EXPANSIONS AND DISCUSSIONS OF THE PHASE GRADIENT ALGORITHM

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Abstract—The Phase Gradient Autofocus algorithm (PGA) is widely used in spotlight mode SAR for motion compensation. The PGA algorithm has been proven to be a superior autofocus method. The original development needed high signal-to-scatter ratios. An improvement was made easing that restriction, but the narrow beam assumption (range dependant errors) remained. This paper progresses one step further by eliminating the narrow beam assumption. Lower altitude SARs have large range dependencies that cannot be ignored. A new phase estimator for PGA is introduced and extended to allow range dependence. An ERS-1 image of Death Valley is used in simulations comparing the new estimator to the maximum likelihood approach and in demonstrating the range-dependent PGA algorithm. Future approaches to the problem of strip-map SAR are also introduced.

INTRODUCTION

Synthetic Aperture Radar (SAR) has become a very important tool in many research areas. These areas, such as archeology, require very high resolution. To obtain a high resolution image the vibrations and movements of the platform must be removed. In most systems an inertial navigational unit (INU) is used to measure movement of a the platform. These devices do not account for all of the movement and vibration that take place at the antennas. In fact there are many errors that are unmeasurable with an INU such as atmospheric errors. As the applications require higher and higher resolution a need has come to estimate the error from another source. This process is called auto-focus because the focusing is done with the data itself. This paper will review the derivation of the PGA for the spotlight mode SAR and discuss the discoveries for the derivation of the range dependent PGA.

The Phase Gradient Autofocus (PGA) algorithm has proven to be a superior method for higher order auto-focus because it does not assume a model for the phase error. The standard PGA model assumes a small beamwidth in range, which results in a phase error constant in the range direction. Satellites and other high altitude systems fit this model. However, a low-altitude SAR like YSAR [Thompson et al., 1996] will have range-dependent phase errors. In this paper we extend the algorithm by dropping the narrow beam assumption and introducing range dependencies in the phase error.

The four main steps in the PGA algorithm are all used in the range dependent model. The four steps defined in this paper are center shifting, windowing, phase estimation and iteration.
[Eichel and C. V. Jakowatz, 1989]. The Phase Weighted Estimation PGA (PWE-PGA) proposed here differs from the other algorithms in the phase estimation step.

SAR BASICS AND DEFINITIONS

Synthetic Aperture Radar (SAR) was originally developed as an all weather day or night imaging system. In the late 60's the military funded the original development of the radar imaging system for the reconnaissance, surveillance and targeting of weapons. Throughout the development many different types of SAR have been developed, each with its own advantages and disadvantages. The requirement of modern day applications have developed the need for a system to have high resolution. In this section the processing for the spotlight and strip-map mode SAR will be briefly discussed.

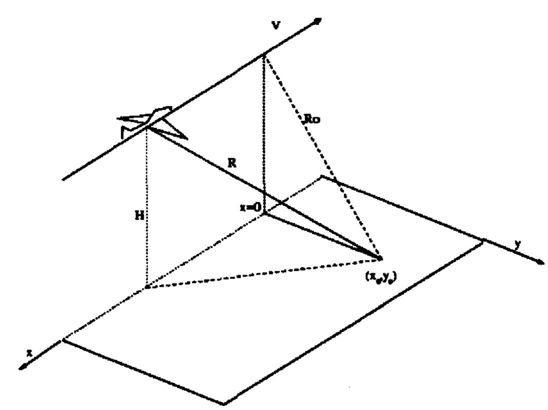

Figure 1: SAR Geometry

SPOT-LIGHT SAR

Spotlight SAR is similar to Strip-map SAR in many ways. The main difference is that the antenna tracks a single spot on the ground. This same spot is imaged in many different directions. In Figure 1, a plane at height, H, above the earth's surface is traveling in the direction ,-x, at velocity V. The range resolution is found by the same matched filter method as the strip-map. A matched filter with the sent signal is used in the range direction. The azimuth is resolved by taking the Fourier transform because each target has its own individual frequency characteristics.

RANGE RESOLUTION FOR SPOT AND STRIP

The range resolution is determined by the time it takes for the chirp to bounce off the targets and return to the antenna. The speed of the chirp is determined from the speed of light. The equation for the scatter return is

$$r_b(t) = A|g(u)|cos(\omega(t - \tau_0 - \tau(u)) + \angle g(u)) \\ (t - \tau_o - \tau(u)) \quad (1)$$

g(u) is the microwave reflectivity of the scatterer with a slant range u. A is the reflectivity factor used for the attenuation for each scatterer. $\angle g(u)$ is the phase change of the scatterer. $\tau(u) = \frac{2u}{c}$ and $\tau_0 = \frac{2u_0}{c}$

RANGE-DEPENDENT PGA

The traditional PGA algorithm described above assumes that the phase error is constant with range and estimates the error in the azimuth direction. A low-altitude SAR system with highly varying incidence angles will exhibit range-dependent effects in the phase errors. This section describes the cause of these range dependencies.

Assume the instrument platform is flying with constant velocity in the direction of increasing z, with the nominal trajectory following $x = y = 0$, as shown in Fig. 2. Then the phase error due to the trajectory errors in the x and y directions can be written as

$$\phi(t,\theta) = \frac{4\pi}{\lambda}(-x(t)\sin(\theta) + y(t)\cos(\theta)), \quad (2)$$

where θ is the incidence angle. The data is stored by range bin instead of incidence angle, so we write the incidence angle for the kth range bin as

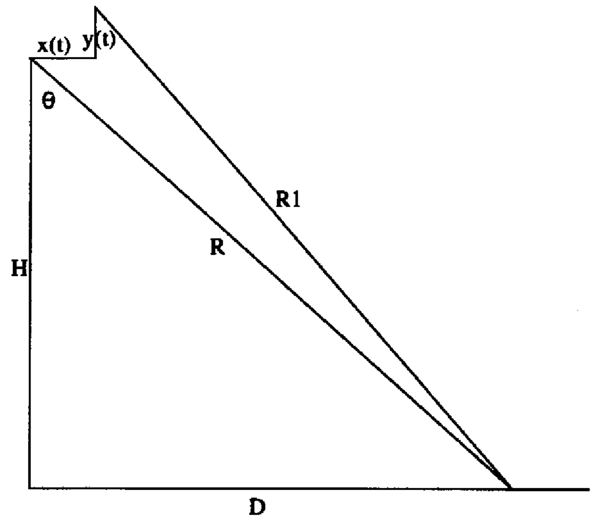

Figure 2: Transverse Motion Geometry

$$\theta_k = \cos^{-1}\left(\frac{H}{R_0 + kR}\right). \quad (3)$$

Here H represents the height of the instrument above the topography, R_0 is the range to the zeroth sample, and R is the range bin size. Now we have two parameters of phase error to estimate for each azimuth position, $\hat{\phi}_x = -\frac{4\pi}{\lambda}x(t)$ and $\hat{\phi}_y = \frac{4\pi}{\lambda}y(t)$. There is still a large amount of redundancy in the data, so one should be able to effectively estimate these two parameters by adding some kind of range-dependent weighting in the PGA phase estimator. One possibility for the range-dependent weighting is developed in this paper.

SPOTLIGHT PGA ALGORITHM DERIVATION

The Phase Gradient Autofocus (PGA) algorithm was first developed for the spotlight mode SAR and later theoretically extended to strip-map. In this paper we further extend the algorithm by dropping the narrow beam assumption. This introduces range dependencies in the phase error. A PGA algorithm that can estimate aircraft movement is necessary. Even though all spotlight and strip-map SARs have range dependent chirps associated with them, because of their altitude most satellite based system s range dependent errors are assumed negligible. This paper develops a more complete PGA for general applications. [Charles V. Jakowatz and Wahl, 1993]

There are four main steps in spotlight mode autofocus processing The four steps are: center shifting, windowing, phase estimation and iteration. (Fig 3)

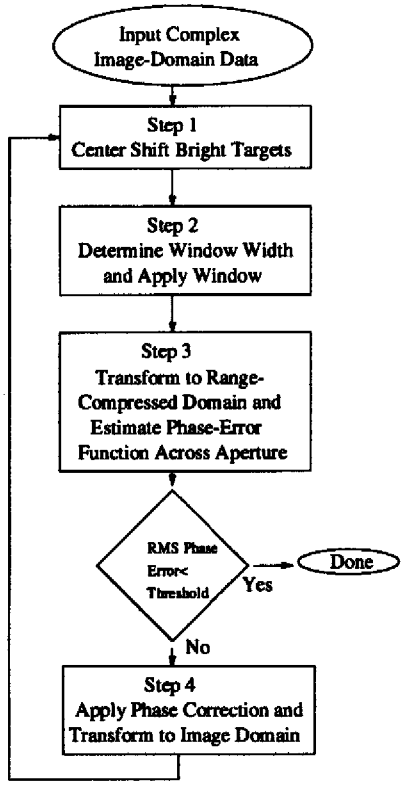

Figure 3: PGA Steps

The frequency of each individual scatterer is the proportional to the distance from the image center. The scatterer at the scene center has a constant phase and a frequency of zero. Constant phase lines mirror scatterer angular position. The system impulse response is the two dimensional Fourier transform of the aperture shape.

The simplest development is the point target. In the image domain it is represented as

$$G(\omega) = A_t T_0 sinc[T_0(\omega - \omega_0)]e^{j\phi_0} \otimes E(\omega). \quad (4)$$

$E(\omega)$ is the Fourier transform of the phase error function that is convolved with the rest of the image.

To see the phase history the inverse Fourier transform is taken. The point target can be represented in the time domain by

$$\begin{aligned} g(t) &= A_k e^{-j(\omega_0 t + \phi_E(t) + \phi_0)} rect \\ rect &= (u(T_0/2 + (t - t_0)) \\ &\quad -u(-T_0/2 + (t - t_0)). \end{aligned} \quad (5)$$

A_t, ω_0 and ϕ_0 are the magnitude, frequency and phase of the signal respectively. $\phi_E(t)$ is the phase error from the motion of the aircraft. The step functions represents the time in the range gate where the signal is valid.

COMPLEX SHIFT

In the image domain the important phase history is evident in all pixels because spotlight mode focuses on one spot during the whole data requisition. As a target scatters in each position of the platform it retains the exact position of the aircraft at that time. Therefore the target with the biggest signal to noise ratio in each constant range line will contain the best phase history. The purpose of phase shifting is to take the modulation out of the signal.

A circular shift is done to bring all points to the same reference point. This shift in the image domain does not effect the phase history.

Looking at this equation we see that we are really interested in the $\phi_E(t)$ term which is the phase error from the movement of the aircraft. However, we still have the function in the first exponential. If each maximum amplitude ,A_k, in each constant range line is shifted to frequency zero then we can eliminate the $e^{\omega_0 t}$ phase term.

$$\begin{aligned} g(t) &= A_k e^{-j(\phi_E(t) + \phi_0)} rect \\ rect &= (u(T_0/2 + (t - t_0)) \\ &\quad -u(-T_0/2 + (t - t_0)). \end{aligned} \quad (6)$$

WINDOWING

The windowing function is done in the image domain derived above. After the shifting of the brightest point on each constant range line the rest of the points appear as noise because all other points contain modulation phase data. The convolution of the phase with the whole line implies that the phase history is contained in each line. Thus the windowing has know effects. All other targets left in the image create noise in the phase when the inverse Fourier transform is taken. As the window is narrowed care must be taken not to window out some of the phase blurring seen by the the chosen pixels.

The next step of windowing can be represented by a weighting function of $W(\omega)$ When this is multiplied in the frequency domain then we zero out the compressed pixels that create the high frequency noise. As the window length decreases the high frequency components are decreased creating a more accurate phase estimation. The inverse Fourier transform of the shifted windowed image domain gives

$$g(t) = A_t e^{j(\phi_0 + \phi_E(t)} \otimes w(t). \tag{7}$$

w(t) represents the inverse Fourier transform of the window function. We notice that the function g(t) represents a magnitude multiplied by a complex exponential. This complex exponential contains the motion phase error which is a function of time and a constant phase from the position.

PHASE GRADIENT ESTIMATION

There are many options for phase estimation. Three methods will be discussed: The original PGA (Minimum Variance), Maximum likelihood, and Phase weighted estimation.

Unbiased Minimum Variance

Equation 7 shows that the phase error and a phase constant are left in each pixel. To isolate the desired error the derivative or gradient is found and then integrated to derive the phase error.

Using the derivative property of the Fourier transform we calculate the phase Gradient estimation by computing the inverse transform of $j\omega G(\omega)$.

$$\dot{g(t)} = j\dot{\phi}_E(t)g(t) \tag{8}$$

Assuming the effect of windowing is small compared to the error an estimate of the phase gradient error is

$$\hat{\dot{\phi}}_E(t) = \frac{Im[\dot{g}(t)g^*(t)]}{|g(t)|^2}. \tag{9}$$

If we integrate the above equation we can obtain the phase error with a unknown constant. Statistically the best way to find a unknown function is to average over many replicas of the same function. Because the noise is uncorrelated with itself than it will average out. When range phase variances are ignored we can assume

$$\hat{\dot{\phi}}_E(t) = \frac{\sum_N Im[\dot{g}(n,t)g^*(n,t)]}{\sum_N |g(n,t)|^2}. \tag{10}$$

As the phase error is different in each range bin the the indices g(n,t) represents the different phase in each constant range line. This derivation is the weighted least squares estimate of the gradient of the phase error.

The last step mentioned earlier was iteration. As was discussed, the wider the window the higher frequency noise that is entered into the phase. We want to narrow the window to eliminate noise, but don't want to eliminate important phase history. As we make one estimate of the phase and correct it the phase error decreases and then we can lower the noise in the windowing process and receive a better phase estimate. This process is repeated until the phase error derived reaches a certain threshold.

This method was the original PGA developed in 1988. This is the least squares estimate of the phase error. The problem with this estimate is that it is biased. This has two explanations. Because we are taking the derivative directly then the estimate is especially susceptible to high frequency noise. If there is a great change than the derivative is very large. If it is noise causing this then we have a greater error.

Maximum Likelihood

In 1993 a maximum likelihood estimation of phase errors was published. This process is superior because it does not assume a high signal to clutter ratio ridding itself to the bias problems.
[Charles V. Jakowatz and Wahl, 1993]

We will start with the same point target that we have

$$\begin{aligned} &g(t) = A_t e^{-j(\omega_0 t + \phi_E(t) + \phi_0)} rect \\ &rect = (u(T_0/2 + (t - t_0)) - \\ &u(-T_0/2 + (t - t_0)) \end{aligned} \tag{11}$$

as the function of the scatterer before the taking the Fourier transform to get to the image domain.

The same process is used to shift and and center the brightest targets. The result is

$$g(m) = A_m e^{j(\phi_0 + \phi_E(m)} \otimes w(m). \tag{12}$$

If window long enough to encompass the whole phase error the effect of the window is neglected. The resultant equation is

$$g(m) = A_m e^{j(\phi_0 + \phi_E(m)} + \eta(m). \tag{13}$$

Where $\eta(m)$ is the noise of the other pixels returns.

The goal is to get the phase difference between two steps to estimate the error. We take the conjugate of a time delayed copy of the scatter $g(m)$ and multiply it by the original This gives

$$
\begin{aligned}
&g(m)g^*(m-1) = \\
&\quad (A_k e^{j(\phi_0+\phi_E(m)} + \eta(m))((A_k^* e^{-j(\phi_0+\phi_E(m-1)} \\
&\quad +\eta^*(m-1)) \qquad (14) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (15)
\end{aligned}
$$

Averaging all the constant range lines eliminates the noise because it is uncorrelated with itself and the phase error. The cross correlation of the noise with the scatterer is the expectation of the scatterer times the expectation of the noise. The auto correlation of the noise is a delta function of the noise is a delta function since we assume that the noise can be modeled as white. Averaging the samples will cause η^2 to go to zero and the cross terms will also become negligible. This is useful because the only signal left will be the difference of the phase with its importance weighted as a function of the magnitude of the signal.

The result is

$$
< |A_k|^2 e^{j(\phi_E(m)-\phi_E(m-1)} > \qquad (16)
$$

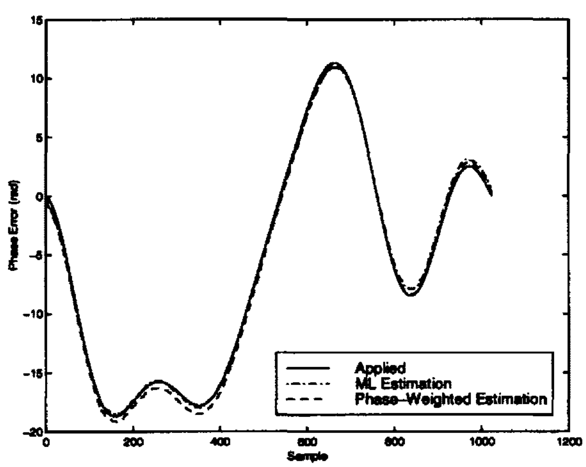

Figure 4: Phase error comparison for non-range-dependent case

Phase Weighted Estimation

The two methods proposed earlier in this paper for the phase estimation step each has its own criteria for optimality. The original PGA algorithm used a linear unbiased minimum variance.

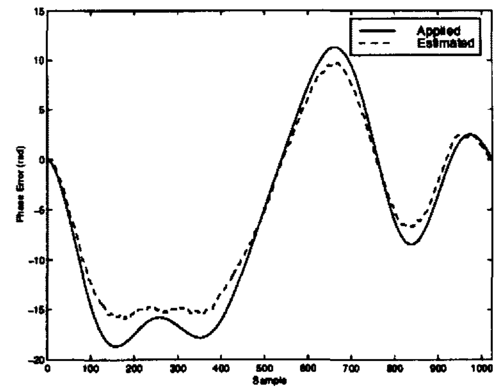

Figure 5: Phase error, x component

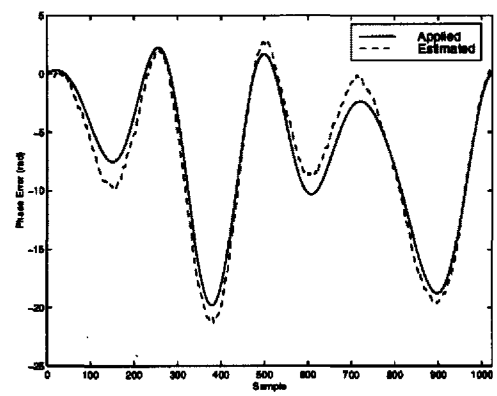

Figure 6: Phase error, y component

[Eichel and C. V. Jakowatz, 1989] The same authors later proposed a method using a maximum likelihood (ML) estimator [Charles V. Jakowatz and Wahl, 1993]. This new phase estimation technique which allows extension to a range-dependent algorithm will now be introduced.

To apply PGA, the gradient of the phase error must be found. The maximum likelihood method is known to be optimal; thus, a first approach would be to apply this method to the range-dependent problem. However, we are not aware of a closed form for the phase estimate. We thus introduce a new algorithm to estimate the phase gradient which allows a simple closed form for a range-dependent version. The phase noise of a sample depends inversely on the magnitude. Thus, our new method weights the phase measurements by the magnitude of the corresponding pixel.

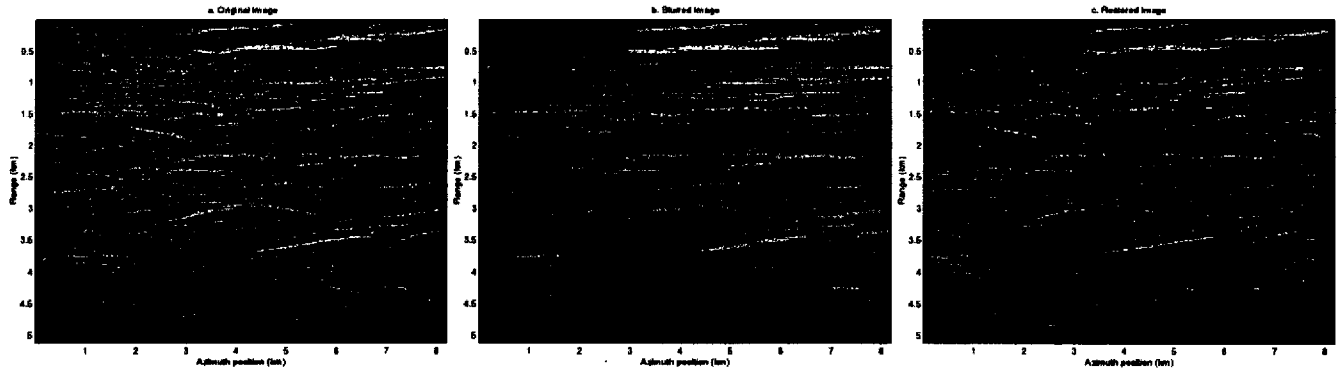

Figure 7: The test image used for range-dependent phase error. (a) The original, focused image of Death Valley from the ERS-1 C-band SAR. (b) The image blurred by a range-dependent phase error. (c) The restored image using the new Phase Weighted Estimation PGA algorithm.

Let g_{kn} denote the image in the range-compressed domain, with k indicating the range bin and n the azimuth bin. Then the estimated phase gradient, denoted $\hat{\phi}_n$, is

$$\hat{\phi}_n = \frac{\sum_{k=0}^{M-1} \left(|(g_{kn} g^*_{k(n-1)})| \angle (g_{kn} g^*_{k(n-1)}) \right)}{\sum_{k=0}^{M-1} |(g_{kn} g^*_{k(n-1)})|}. \tag{17}$$

This algorithm can easily by extended to the range-dependent case. The phase weighting remains the same. We add range weighting and write a vector of equations indexed by range bin k.

$$\hat{\phi}_{xn} \sin(\theta_k) + \hat{\phi}_{yn} \cos(\theta_k) = |g_{kn} g^*_{k(n-1)}| \angle (g_{kn} g^*_{k(n-1)}). \tag{18}$$

This equation is separated into vectors and matrices and written as

$$\mathbf{A}\hat{\phi}_\mathbf{n} = \tilde{\phi}_\mathbf{kn} \tag{19}$$

where $\mathbf{A}$ is the Mx2 matrix made up of the sine values in the first column and the cosine values in the second, $\hat{\phi}_\mathbf{n}$ is the 2x1 vector of phase estimates, and $\tilde{\phi}_\mathbf{kn}$ is the Mx1 vector of weighted image phase gradients. This equation is solved using the pseudoinverse of $\mathbf{A}$ to obtain the range-dependent phase gradient estimate. This gradient is then integrated and applied in the same way as in the original PGA algorithm.

RESULTS

The new PWE-PGA algorithm was tested using synthetic phase errors on an ERS-1 image of Death Valley. First a non-range dependent phase error was applied to the image. The resulting blurred image was corrected using the original ML-PGA algorithm and using the new Phase Weighted Estimation PGA. In Fig. 4 the applied phase error is compared with the maximum likelihood and the phase weighted estimation. For this test, the new method is comparable to the ML algorithm but has a slightly larger error.

Figs. 7-6 show a test using range dependent phase errors on the Death Valley image. The original image is shown in Fig. 7(a). The image was then blurred with the range-dependent phase error, resulting in the image in Fig. 7(b). The image was then corrected using the PWE-PGA algorithm, resulting in the restored image in Fig. 7(c). The estimated and applied phase errors for ϕ_x and ϕ_y are shown in Figs. 5 and 6 respectively.

CONCLUSIONS AND FUTURE WORK

The PGA algorithm has been widely used in spotlight SAR to remove motion-induced blur in the images. PGA has been proven to be both a robust, computationally superior autofucus algorithm. The conventional PGA uses a narrow beam approximation to avoid range dependencies. We have introduced a new phase estimator for use in PGA and have extended it to the range-dependent case. Several tests have shown that this algorithm can be successful at removing range-dependent phase errors.

Our planned future work includes a statistical analysis of the new estimator to determine its optimality. We will then further extend the algorithm for application to stripmap mode SAR.

The shifting and windowing in the strip-map is done by selecting the brightest point in each portion of the image, windowing and convolving with the chirp about the points.

The original PGA phase estimation step used a maximum likelihood averaging technique. A similar approach with a range dependent weighing function for the phase error will be derived. The iterations step is the same as the spotlight mode.

Using a progressive window in the PGA algorithm, the simulated phase errors are removed comparably to the original spotlight mode algorithm.

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