^{j+1} + \left( \begin{array}{l} 2g S_{f}^{j} \left( \frac{2}{3} R \frac{\partial P}{\partial y} - \frac{5}{3} T \left( \begin{array}{l} 2 \frac{\partial \left( \begin{array}{l} (V-c)^j \end{array} \right)}{\partial x} \end{array} \right) Q_{n}^{j+1} + 2g \left( \begin{array}{l} \frac{\partial \left( \begin{array}{l} \partial A_f \end{array} \right)}{\partial x} \end{array} \right) n^{j} + 2g S_{f}^{j} \left( \frac{2}{3} R \frac{\partial P}{\partial y} - \frac{5}{3} T \right) n^{j} y_{n}^{j} \end{array} \right) . \]

For the upstream boundary, differences of Eq. 45 gives,

\[ \left[ 2g S_{f}^{j} \left( \frac{2}{3} R \frac{\partial P}{\partial y} - \frac{5}{3} T \right) + T_{1}^{j} \left( \frac{V - c^2}{\Delta x} \right) + T_{1}^{j} \left( \frac{V+c^2}{\Delta x} \right) \right] y_{1}^{j+1} + \left[ \frac{1}{\Delta t} - \left( \begin{array}{l} \frac{\partial A_f}{\partial x} \end{array} \right) Q_{1}^{j+1} - T_{1}^{j} \left( \frac{V - c^2}{\Delta x} \right) y_{2}^{j+1} \right] + \left( \begin{array}{l} \frac{\partial C^2}{\partial x} \end{array} \right) Q_{2}^{j+1} + gA_{1}^{j} \left( S_{o} - F_{q} + S_{f} \right) y_{1}^{j+1} + gA_{1}^{j} \left( S_{o} - F_{q} + S_{f} \right) y_{1}^{j+1} + gA_{1}^{j} \left( S_{o} - F_{q} + S_{f} \right) y_{1}^{j+1} + \left( \begin{array}{l} 2g S_{f}^{j} \left( \frac{2}{3} R \frac{\partial P}{\partial y} - \frac{5}{3} T \right) y_{1}^{j} + \left( \begin{array}{l} \frac{\partial (V-c)^j}{\partial x} \end{array} \right) y_{1}^{j} - \left( q \right) y_{1}^{j} \right] . \]

Equations 46 and 47 are used to determine one of the unknowns at each boundary.
If the depth \( y \) is specified downstream then the coefficient in the term containing \( y_{n}^{j+1} \) in Eq. 46 is placed on the right of the equal sign.

On the other hand if the flowrate \( Q \) is specified at the downstream boundary the term containing \( Q_{n}^{j+1} \) in Eq. 46 is known and it is placed on the right of the equal sign. Likewise at the upstream boundary either the term containing \( y_{1}^{j+1} \) or \( Q_{1}^{j+1} \) in Eq. 47 becomes part of the known on the right of the equal sign. Thus either Eq. 46 or 47 provides an additional equation for the variable whose value is unknown on that boundary for those boundary conditions for which either \( y \) or \( Q \) is specified as some function of time.

Equations 46 and 47 also supply one equation for the two unknowns on those boundaries on which the stage-discharge relation is specified. The second equation needed to determine the second unknown on these boundaries is obtained by expressing the stage-discharge relation in the form,

\[
b \ y_{n}^{j+1} + Q_{n}^{j+1} = a \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (48)
\]

for the downstream boundary, and

\[
b \ y_{1}^{j+1} + Q_{1}^{j+1} = a \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (49)
\]

for the upstream boundary. The values for \( a \) and \( b \) in Eqs. 48 and 49 are determined so that the specified stage-discharge relation is approximated by a straight line between the two input depth values which bracket \( y_{1} \).

In the event the depths becomes greater (or less) than the largest (smallest) depth given in defining the stage-discharge relation extrapolation based on a straight line is used to define \( a \) and \( b \).

Including Eqs. 46 and 49 for the boundary grid points as required by the particular boundary condition with the finite difference equations for
Table 1  Non-zero elements at the beginning and end of coefficient Matrix for the 8 cases of boundary conditions.

<table>
<thead>
<tr>
<th>Boundary Condition Case</th>
<th>Upstream Coef. Matrix</th>
<th>Unknown Variables</th>
<th>Downstream Coef. Matrix</th>
<th>Unknown Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1 upstream Y1=const. Q1=const. downstream Yn=f(t)</td>
<td>Eq. 37</td>
<td>Y2</td>
<td>Eq. 37</td>
<td>( Y_n-2 )</td>
</tr>
<tr>
<td>#2 upstream Y1 &amp; Q1 satisfy stage-discharge relation downstream Yn=f(t)</td>
<td>Eq. 49</td>
<td>Y1</td>
<td>Eq. 37</td>
<td>( Y_n-2 )</td>
</tr>
<tr>
<td>#3 upstream Y1=f(t) downstream Yn=const. Qn=const.</td>
<td>Eq. 47</td>
<td>Q1</td>
<td>Eq. 37</td>
<td>( Q_n-2 )</td>
</tr>
<tr>
<td>#4 upstream Y1=f(t) downstream Yn &amp; Qn satisfy stage-discharge relation</td>
<td>Eq. 47</td>
<td>Q1</td>
<td>Eq. 37</td>
<td>( Q_n-1 )</td>
</tr>
<tr>
<td>#5 upstream Y1=const. Q1=const. downstream Qn=f(t)</td>
<td>Eq. 37</td>
<td>Q2</td>
<td>Eq. 37</td>
<td>( Q_n-2 )</td>
</tr>
<tr>
<td>#6 upstream Y1 &amp; Q1 satisfy stage-discharge relation downstream Qn=f(t)</td>
<td>Eq. 49</td>
<td>Y1</td>
<td>Eq. 37</td>
<td>( Q_n-2 )</td>
</tr>
<tr>
<td>#7 upstream Q1=f(t) downstream Yn=const. Qn=const.</td>
<td>Eq. 47</td>
<td>Y2</td>
<td>Eq. 37</td>
<td>( Q_n-1 )</td>
</tr>
<tr>
<td>#8 upstream Q1=f(t) downstream Yn &amp; Qn satisfy stage-discharge relation</td>
<td>Eq. 47</td>
<td>Y2</td>
<td>Eq. 37</td>
<td>( Q_n-1 )</td>
</tr>
</tbody>
</table>

1/ The diagonal element is circled.
the interior grid points produces a system of equations for each of the 8 cases described above. Each such system contains as many equations as unknowns. Table 1 illustrates for each of these 8 cases which elements at the beginning and end of the coefficient matrix of these system are nonzero, and from which equation these elements are obtained.

ILLUSTRATIVE EXAMPLES

Example one

To help illustrate the nature of data needed to define a problem, and the flow characteristics determined by the computations described earlier in this report, several examples are given here. The first example gives the channel geometry at 5 sections and specifies that the geometry and computed flow characteristics be given at 20 sections (including the ends) each 300 - ft apart. This is a trapezoidal channel. Table 2 gives the channel specifications at the 5 sections. Table 3 gives the geometry of the channel, the Gauckler-Manning roughness coefficients the slope of the channel bottom, and the flowrate at each of the 20 sections designated as stations at which the flow characteristics are to be given. These values are the basis for the subsequent computations.

The results of the computations based on the steady-state portion of the program are given in table 4. The critical depth in column 3 of table 4 is obtained by the Newton iterative method to satisfy Eq. 4. The critical slope in column 4 is defined as the slope of the channel bottom that would cause flow to be at critical depth. This critical slope is computed from

\[ S_c = \frac{Q^2 n^2 P_c^{4/3}}{2.22 A_c^{10/3}} \]

in which the wetted perimeter \( P_c \) and the cross-sectional area \( A_c \) correspond
Table 2. Specifications of channel properties for Example Problem One

<table>
<thead>
<tr>
<th>Sec. No.</th>
<th>Dist. from Beginning (ft)</th>
<th>bottom width (ft)</th>
<th>Side Slope m</th>
<th>Roughness coeff., n</th>
<th>Slope of channel bottom</th>
<th>Inflow between sections 1/ cfs</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
<td>10</td>
<td>1.50</td>
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<td>.0008</td>
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</tr>
</tbody>
</table>

1/ Flowrate at section 1 specified equal to 120 cfs.

Table 3. Geometric properties of channel and flowrates obtained by interpolation or extrapolation of data in table 1.

<table>
<thead>
<tr>
<th>Sec. No.</th>
<th>Dist. from Beg. (ft) x</th>
<th>bottom width (ft)</th>
<th>Side slope m</th>
<th>Roughness Coefficient n</th>
<th>Slope of channel bottom $S_0$</th>
<th>Flowrate at section Q (cfs)</th>
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<td>.00069</td>
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<td>10.283</td>
<td>1.539</td>
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</table>
Table 4. Characteristics of steady state flow.

<table>
<thead>
<tr>
<th>Sec. No.</th>
<th>x (ft)</th>
<th>Critical depth (ft)</th>
<th>Critical slope $S_C$</th>
<th>Normal depth, $y_o$ (ft)</th>
<th>Varied flow depth $\frac{1}{4}$ (ft)</th>
<th>Cross-sectional area $A$ ($\text{ft}^2$)</th>
<th>Wetted Perimeter $P$ (ft)</th>
<th>Top width $T$ (ft)</th>
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</thead>
<tbody>
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<td>80.6</td>
<td>27.3</td>
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<tr>
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<td>5400</td>
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<td>.00293</td>
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<td>85.2</td>
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<td>5.00</td>
<td>89.9</td>
<td>28.6</td>
<td>25.67</td>
</tr>
</tbody>
</table>

1/ Specification set the depth at downstream section equal to 5.0 ft. Since depths are above both critical and normal depths, these values represent an $M_1$-profile.
to the critical depth $y_c$. The normal depth, $y_o$, in column 5 is obtained to satisfy the Gauckler-Manning Eq. 9 by means of the Newton iteration as described in the section "normal depth." Column 6 contains the varied flow depths which are obtained in this example by assuming a gate, reservoir, or other downstream control has backed the water up at the downstream end to a depth of 5.0 ft. This downstream depth was specified and is the downstream boundary condition needed to solve the ordinary differential Eq. 20, which describes spatially varied steady flow. The values in this column 6 were obtained by the procedure described in the section "Solution to Steady-State Flow" using the Euler Method to begin and continue the solution. Because the downstream specified depth was given as 5.0 ft, the profile represented by the depths in column 6 define an $M_1$ - backwater curve. Had the downstream depth been specified less than the normal depth 2.54 ft, but greater than the critical depth 1.68 ft then an $M_2$ - profile would have resulted. A depth less than the critical depth 1.68 ft at the downstream section is not possible in this mild channel (mild because $y_o$ is greater than $y_c$). Such a depth could have been specified upstream but this specification likely would result in a hydraulic jump occurring somewhere in the channel.

The last three columns in table 4 give the cross-sectional areas, the wetted perimeters and the top widths associated with the depths in column 6.

The spatially varied flow solution, column 6 of table 4 is used as the initial condition for the transient problem. In this example, the downstream depth has been specified to vary with time and the solution has been obtained from imposing the case 1 boundary condition option. The depth at the downstream end $y(\ell, t) = y_b(t)$ has been specified to vary with time as given by the values of $y$ in the second column of table 5. Such
a condition could occur by opening a gate, and/or lowering the elevation of the reservoir or receiving body of water.

The solution provides values for the following at each grid point in the x-t planes: (1) the depth, (2) the flow rate, (3) the velocity, (4) the slope of the energy line (or the friction slope) \( S_f \), (5) the area, (6) the wetted perimeter, and (7) the top width. In this example, 20 grid lines along the x-axis are used and the solution for 28 time steps each of 20 second duration giving a total of 560 grid points for which each of these values are computed. Obviously it is not practical to present all of these results herein. Table 5 summaries the depths and flow rates at 3 different sections along the channels, however. An examination of the depths in table 5 indicates how the dropping water surface proceeds upstream. The solution only goes to the time when the upstream water surface begins to fall.

**Example two**

As a second example, the boundary conditions described previously as case 8 have been specified for a situation in which the channel increases in width but decreases in slope in the downstream direction. A statement of this problem is:

The flow in a 190-ft (57.91 m) reach of river just upstream from a small diversion structure is to be analyzed in detail by giving the flow rate, the depth and velocity through the reach during the time in which a variable quantity of water is being withdrawn at the upstream end of the 190-ft (57.91 m) reach. The geometry and hydraulic properties of this reach of river are defined by the parameter values \( b \), \( m \), \( S_0 \) and \( n \) as given at the 8 sections on Figure 7. The stage-discharge relation at the downstream end of the reach immediately in front of the diversion structure is given by,
Table 5. Summary of transient solution at three sections.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Downstream (sec. 20)</th>
<th>Section No. 19 (x = 5,400 ft)</th>
<th>Section No. 10 (x = 2,700 ft)</th>
</tr>
</thead>
<tbody>
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<td>Q (cfs)</td>
<td>y (ft)</td>
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Stage-discharge relation

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The changing rate of flow diversion causes the following flows to enter at the upstream end of the reach:

*Time dependent upstream flowrate.*

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<th>Time (sec)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q₁ (cfs)</td>
<td>100</td>
<td>110</td>
<td>120</td>
<td>130</td>
<td>135</td>
<td>130</td>
<td>120</td>
<td>110</td>
<td>100</td>
<td>90</td>
<td>82</td>
<td>75</td>
<td>70</td>
</tr>
</tbody>
</table>

Ground water and other accretion flows contribution to the flow in this reach of river as shown in Figure 7 by the amounts given by the lateral arrows. This accretion flow is assumed uniformly distributed throughout the sections shown on Figure 7.

The solution to this problem has been obtained by computing the flow characteristics at twenty sections each at a spacing of 10-ft (3.05 m) along the reach of river being considered. Since the accretion flows when added to the incoming flow of 100 cfs (2.83 cms) give a flowrate at the downstream section of 107.76 cfs (3.05 cms), the stage-discharge relation indicates that at time 0, when the flow in the reach is assumed to be steady, the downstream depth equals 3.7 ft (1.13 m). The depths under steady flow are shown on the profile portion of Figure 7. These depths result from the solution of the gradually varied flow Eq. 20. Also shown on the profile view are lines which represent the normal and critical depths respectively as defined previously.

The transient solution has been obtained for the 28 time intervals each 10 seconds apart for which the flowrate at the upstream end of the reach has been specified. The incoming flow first increases and then
FIG. 7. PLAN AND PROFILE VIEWS OF CHANNEL.
decreases, as if the operator made a mistake of first shutting the gates, but corrected the mistake and eventually opened the gates until only 30 cfs (.85 cms) remained in the river. The variations in flowrate, depth and velocity throughout the reach are plotted on Figures 8, 9, and 10 respectively. The separate curves on these figures show the conditions throughout the reach at the time denoted for that curve. Thus the curve on Figure 8 for \( t = 0 \) gives the variation of discharge under the steady flow conditions prior to changing the upstream diversion. In following the consecutive time lines on Figure 8, it can be noted that the flowrate at the downstream end of the reach continues to increase for some time after the flowrate at the upstream end is reduced. In other words the water storage in the reach cause the response in flowrate at the downstream end to be delayed from that which occurs at the upstream end. This delayed reaction is also apparent during later times, after the upstream end flowrate is constant at 30 cfs (.85 cms). Under the final steady-state conditions the downstream flow rate will equal 37.76 cfs, however this condition is approached asymptotically in time as the excess water in storage within the reach is discharged at a even decreasing rate. These same effects of water storage within the reach are evident from the variations of depth and velocity throughout the reach as given by Figures 9 and 10.

Example Three

The solution to a third hypothetical problem is given in which the properties of the channel vary considerably. The downstream depth in this example is controlled. This control may be by gates or this downstream end may represent a channel discharging into a reservoir. The downstream
FIG. 8. VARIATIONS OF FLOW RATE WITH POSITION AND TIME.
FIG. 9. VARIATIONS OF WATER DEPTH WITH POSITION AND TIME.
FIG. 10. VARIATIONS OF VELOCITY WITH POSITION AND TIME.
control backs up the water initially to a depth of 34.78 ft (10.6 m),
a depth several times the normal depth. Upstream the water discharges
into the channel from a reservoir with a constant water surface elevation
2.24 ft (.683 m) above the channel bottom. The reach is 4,180 ft (1,274 m)
long. The first portion of the reach has a steeply sloping channel bottom,
with a maximum slope of 0.019. The next portion of the reach is flat with a
slope of 0.00002; and before the end of the reach the slope is increased
sharply, but just upstream from the downstream end the slope again diminishes
to 0.00005. Over the flat center portion of the channel the bottom width
increases substantially. Also over this central portion lateral inflow con­
tributes 10 cfs (.283 cms) of water to the channel. The plan and profile views
of the channel on Figure 11 shows its geometry and the specifications used to
describe the channel. The top width of the channel shown on the plan view
of Figure 11 represents the steady flow obtained from solving the gradually
varied flow equation with the boundary condition at the downstream end specifying a depth of 37.78 ft (10.6 m).

The solution to the gradually varied profile, as well as the unsteady
flow characteristics described later, were obtained using 20 nodes. Thus the
space increment $\Delta x$ used in the solution equals 220 ft (67.06 m). The upstream
boundary condition specifies the stage-discharge that would result from a
constant reservoir level and an entrance head loss coefficient $K_L = 0.3$ if the
flow moves into the channel, and $K_L = 1.0$ if the flow reverses itself going
from the channel into the reservoirs. Values of depth and corresponding dis­
charge resulting from these conditions are given below.

Stage-discharge relation

<table>
<thead>
<tr>
<th>Depth, $y_1$ (ft)</th>
<th>1.0</th>
<th>1.49</th>
<th>1.5</th>
<th>1.75</th>
<th>1.9</th>
<th>2.0</th>
<th>2.06</th>
<th>2.1</th>
<th>2.2</th>
<th>2.22</th>
<th>2.235</th>
<th>2.24</th>
<th>2.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flowrate, $Q_1$ (cfs)</td>
<td>116.0</td>
<td>116.0</td>
<td>115.8</td>
<td>111.0</td>
<td>101.0</td>
<td>89.7</td>
<td>80.0</td>
<td>66.8</td>
<td>40.6</td>
<td>29.0</td>
<td>14.6</td>
<td>0.0</td>
<td>-95.8</td>
</tr>
</tbody>
</table>
Fig. 11. Plane area profile views of initial flow conditions for Problem No. 3.
Fig. 12. Variations of flowrate with position and time for Problem Three.
REFERENCES


**NOTATION**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>cross-sectional area.</td>
</tr>
<tr>
<td>$A$</td>
<td>coefficient matrix.</td>
</tr>
<tr>
<td>$A_C$</td>
<td>cross-sectional area corresponding to critical flow.</td>
</tr>
<tr>
<td>$a_y$, $a_Q$</td>
<td>Elements of coefficient matrix.</td>
</tr>
<tr>
<td>$B$</td>
<td>vector</td>
</tr>
<tr>
<td>$b$, $b_1$</td>
<td>bottom width</td>
</tr>
<tr>
<td>$b_1$</td>
<td>elements of vector $B$.</td>
</tr>
<tr>
<td>$c$</td>
<td>celerity of small amplitude gravity wave.</td>
</tr>
<tr>
<td>$e$</td>
<td>equivalent sand roughness of channel wall.</td>
</tr>
<tr>
<td>$F_r$</td>
<td>Froude number</td>
</tr>
<tr>
<td>$F_d$</td>
<td>lateral inflow parameter.</td>
</tr>
<tr>
<td>$f$</td>
<td>function of</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration of gravity</td>
</tr>
<tr>
<td>$H$</td>
<td>elevation of energy line.</td>
</tr>
<tr>
<td>$K$</td>
<td>conveyance.</td>
</tr>
<tr>
<td>$m_1$, $m_1$</td>
<td>slope of channel side</td>
</tr>
<tr>
<td>$n_1$, $n_i$</td>
<td>Gauckler - Manning roughness coefficient.</td>
</tr>
<tr>
<td>$P_1$, $P_i$, $P_{ij}$</td>
<td>wetted perimeter</td>
</tr>
<tr>
<td>$P_c$</td>
<td>wetted perimeter corresponding to critical depth</td>
</tr>
<tr>
<td>$Q_1$, $Q_{ij}$</td>
<td>flowrate</td>
</tr>
<tr>
<td>$q$</td>
<td>lateral inflow</td>
</tr>
<tr>
<td>$Re$</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>$S_f$</td>
<td>slope of energy line</td>
</tr>
<tr>
<td>$S_0$</td>
<td>slope of channel bottom</td>
</tr>
<tr>
<td>$T$, $t$, $t_{ij}$</td>
<td>top width</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$V_i$, $V_1$</td>
<td>average velocity of flow</td>
</tr>
<tr>
<td>$x$</td>
<td>distance in downstream channel direction</td>
</tr>
<tr>
<td>$y_i$, $y_{ij}$</td>
<td>depth of flow</td>
</tr>
<tr>
<td>$y_1$, $y_1$, $y_{ij}$</td>
<td>critical depth</td>
</tr>
<tr>
<td>$y_C$, $y_0$</td>
<td>normal depth</td>
</tr>
<tr>
<td>$Z$</td>
<td>vector</td>
</tr>
<tr>
<td>$z$</td>
<td>elevation of channel bottom</td>
</tr>
<tr>
<td>$z$</td>
<td>distance between water surface and centroid of cross-sectional area.</td>
</tr>
<tr>
<td>$z_1$</td>
<td>elements of vector $\vec{Z}$</td>
</tr>
</tbody>
</table>
Sample Data

<table>
<thead>
<tr>
<th>No.</th>
<th>NSI</th>
<th>NSO</th>
<th>LBEG</th>
<th>XINC</th>
<th>XBEG</th>
<th>XEND</th>
<th>YINC</th>
<th>YINC</th>
<th>YINC</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>39</td>
<td>1</td>
<td>86.0</td>
<td>0.0</td>
<td>110.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>1</td>
<td>34.7</td>
<td>0.0</td>
<td>159.0</td>
<td>159.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Explanation of Input Variables

**FORTRAN Variable**  
**Description of Data**

<table>
<thead>
<tr>
<th>NSI</th>
<th>No. of section for which channel geometry data will be given.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NSO</td>
<td>No. of section for which solution results will be computered and printed.</td>
</tr>
</tbody>
</table>

IOYES  
If not 0 geometry of this problem will be taken the same as previous problem.

QO  
Flow at upstream end. If QO = 0, then the array Q(I) read in later, represents flows at sections.

XREG  
x distance to section 1.

XINC  
Ax distance between output sections consisting of NSO in number.

ERR  
Error parameter to terminate Newton iteration.

NSTART  
Section number at which gradually varied profile solution begins.

NSIMC  
1 if computation of gradually varied flow proceed downstream; -1 if they proceed upstream.

END  
Ending section number for gradually varied flow computations.

INFLOW  
0 if flowrates are to be given at NSO sections; 1 if at NSI sections.

YSTART  
Depth of flow at NSTART section.

LYSTR  
If greater than 0 a check will be made to determine whether a control exists in channel other than at section NSTART.

INUSTR  
If equal to zero only a steady state solution will be obtained.

LINJEP  
If greater than zero the geometry, etc., at the NSI sections will be obtained from the NSI sections by linear interpolation, otherwise quadratic interpolation will be employed.

Q(I)  
Lateral (accretion) inflows (or actual flows – see QO).

XI(I)  
x - distances for NSI sections.

SII(I)  
Slope of channel bottom So at NSI sections.

BII(I)  
Bottom widths of channel at NSI sections.

PMII(I)  
Side slopes of channel at NSI sections.

FNII(I)  
Gawkler-Manning n at NSI sections. (If channel is not trapezoidal then arrays YI(I,J), AI(I,J), PI(I,J), TI(I,J) are needed for geometry instead of above arrays)

(Transient solution)

NT  
No. of time steps.

MTI  
Case no. for boundary conditions.

NTAPE  
If greater than 0 solution also written on tape disk or drum assigned logic unit 9.

DELT  
Time step size At.

FDIVF  
Ratio of error reduction.

TB(I)  
Depths or flowrates for each time step.

QSTAG(I)  
Stage discharge relationship.