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SCALE-INVARIANT AGGREGATE FLUCTUATIONS
OF DISCRETE INVESTMENTS

by

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This paper proposes a method to analyze endogenous fluctuations of aggregate investment when firm-level investment follows an (S,s) policy and has a spillover effect on other firms' investments. First, we derive the distribution function of aggregate fluctuations in a partial equilibrium of differentiated product markets, under the assumption that a firm's position in its (S,s) band follows a uniform distribution. Second, the variance of the growth rate of average capital is shown to converge to a non-zero value when the number of firms tends to infinity, if the technology exhibits constant returns to scale. Third, we numerically compute the equilibrium paths in which the firms' positions evolve deterministically. The simulations uphold our analytical results as well as exhibit echo effects in the output series. Finally, a case of general equilibrium with imperfect information is presented in which the analytical results continue to hold.

Key words: lumpy investment, (S,s) economy, self-organized criticality, contagion
1 Introduction

This paper analyzes a model of endogenous fluctuations of aggregate investment which arise from the lumpy behavior of investments at the firm level. It demonstrates that aggregate fluctuations occur in a partial equilibrium of product markets even in a deterministic environment with infinitely many agents when discrete investments at the micro level have spillover effects.

The recent development of empirical studies on firm-level investments motivates this paper. Researchers have shown the importance of discrete choice over the course of a firm's capital adjustment and a great deal of heterogeneity across firms by using the longitudinal data. For example, Doms and Dunne (1998) found that establishment level capital is adjusted only occasionally but by a jump. Based on the similar empirical findings, Cooper, Haltiwanger, and Power (1999) stressed the role that lumpy investments played in aggregate fluctuations. Ericson and Pakes (1995) pointed out the important effects of exit and entry behavior of firms on collective industrial dynamics, presenting a framework for empirical research of firm dynamics.

These findings call for an analytical method for a dynamical system in which the discrete behavior of many heterogeneous agents are coupled with each other. We consider a specific situation in which firms' lumpy investments have spillover effects. Due to the discreteness of the investment, a firm's capital exhibits non-harmonic oscillation if the capital is depreciated physically. With the spillover effect, the dynamics of the system is then represented by a collection of coupled oscillators. This paper proposes a method to characterize the aggregate fluctuations in this system.

The literature on (S,s) economies and on non-linear dynamics has tackled the question as to how to analyze the aggregate fluctuations that arise from micro-level discreteness, or more generally, micro-level non-linearity. The theory of (S,s) economies
(Caplin and Spulber (1987); Caballero and Engel (1991)) has developed an analytical method without reducing the dimensions of agent heterogeneity. The theory showed a robust tendency that the distribution of agents in an inaction region converges to a uniform distribution in one-sided (S,s) economies. At the uniform distribution, the adjustment at the extensive margin works exactly like the adjustment at the intensive margin so that the aggregate behavior does not differ from the smoothly-adjusting case (the “neutrality” result). To the contrary, economic models of non-linear dynamics have focused on the possibility of endogenous fluctuations arising from the micro-level non-linearity. Brock and Hommes (1997), for example, demonstrated that the aggregate dynamics may be reduced to a low-dimensional non-linear map when the number of types of agents is small. In fact, numerical studies of weakly-coupled non-linear systems (Pikovsky, Rosenblum, and Kurths (2001)) have widely observed endogenous aggregate fluctuations even in a greatly heterogeneous system.

The studies of weakly-coupled non-linear maps typically employ a homogeneous collection of harmonic oscillators as a benchmark case to analyze the endogenous fluctuations. On the one hand, this “phase-dynamics” approach is useful to determine the periodicity of the aggregate fluctuations. The aggregate fluctuation occurs only if there is a certain degree of comovement across agents. Therefore, the frequency of aggregate fluctuations centers around the average frequency of the natural rate of adjustment of a firm, which in our model is determined by the lumpiness size divided by the depreciation rate. On the other hand, this approach obscures the economic point that the aggregate fluctuation is caused by the synchronized timing of firm’s discrete investment, which is decided optimally rather than exogenously. We thus adopt a version of interaction-based models (Brock and Durlauf (2001)). Previous studies in this class of models have shown the possible comovement or “herding” of agents’ actions (Gul
and Lundholm (1995) for example). In particular, Ellison (1993) and Morris (2000) have shown that a best response dynamic can result in a fast spread of a single action ("contagion") depending on the initial configuration of the actions. Our model endogenizes the configuration by incorporating the dynamics of agents' states. We show that a large size contagion is likely to occur at the configuration to which the system has a tendency to converge. This mechanism is best understood as a globally-coupled case of the self-organized criticality introduced by Bak, Chen, Scheinkman, and Woodford (1993). The dynamics of configuration organizes itself to a critical configuration, and therefore the aggregate variables exhibit recurring fluctuations.

Our results build on the previous studies of one-sided (S,s) economies which show that the distribution of a firm's position in the inaction band converges to a uniform distribution. The results are divided into three parts. Firstly, an asymptotic distribution function of the aggregate fluctuation evaluated at the uniform distribution is derived when the number of firms tends to infinity. Secondly, we show that the variance of the aggregate fluctuation does not vanish at the infinite limit of the number of firms in a partial equilibrium of product markets if the technology exhibits constant returns to scale. Thirdly, we compute the equilibrium path numerically and confirm the emergence of endogenous fluctuations with a certain degree of periodicity.

Our model consists of many monopolistic firms that are linked to each other by the derived factor demand when each firm uses other firms' products as intermediate inputs. Their interaction forms a network of input-output relations with spillover effects in capital adjustments. Suppose that a capital adjustment takes the form of a discrete decision. Then there is a chance of a chain-reaction of investments in which one firm's investment triggers another's. This chain-reaction turns out to be represented by a branching process in a partial equilibrium of product markets. The first result shows
that the size of aggregate investment is sensitive to the detailed configuration of firms' positions in the inaction region. As the depreciation of capital drives the evolution of the configuration, the aggregate investment exhibits deterministic and endogenous fluctuations. The second result provides a case where the neutrality result based on the law of large numbers fails to hold. Even though an industry or an economy consists of an infinitely many firms, the non-linear behavior at the firm level may not cancel out with each other in aggregation. The third result confirms the analytical result of fluctuations and shows the presence of the so-called echo effect. The sensitivity analysis shows that constant-returns-to-scale is an important environment for the fluctuations. When the wage and interest rate are fixed and returns to scale are constant, the equilibrium of the product markets with monopolistic suppliers exhibits a "fragile" property. In this environment, the size of the chain reaction of investments depends crucially on the detailed configuration of the positions in the inaction band.

The rest of this paper is organized as follows: Section 2 presents the model; Section 3 details the analytical and numerical results; Section 4 concludes the paper.

2 Model

The product market consists of $N$ monopolists and a representative household. Each monopolist $j$ produces a differentiated good $y_j$, using capital $k_j$ and labor $h_j$. Let us specify the production technology by a Cobb-Douglas function:

$$y_{j,t} = A k_{j,t}^\alpha h_{j,t}^\gamma.$$  \hspace{1cm} (1)

Capital is accumulated over time as:

$$k_{j,t+1} = (1 - \delta_j)k_{j,t} + \ell_{j,t}.$$  \hspace{1cm} (2)
where $\delta_j$ is a firm-specific depreciation rate. Investment $i_{j,t}$ is a composite good produced by combining all the goods symmetrically as:

$$i_{j,t} = N \left( \sum_{l=1}^{N} (z_{l,j,t}^{1})^{\mu} / N \right)$$

where $\mu - 1 > 0$ is the mark-up rate. The production technology is allowed to exhibit increasing returns to scale as long as $\alpha + \gamma < \mu$ is satisfied.

We assume that the investment rate is chosen from a discrete set. Specifically, we assume that:

$$i_{j,t} / k_{j,t} \in \left\{(1 - \delta_j)(\lambda_j^{\kappa_t} - 1)\right\}_{\kappa_t=0,\pm1,\pm2,...}

where $\lambda_j > 1$. Note that the choice space for $k_{j,t}$ is independent of the path: $k_{j,t+1} \in \{(1 - \delta_j)^t k_{j,0} \lambda_j^{\kappa_t}\}_{\kappa_t=0,\pm1,\pm2,...}$. This assumption implies that the next period capital, $k_{j,t+1}$, has to be either the depreciated level, $k_{j,t}(1 - \delta_j)$, or its multiplication or division by $\lambda_j$. By this assumption, the producer is forced to invest in a lumpy manner. This constraint is a shortcut for modeling the lumpy behavior which typically occurs as optimal behavior when an investment incurs fixed costs. This assumption is the only departure from the usual model of monopolistic product markets. The main objective of this paper is to examine the aggregate consequence of the non-linear behavior of producers induced by the discreteness constraint.

Let $p_{j,t}$ denote the price of good $j$ at $t$ and $w_t$ denote a real wage. Define a price index $P_t \equiv (\sum_{j=1}^{N} p_{j,t}^{1/(1-\mu)} / N)^{1-\mu}$ and normalize it to one. Then the monopolist’s profit at $t$ is written as:

$$\pi_{j,t} \equiv p_{j,t} y_{j,t} - w_t h_{j,t} - \sum_{l=1}^{N} p_{l,t} z_{l,j,t}^{f}$$

The demand function for good $j$ is derived by the usual procedure as in Dixit and Stiglitz (1977). Let us suppose that the representative household has preferences over
the sequence of consumption and hours worked:

$$\sum_{t=0}^{\infty} \beta^t U(C_t, H_t)$$  \hspace{1cm} (6)

where $C_t$ is a composite consumption good produced similarly as the investment good:

$$C_t = \left( \sum_{i=1}^{N} (z_{i,t})^{1/\mu} / N \right)^\mu.$$

The representative household maximizes the utility function subject to the sequence of budget constraints:

$$\sum_{j=1}^{N} p_{j,t} z_{j,t} = w_t H_t + \Pi_t$$  \hspace{1cm} (8)

where $\Pi_t$ is the average dividend from firms: $\Pi_t = \sum_{j=1}^{N} \pi_{j,t} / N$.

Cost minimization of the consumer given the level of consumption $C_t$ implies $z_{j,t} = p_{j,t}^{\mu/(\mu-1)} C_t$ and a relation $\sum_{j=1}^{N} p_{j,t} z_{j,t} / N = C_t$. Similarly, the derived demand for good $j$ by the monopolist $l$ given the level of investment $i_{l,t}$ is obtained as $z_{j,l,t} = p_{j,l,t}^{\mu/(\mu-1)} i_{l,t} / N$ and $\sum_{j=1}^{N} p_{j,l,t} z_{j,l,t} = i_{l,t}$. By combining with $y_{j,t} = z_{j,t}^{C} + \sum_{l=1}^{N} z_{j,l,t}$ for good $j$, these relations yield the demand function for good $j$ as: $y_{j,t} = p_{j,t}^{\mu/(1-\mu)} (C_t + I_t)$ where $I_t = \sum_{j=1}^{N} i_{j,t} / N$. Define a production index $Y_t = (\sum_{j=1}^{N} y_{j,t}^{1/\mu} / N)^{\mu}$. Then we have an equilibrium relation $\sum_{j=1}^{N} p_{j,t} y_{j,t} = y_t$. Combining with the consumer’s budget constraint (8) and the equilibrium condition for labor, $H_t = \sum_{j} h_{j,t} / N$, we obtain the demand function:

$$y_{j,t} = p_{j,t}^{\mu/(1-\mu)} Y_t$$  \hspace{1cm} (9)

The monopolist maximizes its discounted future profits as instructed by the representative household. The discount rate, $r_t^{-1}$, is the marginal rate of intertemporal substitution of consumption. Then the monopolist’s problem is defined as follows:

$$\max_{\{y_{j,t}, h_{j,t+1}, h_{j,t}, e_{j,t}, \pi_{j,t}, \delta_{j,t}\}} \sum_{t=0}^{\infty} (r_1 \cdots r_t)^{-1} \pi_{j,t} = \sum_{t=0}^{\infty} (r_1 \cdots r_t)^{-1} \left( p_{j,t} y_{j,t} - w_t h_{j,t} - \sum_{l=1}^{N} p_{l,t} z_{l,j,t} \right)$$  \hspace{1cm} (10)
subject to the production function (1,3), the capital accumulation (2), the discreteness of the investment rate (4), and the demand function (9).

Let us define the average capital index $K_t$ as follows:

$$K_t \equiv \left( \sum_{j=1}^{N} k_{j,t}^\rho / N \right)^{\frac{1}{\rho}}$$

where $\rho \equiv \alpha / (\mu - \gamma)$. Note that $\rho < 1$ holds by the assumption $\alpha + \gamma < \mu$. By using the optimality condition for $h_{j,t}$, the profit at $t$ is reduced to a function of $(k_{j,t}, k_{j,t+1})$ as:

$$\pi_{j,t} = D_0 w_{t-1}^{-\gamma} K_t^{\frac{\rho(\mu-1)}{\rho+1}} k_{j,t}^\rho - k_{j,t+1} + (1 - \delta_j) k_{j,t}$$

where $D_0 \equiv (1 - \gamma / \mu)(A(\gamma / \mu)^\gamma)^{1/(1-\gamma)}$. The profit is concave in $k_{j,t}$ due to $\rho < 1$. Thus the optimal policy is characterized by an inaction region of $k_{j,t}$ with a lower bound $k_{j,t}^*$ and an upper bound $\lambda_j k_{j,t}^*$. Consider two sequences of $k_{j,s}$ which are identical except at $s = t$. Such sequences can be constructed by assigning a positive investment at $t - 1$ and zero investment at $t$ in one sequence and zero investment at $t - 1$ and a positive investment at $t$ in the other sequence. Then the lower bound of the inaction region is derived by solving for $k_{j,t}$ at which the two sequences yield the same discounted profit. Namely, if $k_{j,t}$ is strictly less than $k_{j,t}^*$, the producer is better off by adjusting it upward rather than waiting. Let the sequence with zero investment at $t - 1$ have $k_{j,t} = k_{j,t}^*$, and let the other sequence have $k_{j,t} = \lambda_j k_{j,t}^*$. Then the both sequences have the same amount of capital at $t - 1$ and $t + 1$: $k_{j,t-1} = (1/(1-\delta_j))k_{j,t}^*$ and $(\lambda_j(1-\delta_j))k_{j,t}^*$. Solving for $k_{j,t}^*$ which equates the discounted profits of the two sequences, we obtain:

$$k_{j,t}^* = \left( \frac{D_0 (\lambda_j^\rho - 1)}{\lambda_j - 1} \right)^{\frac{1}{\rho+1}} (r_t - 1 + \delta_j)^{\frac{1}{\rho+1}} w_t^{(1-\gamma)(1-\rho)} K_t^{\rho}$$

where

$$\varphi \equiv \frac{\rho(\mu - 1)}{(1-\rho)(1-\gamma)} = \left( \frac{\alpha}{1-\gamma} \right) \left( \frac{\mu - 1}{\mu - (\alpha + \gamma)} \right).$$
Equation (13) expresses the strategic complementarity between the average capital level $K_t$ and the threshold $k_{jt}^*$ for an individual capital level $k_{jt}$. The degree of the complementarity is represented by $\varphi$. A percentage change in average capital induces $\varphi$ percent change in the individual threshold $k_{jt}^*$. In particular, the movement in $K_t$ and $k_{jt}^*$ coincides if $\varphi = 1$. A simple manipulation reveals the following property.

**Lemma 1** $\varphi \geq 1$ if and only if $\alpha + \gamma \geq 1$.

Whether the complementarity effect exceeds one is solely determined by the returns to scale, and is not dependent on the competitiveness of the market, $\mu$.

The spillover effect on $k_{jt}$ is non-linear because of the threshold. The average capital level $K_t$ affects the threshold, but it may or may not induce the adjustment of $k_{jt}$. Hence the property of a firm's capital choice may be summarized as local inertia and global strategic complementarity. The individual capital is insensitive to a small perturbation in the average capital level, while it synchronizes with the average capital if the perturbation is large.

## 3 Results

### 3.1 Partial Equilibrium

Let us focus on the network of producers in the product markets while abstracting from the rest of the economy by assuming that the equilibrium wage and interest rate only depend on the average capital level. Suppose that the equilibrium wage and interest are approximated by constantly elastic functions of the average capital $K_t$ in the vicinity of the time-average levels of wage, interest, and average capital $\bar{w}, \bar{r}, \bar{K}$:

$$\frac{r_t - 1 + \delta_j}{\bar{r} - 1 + \delta_j} = \left(\frac{K_t}{\bar{K}}\right)^{\theta_r} \tag{15}$$
\[
\frac{w_t}{\bar{w}} = \left( \frac{K_t}{\bar{K}} \right)^{\theta_w}
\]

This corresponds to a partial equilibrium assumption when \( \theta_r = \theta_w = 0 \). Section 3.3 provides a case in which the pricing functions above hold in a general equilibrium. By assuming (15–16), the threshold (13) is simplified as:

\[
\frac{k_j^*}{k_j^*} = \left( \frac{K_t}{\bar{K}} \right)^{\phi}
\]

where \( k_j^* \) is a threshold corresponding to \((\bar{r}, \bar{w}, \bar{K})\), and \( \phi \) represents the strategic complementarity between the individual and average capital:

\[
\phi = \varphi - \frac{\gamma \theta_w + (1 - \gamma) \theta_r}{(1 - \gamma)(1 - \rho)}
\]

Note that \( \phi \) is less than \( \varphi \) if \( \theta_r, \theta_w > 0 \). This implies that the strategic complementarity between producers is weakened due to the equilibrium response of the wage and interest rate if the response is procyclical. The price response works as a dampening factor in the investment propagation as emphasized by Thomas (2002).

The equilibrium of the product markets is given by a capital profile which satisfies \( k_{j,t} \in [k^*_{j,t}, \lambda_j k^*_{j,t}] \). This condition allows multiple equilibria. Here we employ best response dynamics introduced by Vives (1990) as an equilibrium selection algorithm. Suppose that a predetermined capital \( k_{j,t} \) resides within the inaction region. The capital in the next period \( k_{j,t+1} \) only decreases by depreciation unless adjusted. In the first step of the best response dynamics, the producers adjust capital by \( \lambda_j \) if their capital levels go below \( k^*_{j,t} \) given \( K_t \). Note that, assuming \( \delta_j < \lambda_j \), the adjustment never exceeds \( \lambda_j \). In the second step, \( K_t \) is calculated by a new capital profile, and the producers adjust their capital according to the updated \( K_t \). This procedure is repeated until the capital profile converges. The adjustments after the second step can be upward or downward, depending on whether the upward adjustments in the first step weigh more or less than
the depreciation of overall capital. Let us formally define the best response dynamics as follows. Set the initial point of the dynamics as \( k_{j,t}^0 = k_{j,t} (1 - \delta_j) \) and \( K_t^0 = K_t \). The average capital \( K_t^u \) after \( u = 1 \) is defined by the profile \( k_{j,t}^u \) and definition (11). Then \( k_{j,t}^u, \ u = 0, 1, \ldots \), evolves according to the \((S,s)\) rule:

\[
\begin{align*}
  k_{j,t}^{u+1} = \begin{cases} 
    \lambda_j k_{j,t}^u & \text{if } k_{j,t}^u < k_{j,t}^{*u} \\
    k_{j,t}^u / \lambda_j & \text{if } k_{j,t}^u > \lambda_j k_{j,t}^{*u} \\
    k_{j,t}^u & \text{otherwise}
  \end{cases}
\end{align*}
\]

(19)

We can show that this dynamics converges at a finite stopping time \( T \) with probability one when \( N \to \infty \) and \( \phi \leq 1 \). Thus the best response dynamics is a valid equilibrium selection algorithm. We define the converged point as an equilibrium capital profile at \( t + 1 \), namely, \( k_{j,t+1} = k_{j,t}^T \).

The best response dynamics is a realistic equilibrium selection mechanism in a situation where many agents interact with each other. The only information needed for an agent to make a decision is the prices and the average capital level. This selection mechanism precludes big jumps that occur due to the informational coordination among agents. In this sense, the best response dynamics selects the least volatile among possible equilibrium paths.

The aggregate investment fluctuates along with the evolution of configuration of the capital profile. To evaluate the magnitude of fluctuations analytically, we regard the capital configuration as being a random variable that takes values within the inaction region. Specifically, we assume that the position of an individual capital relative to the lower bound of its inaction region (in log-scale) follows a uniform distribution independent across firms. The uniformity assumption is motivated by the theory of \((S,s)\) economies. Define a producer’s position in an inaction region as \( s_{j,t} = (\log k_{j,t} - \log k_{j,t}^{*}) / \log \lambda_{j} \). If \( s_{j,t} \) is mutually independent across \( j \), then \( s_{j,t+1} \) is
mutually independent asymptotically as \( N \to \infty \) for \( \phi < 1 \). Also the dynamics of \( s_{j,t} \) shifts a uniform measure to itself. Hence the mutually independent uniform distribution is an invariant measure asymptotically for \( \phi < 1 \). Moreover, Engel (1992) shows that \( s_{j,t} \) converges to the uniform distribution if its dynamics contain a random component whose distribution flattens over time. Caballero and Engel (1991) also shows that the heterogeneity of \( \lambda_j \) (as well as \( \delta_j \) in our model) contributes to the convergence of a cross-section distribution of \( s_j \) to the uniform distribution. We will show that the uniformity assumption brings us good insights on the unconditional fluctuations of the aggregates.\(^1\) In Section 3.2, we will see that a departure from this assumption, in particular the correlation of \( s_{j,t} \), provides a rich structure for the time-series properties of the aggregates.

We assume that \( \lambda_j \) and \( \delta_j \) are common across \( j \). Then the periodicity of an individual oscillation is \( q = \log \lambda / |\log(1 - \delta)| \). Also the threshold \( k^*_{j,t} \) becomes constant across \( j \). Define \( m = N(\log K^1_t - \log K_t) / \log \lambda \) where \( K^1_t \) is the average capital at the first step of the best response dynamics. \( m \) indicates the gap between the effects of depreciation and of adjustments at the first step on the average capital. Also define \( W \) as the total number of firms which adjust their capital in the best response dynamics after the first step. Then we obtain the following proposition.

**Proposition 1** Suppose that \( \lambda_j \) and \( \delta_j \) are common across \( j \). Suppose that \( s_{j,t} \) is a random variable which follows a uniform distribution independently across \( j \). Then \( m/\sqrt{N} \) asymptotically follows a normal distribution with mean zero and variance:

\[
\sigma^2_m = \left( \frac{(1 - \lambda^{-2\rho/q}) \log \lambda}{2\rho} - \left( \frac{1 - \lambda^{-\rho/q}}{\rho} \right)^2 \right) \frac{1}{(\log \lambda)^2}.
\]  

If \( \phi \leq 1 \), then \( |W| \) conditional to \( |m| \) follows an infinitely divisible distribution function.

\(^1\)See Nirei (2005) for the case in which the distribution is not uniform.
asymptotically as $N \to \infty$:

$$
\Pr ( |W| = w \mid |m|) = \frac{|m| e^{-\phi w - |m| (\phi w + |m|)^{w-1}}}{w!}
$$

for $w = 0, 1, \ldots$. The unconditional distribution of $W$ is symmetric. The tail is approximated by:

$$
\Pr ( |W| = w \mid |m|) \sim \left( \frac{|m| e^{-(1-1/\phi)|m|}}{\phi \sqrt{2\pi}} \right) \left( \frac{e^{\phi - 1}}{\phi} \right)^{-w} w^{-1.5}
$$

The normalized aggregate capital growth rate $N(\log K_{t+1} - \log K_t)$ converges in distribution to $(m + W) \log \lambda$.

Proof is deferred to Appendix A.

The key to the proof is to embed the best response dynamics in a branching process which has the following recursive property. Let $G(s)$ be the generating function of the subsequent adjustments $W$, provided that the initial deviation from the time average level is $m = 1$. Let $x$ be the number of firms that adjust capital due to $m$, and $F(s)$ be its generating function. Each adjustment of $x$ then has a chance to propagate in the next step just like the initial adjustment $m_2$. Thus the total number of offsprings which are originated from each of $x$ follows $G(s)$. Hence we obtain a functional equation $G(s) = sF(G(s))$, from which we derive the distribution of $W$. A similar functional equation obtains in a generalized model with heterogeneous $\lambda_i$ and $\delta_i$ and with non-uniformly distributed $s_{j,t}$ (see Nirei (2005)). The functional equation characterizes the propagation distribution completely, because any moment can be derived from the functional.

Proposition 1 implies that the capital growth $\log K_{t+1} - \log K_t$ conditional to $m = 1$ is approximated by a power distribution $w^{-1.5}$ truncated by an exponential distribution.

---

2The symbol $\sim$ indicates that the ratio of the both sides converges to one as $w \to \infty$. 
that declines at rate $1 - \phi$. The capital growth conditional to $m = 1$ has an asymptotic mean $\log \lambda / (N(1 - \phi))$ and variance $(\log \lambda / N)^2 (2 - \phi) / (1 - \phi)^3$ for $\phi < 1$. The variance of the capital growth rate declines linearly in $N$, hence the law of large numbers obtains.

The fluctuation of the capital growth exhibits quite a different behavior when $\phi = 1$. The distribution of $W$ becomes a power law distribution. With the exponent 0.5 (in a cumulative distribution), the distribution does not have either mean or variance. That is, the sample moments diverge as the sample size increases. In fact, the variance of the capital growth rate ceases to depend on $N$ for a large $N$ when $\phi = 1$, as we state in the following proposition.

**Proposition 2** When $\phi = 1$, the variance of the aggregate capital growth rate $\log K_{t+1} - \log K_t$ converges to a non-zero constant as $N \to \infty$. The limit standard deviation is:

$$\sqrt{\frac{2}{\pi}} (\sigma_m + 1/3) \sigma_m$$

Formula (23) with (20) gives the standard deviation of growth rates as a function of the lumpiness parameter $\lambda$ and the periodicity $q$ of capital oscillation at the agent level.

Proof is deferred to Appendix B.

This result means that the growth rate fluctuation becomes independent of the number of firms as the number grows. No matter how large the aggregative system is, the non-linearity of the individual behaviors can add up to aggregate fluctuations. Some numerical examples are shown in Table 1. In the table, we observe that the empirically plausible magnitude of lumpiness is large enough to generate the fluctuations in aggregate production observed in the business cycles. Table 1 also shows little dependence of the fluctuation magnitude on the mark-up rate. In fact the standard
deviation is not significantly changed even when the mark-up rate goes to infinity, at which \( \sigma^2_m \) is simplified to \( (1 - 1/q)/q \).

Our analytical results imply two things on the investment propagation. First, it challenges the conventional view that the sectoral propagation does not add up to a large aggregate fluctuation due to the law of large numbers effect. Our result shows that, when the response of wage and interest rate is rigid enough, the sectoral propagation can generate a significant fluctuation in aggregate level. Secondly, our result shows that the large, non-degenerate investment fluctuation can occur endogenously in a deterministic environment. This implies that an interaction of small non-linear behaviors at the micro level may play a crucial role in aggregate investment fluctuations.

The distribution formula (21) exhibits a non-normal, heavy-tailed distribution that converges to a power-law distribution as \( \phi \to 1 \). This property is well understood in the light of critical phenomena. The propagation process is a branching process with mean \( \phi \). Thus the behavior of the propagation process under the uniform distribution of \( s_i \) is essentially identical to a percolation on a Bethe lattice (Grimmett, 1999, page 254) with \( 1 - \phi \) being the difference of the probability from a critical probability. It is known that the cluster volume in the percolation follows a power law with exponent \(-1.5\) at

<table>
<thead>
<tr>
<th>mark-up ((\mu - 1))</th>
<th>(0.02)</th>
<th>(0.2)</th>
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<td>lumpiness ((\log \lambda))</td>
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<tr>
<td>4</td>
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<td>2.29</td>
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<tr>
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<td>0.82</td>
</tr>
<tr>
<td>8</td>
<td>0.75</td>
<td>1.87</td>
</tr>
</tbody>
</table>

Table 1: Limiting standard deviations of capital growth \( g \) (percent)
the criticality $\phi = 1$, which corresponds to our power law in (22) for $\phi = 1$. Also, the second moment of the cluster is proportional to an inverse cube of the difference to the critical probability, just as we have $(1 - \phi)^{-3}$ in the asymptotic variance of the growth rate. We can interpret the criticality condition $\phi = 1$ as the case of perfect strategic complementarity across firms. By perfect complementarity we mean that a proportional increase in capital of all the other firms induces the same proportional increase in capital of a firm, if the increment is larger than the lumpiness. A shock smaller than the lumpiness, however, does not cause a symmetric movement across firms because of the lumpy behavior. The power-law distribution of aggregate growth rates is caused by the perfect complementarity in a global range and no complementarity in a local range.

The possibility of a power-law distribution of sectoral propagation was first pointed out by Bak et al. along the lines of the literature of self-organized criticality. The point of the literature is that the critical phenomena, which are broadly associated with the power-law distributions, can occur at the sink of a class of dynamical systems, whereas such criticality had been believed to require a fine tuning of parameters. The "self-organization" mechanism to arrive at a critical point is expressed in our model as a convergence of $s_i$ to a uniform distribution. The result differs in the exponent of the power-law distribution, which is $1/2$ in our model and $1/3$ in Bak et al. The difference arises from the topology of the network. The latter assumes a two-dimensional lattice network in which two avalanches started from neighboring sites can overlap. This leads to the longer chain of reaction and thus the flatter power-law tail. Our model assumes a global interaction which corresponds to an infinite-dimension case of the lattice models. This setup enables us to treat the two neighboring avalanches as mutually independent and to utilize the recursive structure of the branching process.
3.2 Dynamics

So far, we have focused on the case in which the firms' positions relative to the threshold follows a uniform distribution. Aggregate investment is shown to be sensitive to the configuration of the positions. Aggregate investment will exhibit fluctuations along with the evolution of the configuration which is driven by the depreciation of capital. In this section we investigate the dynamic path of the aggregate fluctuations by numerically computing the perfect foresight equilibrium path of the product market with the wage and interest fixed at the time-average level.

Parameters are specified as follows. The returns-to-scale parameter $\alpha + \gamma$ takes various values close to one. The labor's share of income $\gamma/\mu$ is equal to 0.58. The mark-up rate $\mu - 1$ is set at 0.2. To determine the time-average levels of the aggregate variables, we specify the utility function to be quasi-linear, $U(C_t, H_t) = \log C_t - H_t$. The annual discount rate of utility is set at $\beta = 0.96$. The annual depreciation rate of capital $\delta_j$ is assumed to follow a uniform distribution between 0.01 and 0.2. The lumpiness $\lambda_j$ follows a normal distribution with mean 1.5 and standard deviation 0.2. The equilibrium is computed sequentially for 200 quarters, and the first 100 quarters are discarded in order to focus on the stationary fluctuations. Figure 1 plots a sample path of such an equilibrium for the case $N = 500000$ and $\alpha + \gamma = 0.999$. We observe a considerable fluctuation of the investment and output, as well as some persistence in the both series.

We compute the standard deviation and the autocorrelation of $Y$ for each path, and repeat the procedure for 100 times for each parameter set (except for the case $N = 500000$ in which we repeat for 12 times due to heavy computational load). Table 2 reports the mean and standard deviation of the computed standard deviations and autocorrelations of output for various parameters. The standard errors of the estimates
Figure 1: A simulated path of output ($Y$) and investment ($I$) when $N = 500000$ and $\alpha + \gamma = 0.999$ are reported in parentheses.

Table 2 confirms that the magnitude of the fluctuation is increasing as the returns to scale approach to one. The magnitude decreases as the number of firms increases, but even for the case of 500,000 firms the model generates significant fluctuations. Considering that the number of operating manufacturing plants in the U.S. is about 350,000 (Cooper, Haltiwanger, and Power (1999)) and that over half of the plants experience a 1-year capital adjustment of at least 37% of the capital in the estimate of Doms and Dunne (1998), we find it a quantitative possibility that the aggregate fluctuations with magnitude of empirical business cycles endogenously arise from microscopic discrete investments. We also observe in Table 2 that the model generates considerable autocorrelation in output. The synchronization of oscillating capital alone can generate autocorrelations via echo effects, while the significant heterogeneity in lumpiness and
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<table>
<thead>
<tr>
<th>$\alpha + \gamma$</th>
<th>Standard deviation of $Y$ (%)</th>
<th>Autocorrelation of $Y$ (%)</th>
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</thead>
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<tr>
<td></td>
<td>$0.9$</td>
<td>$0.99$</td>
</tr>
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<td>$500$</td>
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<tr>
<td></td>
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<td>(0.59)</td>
</tr>
<tr>
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<td>1.92</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.23)</td>
</tr>
<tr>
<td>$50000$</td>
<td>0.07</td>
<td>0.66</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.07)</td>
</tr>
<tr>
<td>$500000$</td>
<td>0.02</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>(0.00)</td>
<td>(0.02)</td>
</tr>
</tbody>
</table>

Table 2: Standard deviation and autocorrelation of output
depreciation rates across firms prevents the capital from being completely synchronized and exhibiting perfect phase-locking.

3.3 A case of general equilibrium with imperfect information

This section presents a particular case of general equilibrium in which the approximated pricing formulae (15–16) hold exactly. We consider a model of imperfect information in which the households do not observe the distribution of firms’ positions relative to the thresholds. We assume that the households expect the future investment demand function to be the same as its smoothly-adjusting counterpart as a result of the asymmetric information. It is also assumed that the depreciation rate of capital is one hundred percent.
Let us specify the instantaneous utility function as:

$$U(C_t, H_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{b H_t^{1+\nu}}{1+\nu}.$$  \hspace{1cm} (24)

Then the household's optimal choice must satisfy:

$$w_t = bC_t^\sigma H_t^{\nu}.$$  \hspace{1cm} (25)

The households expect firm $j$ to maximize the discounted value of profits (10) by choosing a stream of capital $k_{t+1}$ without the discreteness constraint (4). By aggregating the first order conditions across $j$, and by applying $\delta_j = 1$, we obtain that for any $t$:

$$\rho D\omega_t^{-\gamma/(1-\gamma)} K_t^{\rho-1} = r_t$$  \hspace{1cm} (26)

Aggregate demand for labor is obtained by aggregating the first order condition for the maximization problem (10) with respect to $h_{j,t}$ as:

$$w_t H_t = (\gamma/\mu) Y_t$$  \hspace{1cm} (27)

Plugging the same first order condition to the production function yields in aggregation:

$$Y_t = (A_t(\gamma/\mu)\gamma)^{1/(1-\gamma)} w^{-\gamma/(1-\gamma)} K^{\alpha/(1-\gamma)}.$$  \hspace{1cm} (28)

Along with the definition of the discount rate for the firm's dynamic optimization, $r_t = (C_t/C_{t-1})^\sigma/\beta$, the above equations (25, 26, 27, 28) determine the growth rates of $Y_t, C_t, H_t, r_t,$ and $w_t$ given that of $K_t$. By solving for $w_t$ and $r_t$, we obtain:

$$\frac{w}{w} = \left(\frac{K}{K}\right)^{(1-\gamma)(\rho-1)+\alpha\nu}$$  \hspace{1cm} (29)

$$\frac{r}{r} = \left(\frac{K}{K}\right)^{(\rho-1)(1-\gamma)/1+\nu}.$$  \hspace{1cm} (30)

These recover our approximation (15,16). The sensitivity of the prices to the capital growth, $(\theta_r, \theta_w)$, is positive when $\nu$ is large. In a special case of quasi-linear utility
function in which $\nu = 0$, we obtain a negative sensitivity $\theta_r = \theta_w = (1 - \gamma)(\rho - 1)$ for $\gamma < 1$.

Consider a general case where $\delta_j$ is not 1 (but still constant across $j$). Then the equation system is reduced to:

$$\frac{\rho D_0 w_t^{-\gamma/(1-\gamma)} K_t^{\rho-1}}{w_t^{1+\frac{\nu}{1-\gamma}} (K_t)^{-\frac{\nu}{1-\gamma}}} = r_t - 1 + \delta \quad (31)$$

$$\left(\frac{w_t}{\bar{w}}\right)^{1+\frac{\nu}{1-\gamma}} (\frac{K_t}{\bar{K}})^{-\frac{\nu}{1-\gamma}} = \frac{r}{\bar{r}} \quad (32)$$

The functional relation is not exactly same as our approximation (15, 16), yet we can calculate the sensitivity parameter $\theta_r, \theta_w$ in the neighborhood of the time-average level:

$$\frac{d \log w_t}{d \log K_t} \mid _{\bar{r}, \bar{w}, \bar{K}} = \frac{(1-\gamma)(\rho - 1)(1 - (1 - \delta)/\bar{r}) + \alpha \nu}{1 + \nu - \gamma(1 - \delta)/\bar{r}} \quad (33)$$

$$\frac{d \log r_t}{d \log K_t} \mid _{\bar{r}, \bar{w}, \bar{K}} = \theta_w \left(1 + \frac{\nu}{1 - \gamma}\right) - \frac{\alpha \nu}{1 - \gamma} \quad (34)$$

We can see that, for the quasi-linear case $\nu = 0$, $\theta_w = \theta_r < 0$ holds for any $\delta$. We obtain $\theta_w \to 0$ and $\theta_r \to 0$ when $\rho \to 1$. Thus, the prices do not respond to capital when, for example, the technology exhibits constant returns to scale and the goods perfectly substitute with each other ($\mu \to 1$). The prices are also irresponsive when $\delta = 0$ and $\beta \to 1$. Namely, the general equilibrium effect by the adjustment of wage and interest rate is small when the unit time is short so that the rates of capital depreciation and utility discounting are small.

These results crucially depend on the particular assumption on the household’s expectation. Since the actual investment generally differs from the expected investment, the equilibrium is different from the household’s expectation. In particular, the equilibrium consumption is determined as output less investment, and thus the marginal rate of intertemporal substitution $r_t$ differs _ex post_ from the one applied by firms.
4 Conclusion

This paper characterizes the aggregate fluctuations arising from spillover effects of discrete investments at the firm level. The theory of one-sided \((S,s)\) economies has shown that the distribution of a firm’s position in an inaction band has a robust tendency to converge to a uniform distribution. Based on the theory, we evaluate the deterministic fluctuation of aggregate investment along the ergodic evolution of the configuration as a stochastic fluctuation whose randomness arises from the stochastic configuration of the capital. For each configuration, one-period aggregate investment is derived by a fictitious best-response dynamics of firms’ investment decisions. The best response dynamics unconditional to the initial configuration can be embedded in a branching process. This enables us to derive the distribution function of the aggregate fluctuation in an explicit form.

In a partial equilibrium of product markets, the aggregate investment follows a power-law distribution with an exponential truncation at the tail. The truncation speed is determined by \(1 - (\alpha + \gamma)\) where \(\alpha + \gamma\) is the returns to scale of production technology. Under the constant returns to scale, the aggregate investment follows a power-law distribution, and its variance is shown to be strictly positive even when there are an infinite number of firms.

The equilibrium path of the model is numerically computed. The simulation confirms the validity of the analysis above, and also finds that the paths of output and investment show persistence and mild periodicity. This expresses the echo effect in which a clustering of investments in a period reappears after several periods. The frequency of the echo effect is determined by the natural frequency of a firm’s capital adjustment, which in our case is equal to the lumpiness \((\log \lambda_j)\) divided by the depreciation rate \((\text{log}(1 - \delta_j))\).
Appendix

A Proof of Proposition 1

Let us rewrite the best response dynamics in \( t \). We use \( u \) to denote the step in the dynamics and suppress \( t \). Let \( K^u \) denote the average capital defined by \( (11) \) with a profile \( k^u_j \) for \( u \geq 1 \). Define \( k^u_j \) by the threshold formula \( (17) \) with \( K^u \) for \( u \geq 1 \). For \( u = 0 \), we define \( K^0 = K_t \) and \( k^0_j = k^t_j \). Define \( s^u_j = (\log k^u_j - \log k^*_j) / \log \lambda \). Then the dynamics of \((k^u_j, s^u_j)\) is written as follows.

\[
k^0_j = k^t_j (1 - \delta) \tag{35}
\]
\[
s^0_j = s^t_j + (\log k^0_j - \log k^t_j) / \log \lambda \tag{36}
\]
\[
k^{u+1}_j = \begin{cases} k^u_j / \lambda & \text{if } s^u_j < 0 \\ k^u_j / \lambda & \text{if } s^u_j > 1 \\ k^u_j & \text{otherwise} \end{cases} \tag{37}
\]
\[
s^{u+1}_j = s^u_j + (\log k^{u+1}_j - \log k^u_j - \phi(\log K^{u+1}_j - \log K^u_j)) / \log \lambda \tag{38}
\]

We consider the case \( m \geq 0 \). Then \( s^u_j > 1 \) never happens in the best response dynamics for \( u \geq 1 \). The case \( m < 0 \) is proved symmetrically by changing the sign of adjustments in logarithm. Define \( H_u \) for \( u \geq 1 \) as the set of \( j \) such that \( \log k^u_j - \log k^{u-1}_j = \log \lambda \). Then \( \log k^u_j = \log k^{u-1}_j \) for \( u \notin H_u \). Define \( m_u \) as the size of \( H_u \).

First we examine \( m = N(\log K^1 - \log K_t) / \log \lambda \). We break \( m \) into two terms as \( m = N(\log K^1 - \log(\sum_{j=1}^{N}(k^0_j)^{\rho})^{1/\rho}) / \log \lambda + N(\log(\sum_{j=1}^{N}(k^0_j)^{\rho})^{1/\rho} - \log K_t) / \log \lambda \). The second term represents the depreciation and is equal to \( N \log(1-\delta) / \log \lambda = -N/q \). The first term represents the first-step adjustments induced directly by the depreciation. By the assumption that \( s_{j,t} \) follows a uniform distribution, we obtain \( \Pr(s^0_j < 0) = 1/q \).
Then $m_1$ follows a binomial distribution $\text{Bin}(N, 1/q)$. Thus:

$$
\log K^1 - \log \left( \sum_{j=1}^{N} \left( \frac{k_{j}^{0} \rho}{N} \right) \right)^{1/\rho} = \frac{1}{\rho} \left( \log \left( \frac{\lambda^\rho \sum_{j \in H_1} \frac{(k_{j}^{0} \rho)^\rho}{N} + \sum_{j \notin H_1} \frac{(k_{j}^{0} \rho)^\rho}{N} }{\log \left( \begin{array}{l} \lambda^\rho \sum_{j \in H_1} \frac{(k_{j}^{0} \rho)^\rho}{N} + \sum_{j \notin H_1} \frac{(k_{j}^{0} \rho)^\rho}{N} \\ \end{array} \right) - \log \frac{\sum_{j=1}^{N} \left( \frac{k_{j}^{0} \rho}{N} \right)^\rho}{N} } \right) \\
= \frac{1}{\rho} \log \left( \frac{\lambda^\rho - 1}{\sum_{j \in H_1} \frac{(k_{j}^{0} \rho)^\rho}{N} + \sum_{j \notin H_1} \frac{(k_{j}^{0} \rho)^\rho}{N} } + 1 \right) \\
= \frac{1}{\rho} \log \left( \frac{\lambda^\rho - 1}{\sum_{j \in H_1} \frac{\lambda_{j}^{0 \rho}}{N} } + \frac{\sum_{j \notin H_1} \frac{\lambda_{j}^{0 \rho}}{N} }{\sum_{j=1}^{N} \frac{\lambda_{j}^{0 \rho}}{N} } + 1 \right) \\
$$

The last line utilized that $k_{j}^{*}$ is constant across $j$ when $\lambda_j$ and $\delta_j$ are constant. Since $\delta_j^0$ is distributed uniformly, the strong law of large numbers implies that, with probability one,

$$
\lim_{N \to \infty} \sum_{j=1}^{N} \frac{\lambda_{j}^{0 \rho}}{N} = \int_{0}^{1} \lambda_{j}^{0 \rho} ds_j^0 = \frac{\lambda^\rho - 1}{\rho \log \lambda} \\
$$

Also, $j \in H_1$ is equivalent to $0 \leq s_j^0 < 1/q$. Thus, by the central limit theorem,

$$
\sqrt{N}(\sum_{j \in H_1} \frac{\lambda_{j}^{0 \rho}}{N} - \int_{0}^{1/q} \lambda_{j}^{0 \rho} ds_j^0) \to \sqrt{N}(\sum_{j \in H_1} \lambda_{j}^{0 \rho}/N - (\lambda^\rho/q - 1)/\rho \log \lambda) \\
$$

converges in distribution as $N \to \infty$ to a normal distribution with mean zero and variance:

$$
\int_{0}^{1/q} \lambda_{j}^{0 \rho} ds_j^0 - \frac{(\lambda^\rho/q - 1)^2}{2 \rho \log \lambda} = \frac{\lambda^\rho/q - 1}{2 \rho \log \lambda} - \frac{(\lambda^\rho/q - 1)^2}{\rho \log \lambda} \\
$$

By regarding (39) as a nonlinear function $F(x)$ of $x = \sum_{j \in H_1} \lambda_{j}^{0 \rho}/\sqrt{N}$, we can use the delta method to obtain the asymptotic distribution of $\sqrt{N}F(x)$ as a normal distribution with mean $F(x_0)$ and variance $F'(x_0)^2\text{Avar}(x)$ where $x_0$ is the asymptotic mean of $x$. The mean $F(x_0)$ is calculated as:

$$
(1\rho) \log \left( \frac{(\lambda^\rho - 1)(\lambda^\rho/q - 1)/\rho \log \lambda)}{(\lambda^\rho - 1)/\rho \log \lambda} + 1 \right) = \frac{\log \lambda}{q} \\
$$

and the variance is:

$$
(1 - \lambda^{-2\rho/q}) \log \lambda/(2 \rho) - (1 - \lambda^{-\rho/q})^2/\rho^2 \\
$$
By collecting the results, it is shown that \( m/\sqrt{N} \) asymptotically follows a normal distribution with mean zero and the variance \((\log \lambda)^2\) divided by \((\log \lambda)^2\).

Next we derive a limit for \( N(\log K^{u+1} - \log K^u) \) for \( u \geq 1 \). The Taylor series expansion yields:

\[
N(\log K^{u+1} - \log K^u) = \sum_{n=1}^{\infty} \sum_{j \in H_{u+1}} \left( \frac{k^u_j}{K^u} \right)^{\rho^{n-1}} \frac{(\log \lambda)^n}{n!} + O(1/N)
\]

\[
= \frac{(k^{\ast u})^\rho}{K^u} \frac{\lambda^\rho - 1}{\rho} \sum_{j \in H_{u+1}} \lambda s_j^{\ast \rho} + O(1/N)
\]

\[
= \frac{\lambda^\rho - 1}{\rho} \sum_{j \in H_{u+1}} \frac{\lambda s_j^{\ast \rho}}{N} + O(1/N)
\]

(44)

The residual term in the first equation is of order \( 1/N \), because it consists of the terms involving \( \partial K^u/\partial k_j^u \) which is of order \( 1/N \), and because the number of terms (the size of \( H_{u+1} \)) is finite with probability one as is shown later. The second equation holds since \( k_j^u \) is constant across \( j \). For the same reason the third equation obtains, since \( K^u = k^{\ast u}(\sum_{j=1}^N \lambda s_j^{\ast \rho}/N)^{1/\rho} \) holds. The average \( \sum_{j=1}^N \lambda s_j^{\ast \rho}/N \) converges to \( E[\lambda s_j^{\ast \rho}] \) as \( N \to \infty \) almost surely. The expectation is equal to \( \int_0^1 \lambda s_j^{\ast \rho} ds_j^u = (\lambda^\rho - 1)/(\rho \log \lambda) \), because the following three facts hold as \( N \to \infty \) as we see later, namely, \( s_j^u \) for \( j \in H_1 \) is uniformly distributed in \([1 - 1/q, 1)\), \( s_j^u \) for \( j \notin \cup_{u=1}^u H_v \) is uniformly distributed in \([0, 1 - 1/q)\), and \( \cup_{v=1}^u H_v \) is finite with probability one. Also, \( \sum_{j \in H_{u+1}} \lambda s_j^{\ast \rho} \) converges to \( m_{u+1} \) in distribution. This is because \( s_j^u < \phi(\log K^u - \log K^{u-1})/\log \lambda \) for \( j \in H_{u+1} \) and the right hand side is of order \( 1/N \) as (44) shows. Thus the summation becomes to follow a binomial distribution. Hence, we obtain for \( u \geq 1 \) a convergence in distribution:

\[
N(\log K^{u+1} - \log K^u) \to m_{u+1} \log \lambda
\]

(45)

Next we examine \( m_u \) conditional to \( m_{u-1} \) for \( u \geq 2 \). We have \( \Pr(j \in H_u | j \notin \cup_{v=1,2,\ldots,u-1} H_v) = (\phi(\log K^u - \log K^{u-1})/\log \lambda)/((N - \sum_{v=1}^{u-1} m_v)/N) \). Thus \( m_u \) follows
$\text{Bin}(N - \sum_{v=1}^{u-1} m_v, (\phi \log K^u - \log K^{u-1})/\log \lambda)/((N - \sum_{v=1}^{u-1} m_v)/N))$. This defines the stochastic process $m_u$ completely. As we let $N \to \infty$, the limit (45) holds and the binomial distribution of $m_u$ converges to a Poisson distribution with mean $\phi m_{u-1}$ for $u \geq 3$. For $u = 2$, $m_2$ converges in distribution to a Poisson with mean $\phi m$ where the distribution of $m$ is defined conditionally on $m_1$.

Since a Poisson distribution is infinitely divisible, the Poisson variable with mean $\phi m_{u-1}$ is equivalent to a $m_{u-1}$-times convolution of a Poisson variable with mean $\phi$. Thus the process $m_u$ for $u \geq 2$ conditional to $m$ is a branching process with a step random variable being a Poisson with mean $\phi$ for $u \geq 3$ and $m_2$ following a Poisson with mean $\phi m$. Since $\phi \leq 1$, the process $m_u$ reaches 0 by a finite stopping time with probability one (see Feller (1957)). Thus the best response dynamics is a valid algorithm of equilibrium selection. Let $T$ denote the stopping time. Using the previous asymptotic results, we have $W \to \sum_{u=2}^{T} m_u$ in distribution. By using the property of the Poisson branching process (Kingman (1993)), we obtain an infinitely divisible distribution called Borel-Tanner distribution for the accumulated sum $W$ conditional to $m_2$ as:

$$\Pr(W = w \mid m_2) = (m_2/w)e^{-\phi w}(\phi w)^{w-m_2}/(w - m_2)!$$

for $w = m_2, m_2 + 1, \ldots$. Using that $m_2$ follows the Poisson distribution with mean $\phi m$, we obtain (21) in the Proposition as follows:

$$\Pr(W = w \mid m) = \sum_{m_2=0}^{w} ((m_2/w)e^{-\phi w}(\phi w)^{w-m_2}/(w - m_2)!e^{-\phi m}m^{m_2}/m_2!$$

$$= (me^{-\phi w-m}/w) \sum_{m_2=1}^{w} (\phi w)^{w-m_2}m^{m_2-1}/((w - m_2)!(m_2 - 1)!)$$

$$= (me^{-\phi w-m}/w)(\phi w + m)^{w-1}/(w - 1)!$$

$$= me^{-\phi w-m}(\phi w + m)^{w-1}/w!$$

(47)
Approximation (22) is obtained by applying the Stirling’s formula $w! \sim \sqrt{2\pi}e^{-w}w^{w+0.5}$.

\[
me^{-\phi w - m}(\phi w + m)^{w-1}/w! \\
\sim me^{-\phi w - m}(\phi w + m)^{w-1}/(\sqrt{2\pi}e^{-w}w^{w+0.5}) \\
= me^{(1-\phi)w - m}(\phi w + m)/(\phi w))^{w}((\phi w + m)/(\phi w))^{-1}\phi^{w-1}w^{-1.5}/\sqrt{2\pi} \\
\sim (me^{(1/\phi - 1)m}/(\phi \sqrt{2\pi}))(e^{\phi - 1}/\phi)^{-w}w^{-1.5}
\]

(48)

This completes the proof.

B Proof of Proposition 2

We focus on $(m + W)/N$, provided with the relation $(\log K_{t+1} - \log K_t)/\log \lambda \sim (m + W)/N$ shown in the previous proof. The unconditional variance is decomposed as follows:

\[
\text{Var}\left(\frac{m + W}{N}\right) = \mathbb{E}\left(\text{Var}\left(\frac{W}{N} \mid m\right)\right) + \text{Var}\left(\frac{m}{N} + \mathbb{E}\left(\frac{W}{N} \mid m\right)\right) \\
= \mathbb{E}\left[\text{Var}\left(\frac{W}{N} \mid m, m_2\right) \mid m\right] + \text{Var}\left(\mathbb{E}\left(\frac{W}{N} \mid m, m_2\right) \mid m\right) \\
+ \text{Var}\left(\frac{m}{N} + \mathbb{E}\left(\frac{W}{N} \mid m, m_2\right) \mid m\right).
\]

(49)

$W$ asymptotically follows a branching process when $N \to \infty$, and by the nature of the branching process, $|W|$ conditional to $|m_2|$ is equivalent to the $|m_2|$-times convolution of $W$ conditional to $m_2 = 1$. Using these facts, we obtain that:

\[
\text{Var}(W/N \mid m, m_2) \sim |m_2|\text{Var}(W/N \mid m_2 = 1) \quad (50) \\
\mathbb{E}(\mathbb{E}(W/N \mid m, m_2) \mid m) \sim \mathbb{E}(m_2 \mid m)\mathbb{E}(W/N \mid m_2 = 1) \\
\sim m\mathbb{E}(W/N \mid m_2 = 1). \quad (51)
\]
Also, $|m_2|$ conditional to $m$ asymptotically follows a Poisson distribution with mean $|m|$ and the unconditional distribution of $m_2$ is symmetric. Since $m/\sqrt{N}$ asymptotically follows $N(0, \sigma_m^2)$ by Proposition 1, we can use the formula $E(|m|/\sqrt{N}) \to \sigma_m \sqrt{2/\pi}$.

Applying these, we obtain:

$$
\text{Var} \left( \frac{m + W}{N} \right) \\
\sim E \left[ E \left( |m_2| | m \right) \text{Var} \left( \frac{W}{N} | m_2 = 1 \right) + \text{Var}(m_2 | m) \left( E \left( \frac{W}{N} | m_2 = 1 \right) \right)^2 \right] \\
+ \text{Var} \left( \frac{m}{N} + E \left( \frac{W}{N} | m_2 = 1 \right) E(m_2 | m) \right) \\
\sim (\sigma_m \sqrt{2/\pi}) E \left( \frac{W^2}{N} | m_2 = 1 \right) + \sigma_m^2 \left( \frac{1}{\sqrt{N}} + E \left( \frac{W}{\sqrt{N}} | m_2 = 1 \right) \right)^2 \quad (52)
$$

Next we calculate $\lim_{N \to \infty} E(W/\sqrt{N} | m_2 = 1)$, provided that the best response dynamics reaches an equilibrium before all the $N$ firms adjust. Namely, we take the expectation conditional to $W \leq N$ for a fixed $N$ by using the asymptotic probability function taken from (46):

$$
\Pr(W = w | m_2 = 1, W \leq N) \Pr(W \leq N) = e^{-w} w^{w-1} / w!.
\quad (53)
$$

Proof of Proposition 1 shows that the probability of the event $W \leq N$ converges to one as $N \to \infty$. By using the inequality (see Feller (1957)):

$$
\sqrt{2\pi} w^{w+0.5} e^{-w+1/(12w+1)} < w! < \sqrt{2\pi} w^{w+0.5} e^{-w+1/(12w)}
\quad (54)
$$

we can compute the upper and lower bounds of the probability of $W$ for a fixed $N$ as follows.

$$
E(W/\sqrt{N} | m_2 = 1, W \leq N) \Pr(W \leq N) = \sum_{w=1}^{N} e^{-w} w^w / (w! \sqrt{N}) \\
< \sum_{w=1}^{N} e^{-w} w^w / (\sqrt{2\pi} w^{w+0.5} e^{-w+1/(12w+1)} \sqrt{N})
$$
The second to last line holds because \( e^{-1/(12w+1)} \) is bounded by one. Similarly, the lower bound turns out to converge to the same value. Let us note that the function \( e^{-1/(12w)w^{-0.5}} \) is decreasing for \( w > 1/6 \). Then we obtain:

\[
\sum_{w=1}^{N} e^{-w}w^{w}/(w!\sqrt{N}) > \sum_{w=1}^{N} e^{-w}w^{w}/(\sqrt{2\pi w^{w+0.5}} e^{-w+1/(12w)}\sqrt{N})
\]

\[
= \sum_{w=1}^{N} e^{-1/(12w)}w^{-0.5}/\sqrt{2\pi N}
\]

\[
> \int_{1}^{N+1} e^{-1/(12w)}w^{-0.5}dw/\sqrt{2\pi N}
\]

\[
= \left( e^{-1/(12(N+1))}\sqrt{N+1} - e^{-1/12} + \int_{1}^{N+1} w^{-1.5}e^{-1/(12w)}/12dw \right) / (0.5\sqrt{2\pi N})
\]

\[
\rightarrow_{N \rightarrow \infty} \sqrt{2/\pi}
\]

Hence, \( E(W/\sqrt{N} \mid m_2 = 1, W \leq N) \rightarrow \sqrt{2/\pi} \). Similarly, \( E(W^2/N^{1.5} \mid m_2 = 1) \) is calculated as follows.

\[
E(W^2/N^{1.5} \mid m_2 = 1, W \leq N) = \sum_{w=1}^{N} e^{-w}w^{w+1}/(w!N^{1.5})
\]

\[
> \sum_{w=1}^{N} e^{-w}w^{w+1}/(\sqrt{2\pi w^{w+0.5}} e^{-w+1/(12w)}N^{1.5})
\]

\[
= \sum_{w=1}^{N} e^{-1/(12w)}\sqrt{w}/(\sqrt{2\pi N^{1.5}})
\]

\[
> \left( \int_{1}^{N} e^{-1/(12w)}\sqrt{w}dw \right) / (\sqrt{2\pi N^{1.5}})
\]

\[
= \left( (e^{-1/(12N)}N^{1.5} - e^{-1/12})/1.5 + \int_{1}^{N} (w^{1.5}/1.5)e^{-1/(12w)}(1/(12w^2))dw \right) / (\sqrt{2\pi N^{1.5}})
\]
\[
\begin{align*}
\int_1^N w^{-0.5} e^{-1/(12w)} dw /
\int_1^{N+1} w^{-0.5} e^{-1/(12w+1)} dw
\end{align*}
\]
\[\sim 1/(1.5\sqrt{2\pi})
\]

Where the inequality in the fourth line holds since the function \(e^{-1/(12w)}\sqrt{w}\) is increasing
in \(w\). Similarly, the upper bound is obtained as follows.

\[
\sum_{w=1}^{N} e^{-w^{w+1}}/(w!N^{1.5}) < \sum_{w=1}^{N} e^{-w^{w+1}}/(\sqrt{2\pi}w^{w+0.5} e^{-w+1/(12w+1)N^{1.5}})
\]
\[= \sum_{w=1}^{N} e^{-1/(12w+1)} \sqrt{w}/(\sqrt{2\pi}N^{1.5})
\]
\[< \int_1^{N+1} e^{-1/(12w+1)} \sqrt{w}dw/(\sqrt{2\pi}N^{1.5})
\]
\[= \left(e^{-1/(12N+13)}(N + 1)^{1.5} - e^{-1/13} + \int_1^{N+1} w^{-0.5} e^{-1/(12w+1)}(12/(12 + 1/w)^2)dw\right)/(1.5\sqrt{2\pi}N^{1.5})
\]
\[\sim 1/(1.5\sqrt{2\pi})
\]

Hence, we obtain that \(E(W^2/N^{1.5} | m_2 = 1) \rightarrow 1/(1.5\sqrt{2\pi})\). Collecting the results, we
obtain that:

\[
\text{Var}\left(\frac{m + W}{N}\right)
\]
\[\sim \sigma_m \sqrt{2/\pi \text{E} (W^2 | m_2 = 1)} + \text{Var}\left(\frac{m}{\sqrt{N}}\right) \sigma_m^2 \left(\frac{1}{\sqrt{N}} + \text{E} (\frac{W}{\sqrt{N}} | m_2 = 1)\right)^2
\]
\[\rightarrow_{N \to \infty} (2/\pi)(\sigma_m + 1/3)\sigma_m
\]

Hence, the capital growth rate asymptotically has variance (59) times \((\log \lambda)^2\).

References

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Scale-Invariant Aggregate Fluctuations of Discrete Investments

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Abstract

This paper proposes a method to analyze endogenous fluctuations of aggregate investment when firm-level investment follows an (S,s) policy and has a spillover effect on other firms’ investments. First, we derive the distribution function of aggregate fluctuations in a partial equilibrium of differentiated product markets, under the assumption that a firm’s position in its (S,s) band follows a uniform distribution. Second, the variance of the growth rate of average capital is shown to converge to a non-zero value when the number of firms tends to infinity, if the technology exhibits constant returns to scale. Third, we numerically compute the equilibrium paths in which the firms’ positions evolve deterministically. The simulations uphold our analytical results as well as exhibit echo effects in the output series. Finally, a case of general equilibrium with imperfect information is presented in which the analytical results continue to hold.

Keyword: Lumpy investment, (S,s) economy, self-organized criticality, contagion

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1 Introduction

This paper analyzes a model of endogenous fluctuations of aggregate investment which arise from the lumpy behavior of investments at the firm level. It demonstrates that aggregate fluctuations occur in a partial equilibrium of product markets even in a deterministic environment with infinitely many agents when discrete investments at the micro level have spillover effects.

The recent development of empirical studies on firm-level investments motivates this paper. Researchers have shown the importance of discrete choice over the course of a firm's capital adjustment and a great deal of heterogeneity across firms by using the longitudinal data. For example, Doms and Dunne (1998) found that establishment level capital is adjusted only occasionally but by a jump. Based on the similar empirical findings, Cooper, Haltiwanger, and Power (1999) stressed the role that lumpy investments played in aggregate fluctuations. Ericson and Pakes (1995) pointed out the important effects of exit and entry behavior of firms on collective industrial dynamics, presenting a framework for empirical research of firm dynamics.

These findings call for an analytical method for a dynamical system in which the discrete behavior of many heterogeneous agents are coupled with each other. We consider a specific situation in which firms' lumpy investments have spillover effects. Due to the discreteness of the investment, a firm's capital exhibits non-harmonic oscillation if the capital is depreciated physically. With the spillover effect, the dynamics of the system is then represented by a collection of coupled oscillators. This paper proposes a method to characterize the aggregate fluctuations in this system.

The literature on (S,s) economies and on non-linear dynamics has tackled the question as to how to analyze the aggregate fluctuations that arise from micro-level discreteness, or more generally, micro-level non-linearity. The theory of (S,s) economies