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Interactive Geometry for Surplus Sharing in Cooperative Games

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Interactive Geometry for Surplus Sharing in Cooperative Games

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ABSTRACT

This paper presents interactive geometrical depictions of the Shapley value, nucleolus, and per-capita nucleolus surplus-sharing rules for cooperative games with three players. The program graphically demonstrates how the simplexes corresponding to a host of characteristic functions are "shrunk" to their corresponding cores, calculates allocations using the Shapley Value, nucleolus, and per-capita nucleolus surplus-sharing rules, and graphically depicts the locations of these allocations in the corresponding cores.

Key Words: core, Shapley value, nucleolus, simplex, characteristic function, graphical user interface

JEL Classification: A22, A23, C71

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Interactive Geometry for Surplus Sharing in Cooperative Games

1. Introduction

The need to understand surplus- and cost-sharing rules in cooperative game theory is prevalent at both the undergraduate and graduate levels (Moulin, 1988 and 2004). However, the rules are generally depicted as raw mathematical problems that entail the calculation of a vector of allocations. This paper presents interactive geometrical depictions of the rules for games with small numbers of players (the executable program in win32 platform is available at <http://cc.usu.edu/~slklr/teaching/tugames/tugame.htm>). The program (i) demonstrates how the simplexes associated with a host of characteristic functions are "shrunk" to their corresponding cores, (ii) calculates the corresponding allocations using the Shapley Value, nucleolus, and per-capita nucleolus surplus-sharing rules, and (iii) graphically depicts the locations of these allocations in the corresponding cores. It is therefore a supplementary tool that can be used to enhance the instruction of surplus-sharing rules in either undergraduate- or graduate-level cooperative game theory courses.

The next section discusses the relevant preliminaries of cooperative game theory, focusing on derivations of the Shapley Value and nucleolus surplus-sharing algorithms. Section 3 discusses the interactive geometry for these rules. Section 4 concludes.

2. Preliminaries of Cooperative Games

Let N be a set of n players with 2^N denoting the corresponding power set of N (i.e., the set of all possible coalitions based on N , including the empty and grand coalitions) and $v: 2^N \rightarrow \mathbb{R}$ denoting a characteristic function that allocates some outcome (e.g., coalitional cost or surplus) among each element of 2^N . An allocation in the grand coalition of the n players is represented by $\mathbf{x} = (x_1, \dots, x_n)$. The core of the game (N, v) is the set of all allocations that induces each of the n players to rationally join in the formation of the grand coalition, i.e.,

$$\text{core}(N, v) = \left\{ \mathbf{x} : \sum_{i \in N'} x_i \geq v(N') \text{ for all coalitions } N' \in 2^N \right\}.$$

Provided that v generates solely positive values (i.e., surplus as opposed to cost), three common types of games are typically considered:

- (i) *Additive*: for all disjoint coalitions $N_1, N_2 \in 2^N$, $v(N_1 \cup N_2) = v(N_1) + v(N_2)$
- (ii) *Superadditive*: for all disjoint $N_1, N_2 \in 2^N$, $v(N_1 \cup N_2) \geq v(N_1) + v(N_2)$
- (iii) *Convex*: for all $N_1, N_2 \in 2^N$, $v(N_1 \cup N_2) + v(N_1 \cap N_2) \geq v(N_1) + v(N_2)$

The core for games of type (i) is a singleton, i.e., $x_i = v(\{i\})$ for each player i , corresponding to the Shapley value. Games of type (iii) guarantee a non-empty core that includes the Shapley value. Games of type (ii) are a parent class of types (i) and (iii), but unlike types (i) and (ii) the existence of the Shapley value in the core is not ensured.¹

We can represent the marginal contribution of agent i to coalition N' as $v(N' \cup \{i\}) - v(N')$ where $i \notin N'$. We have as many as $2^{|N'|-1}$ such values relative to all possible coalitions in $2^{N \setminus \{i\}}$, where $|N'|$ represents the cardinality of coalition N' . An allocation obtained by a weighted average of these marginal contributions therefore accounts for the "marginality" of this agent. A special set of weights yielding allocations

$$x_i = \sum_{N' \in 2^{N \setminus \{i\}}} \frac{|N'|!(n - |N'| - 1)!}{n!} [v(N' \cup \{i\}) - v(N')], \quad (1)$$

is known as the Shapley value (SV) of game (N, v) .

Calculation of the SV is straight-forward, particularly for games with small numbers of players. For example, in the three-player game with coalitional surpluses depicted in Table 1,

$$x_1 = \frac{2!0!}{3!}(9 - 6) + \frac{1!1!}{3!}(5 - 3) + \frac{1!1!}{3!}(4 - 2) + \frac{0!2!}{3!}(1 - 0) = 2.$$

Similar calculations for players 2 and 3 yield allocations $x_2 = 3$ and $x_3 = 4$, respectively.

[INSERT TABLE 1 HERE]

The nucleolus provides an alternative method of obtaining an allocation $\mathbf{x}$. Begin by noting that the net benefit, or "excess" for the members of coalition N' that results from

¹ See Moulin (1988) for further details on the relationships between additivity, superadditivity, convexity, the core, and the Shapley value. Examples of each of these outcomes for game types (i) – (iii) can be shown using the interactive geometry presented in Section 3.

joining the grand coalition is given by $\sum_{i \in N'} x_i - v(N')$. Also note that there are 2^n values of such coalition-wise net benefits associated with forming the grand coalition (including the empty and grand coalitions). The basic concept of the nucleolus is to choose an allocation $\mathbf{x}$ in such a way that the best *leximin* ordering of these values is obtained.² Here, a leximin ordering $\tilde{\mathbf{w}} = (\tilde{w}_1, \dots, \tilde{w}_m)$ of vector $\mathbf{w} = (w_1, \dots, w_m)$ is obtained by sorting $w_1, \dots, w_m$ in ascending order, resulting in $\tilde{w}_1 \leq \dots \leq \tilde{w}_m$. For example, if $\mathbf{w} = (3, 2, 1)$, then $\tilde{\mathbf{w}} = (1, 2, 3)$. Vector $\mathbf{w}$ is "leximin preferred" to $\mathbf{u}$ if there exists an m' such that

$$\tilde{w}_{m''} = \tilde{u}_{m''} \quad \text{for all } m'' = 1, \dots, m' - 1 \quad \text{and} \quad \tilde{w}_{m'} > \tilde{u}_{m'}. \quad (2)$$

In other words, $\mathbf{w}$ is leximin preferred to $\mathbf{u}$ if the smallest element of $\mathbf{w}$ is greater than the smallest element of $\mathbf{u}$, or they are equal but the second smallest element in $\mathbf{w}$ is greater than the second smallest element in $\mathbf{u}$, and so on (Maniquet, 2002).

If we set $\mathbf{e}(\mathbf{x}) = \left(\sum_{i \in N'} x_i - v(N') \right)_{(2^n \times 1)}$, which is a $2^n \times 1$ vector accounting for each element (i.e., coalition N') in 2^N , then the nucleolus, $\mathbf{x}^*$, is such that $\mathbf{e}(\mathbf{x}^*)$ is leximin preferred to $\mathbf{e}(\mathbf{x})$ for any $\mathbf{x}$. To see how the nucleolus is determined, consider the three-player game with coalitional surpluses depicted in Table 1 (Serrano, 1999).

[INSERT TABLE 2 HERE]

Following Serrano (1999), begin by considering the "equal-split" vector of surpluses $\mathbf{x} = (14, 14, 14)$, where the individual surpluses sum to $v(N) = 42$. Thus, $\mathbf{e}(\mathbf{x}) = (-12, -2, 8, 14, 14, 14)$, where the worst-treated coalition is $\{2, 3\}$ with an excess of -12. From an egalitarian perspective, it therefore seems to make sense to transfer some of player 1's surplus to players 2 or 3. Let the transfer occur from player 1 to player 3, resulting in the surplus vector $\mathbf{z} = (4, 14, 24)$ and the corresponding excess vector $\mathbf{e}(\mathbf{z}) = (-2, -2, -2, 4, 14, 24)$. By (2) we see that $\mathbf{e}(\mathbf{x})$ is leximin preferred to $\mathbf{e}(\mathbf{z})$. Through iterative pairwise comparisons of each leximin-ordered excess vector, it can then be shown that $\mathbf{z}$ is indeed the nucleolus for this particular game. Hence, the nucleolus provides an egalitarian

² See Moulin (1988) and Serrano (1999) for further details on the nucleolus.

solution in terms of the coalition-wise net benefits associated with forming the grand coalition.

The per-capita nucleolus is obtained in the same way as the nucleolus, except that $\sum_{i \in N'} x_i - v(N')$ is replaced with $\sum_{i \in N'} (x_i - v(N')) / |N'|$ to account for the cardinality of each coalition. We present the pseudocode used to calculate the nucleolus in Section 3.2. (The actual C++ source code is available upon request from authors.)

3. The Interactive Geometry

For graphical purposes, our interactive program specializes in three-player games. The interface, depicted in Figure 1 for the SV, consists of a view which draws a simplex and lines representing coalition-wise values of the characteristic function. The user can set the values for the characteristic function via a dialog box that appears by clicking the "Set" button on the toolbar (discussed further in Section 3.3). The toolbar provides three additional buttons to calculate the corresponding SV, nucleolus, and per-capita nucleolus. In the process of determining these surplus-sharing values, the core is also determined by "shrinking" the simplex according to the constraints on surpluses for each proper coalition. In the case of Figure 1, the core represents 7.4% of the total area of the simplex.

[INSERT FIGURE 1 HERE]

The program is built by C++ with object-oriented capabilities for graphical user interface. Fig.2 describes the overall system relations.

[INSERT FIGURE 2 HERE]

3.1. Construction of the Simplex and Determination of the Core

To see how the simplex is constructed and the core determined in Figure 1, consider its partial construction in Figure 3. In this figure, the simplex is shown from its three-dimensional perspective with the coordinates of each of its vertices normalized by the surplus value of the grand coalition. Recalling from Figure 1 that the allocation for the singleton coalition $\{1\}$ is $x_1 = 1$, or one-ninth of the value of the grand coalition, the area

denoted **A** is effectively "cut away" from the simplex. Similarly, based on the allocation for the coalition $\{2,3\}$ of $x_2 + x_3 = 6$, or two-thirds of the value of the grand coalition, the area denoted **B** is also cut away from the simplex. This cutting away process is performed for each of the proper coalitions listed in Figure 1 until the simplex has shrunk to its core, depicted as the shaded area in Figure 1.

[INSERT FIGURE 3 HERE]

3.2. Computation of the Surplus-Sharing Values

As discussed in Section 2, the interactive program computes and graphically depicts the SV, nucleolus, and per-capita nucleolus allocations. Computation of the SV is performed using (1). With respect to the nucleolus and per-capita nucleolus, recall that both of these surplus-sharing allocations are derived using the same algorithm, except that the latter divides the coalitional net benefits by their respective cardinalities. We therefore focus solely on the computation of the nucleolus.

Although the search for the nucleolus involves finding a maxima, gradient-based approaches (e.g., the Newton method) are not relevant since the values assigned by the leximin ordering are in general non-differentiable. We therefore use the following pseudocode to conduct a search in the two-dimensional simplex:

```

precision = 0.1
loop precision ≥ 0.001
    loop 0 ≤ x1 ≤ v(N)
        loop 0 ≤ x2 ≤ v(N) - x1
            "compare"
            x2 ← x2 + precision
        end loop
        x1 ← x1 + precision
        x2 ← 0
    end loop
    precision ← precision / 10
end loop

```

where the “*compare*” command initiates comparisons of the leximin orderings. To sort the comparisons, we then employ the following "Bubble Sort" algorithm.³

```

loop swapFlag = true
  swapFlag = false
  loop 1 ≤ m' ≤ m - 1
    if wm' > wm'+1 then
      swap(wm', wm'+1)
      swapFlag = true
    end if
  end loop
end loop

```

The sorting procedure enables a straightforward comparison of the leximin ordered vectors.

3.3. Construction and Representation of Characteristic Function

Characteristic functions are represented by an n -dimensional array of size two for each dimension. This is because the power set 2^N literary contains 2^n elements. If we denote a characteristic function by $v[i][j][k]$ for three-player case, the representation is given by identifying

$v(\emptyset) \equiv 0$	with	$v[0][0][0]$,
$v(\{1\})$	with	$v[1][0][0]$,
$v(\{2,3\})$	with	$v[0][1][1]$,
$v(N)$	with	$v[1][1][1]$,

and so on.

Our program provides two options for specifying the characteristic function. One option is to directly set the values of a characteristic function using the Set button. The corresponding dialogue box is provided in panel (a) of Figure 4. The other option is to select a predetermined characteristic function from the list presented in panel (b) of Figure 4. The three main types of games presented in Section 2 (additive, superadditive,

³The Bubble Sort algorithm is not the most time efficient algorithm. However, given our fixed dimension of three-player games, this loss in efficiency is a non-issue.

convex) are provided in both symmetric and asymmetric forms.⁴ The Elongated Core assigns surpluses of 1 to each of the singleton coalitions $\{1\}$, $\{2\}$, and $\{3\}$, surpluses of 2 to coalitions $\{1,2\}$ and $\{1,3\}$, and a surplus of 7 to coalition $\{2,3\}$. The Trapezoidal Core also assigns surpluses of 1 to each of the singleton coalitions, but then assigns surpluses of 2 to coalitions $\{1,3\}$ and $\{2,3\}$ and 5 to coalition $\{1,2\}$. The Wide Core is same as the Elongated Core, except that coalition $\{2,3\}$ is also assigned a surplus of 2. The Boundary-Intersecting Core sets each of the singleton surpluses in the Wide Core to zero.

[INSERT FIGURE 4 HERE]

As an example, Figure 5 presents the core and corresponding Shapley value for the Elongated Core characteristic function.

[INSERT FIGURE 5 HERE]

4. Conclusion

This paper has demonstrated an interactive program that can be used to supplement undergraduate- and graduate-level cooperative game theory courses by graphically demonstrating (i) the construction of the core and (ii) the determination and location of the common Shapley value, nucleolus, and per-capita nucleolus surplus-sharing allocations for games with three players. Because the geometry for the construction of the core is common across all cooperative games, the program can conceivably be augmented with new surplus-sharing rules as they are introduced in the literature. As a result, the program can be thought of as a "one-stop" graphical interface for surplus-sharing games.

⁴ Symmetric(asymmetric) in this context means that individual surpluses assigned to the singleton coalitions are equal(unequal). For example, the asymmetric convex characteristic function depicted in panel (a) of Figure 4 assigns surpluses of 1, 2, and 3 to the singleton coalitions $\{1\}$, $\{2\}$, and $\{3\}$, respectively.

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<u>Coalition</u>	<u>Coalitional Surplus</u>
{1}	1
{2}	2
{3}	3
{1,2}	4
{1,3}	5
{2,3}	6
{1,2,3}	9

Table 1. Coalitional Surpluses for Calculating the Shapley Value.

<u>Coalition</u>	<u>Coalitional Surplus</u>
{1}	0
{2}	0
{3}	0
{1,2}	20
{1,3}	30
{2,3}	40
{1,2,3}	42

Table 2. Coalitional Surpluses for Calculating the Nucleolus.

Figure 1. The graphical interface.

Figure 2. The overall systems relations.

Figure 3. Partial construction of the core in Figure 1.

Set Coalition Payoffs

Payoffs for Coalitions

(1)	1.000000	(2)	2.000000	(3)	3.000000
{2,3}	6.000000	{1,3}	5.000000	{1,2}	4.000000
Grand Coalition {1,2,3}			9.000000		

OK Cancel

(a) Defining a characteristic function.

Load Game Property

Example Types of Games

- ☐ Symmetric Additive
- ☐ Asymmetric Additive
- ☐ Symmetric Superadditive
- ☐ Asymmetric Superadditive
- ☐ Symmetric Convex
- ☒ Asymmetric Convex
- ☐ Elongated Core (Convex)
- ☐ Wide Core (Convex)
- ☐ Boundary-Intersecting Core (Convex)
- ☐ Trapezoidal Core (Convex)

OK Cancel

(b) Loading predetermined example for characteristic functions

Figure 4. Interface for the characteristic functions.

Figure 5. The Elongated Core Characteristic Function.