

STAGNATION FLOW AGAINST CONCAVE SURFACES

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Abstract

A review of previous work relative to stagnation flow against curved surfaces is presented. The unsteady, three-dimensional nature of stagnation flow against concave surfaces has motivated much research in the context of turbulence. However, an alternative approach, which could yield valuable insight into the influence of curvature on heat transfer at the surface, is that taken by Kirchhoff in analyzing the flat plate with flow separation. An analytical solution for the inviscid velocity profile along a flat plate with separation is presented. The velocity and pressure distributions are compared with those for attached inviscid flow. The application of the method to curved surfaces is discussed.

An important issue in aerospace engineering problems is thermal control, involving a broad range of applications ranging from atmospheric re-entry to the challenge of turbine blade cooling. Currently, the constraint on the turbine inlet temperature imposed by the turbine blade material is the limiting factor in improvements in gas turbine efficiency. One of the key mechanisms which influence the estimate of heat transfer to gas turbine blades is the acceleration of the fluid along the surface, due in part to the curvature of the surface¹. The problem of inviscid stagnation flow past a finite flat plate with separation occurring behind the plate was first studied by Helmholtz². The study was motivated by the inability of the continuous potential flow solution (see Fig. 1) to give agreement with known values of pressure coefficient. Kirchhoff³ then developed a general method for handling flows past flat surfaces using the hodograph plane.^{4,5} The difficulty with applying the method to curved surfaces will be discussed shortly. The potential flow solution provides an outer boundary condition for the subsequent boundary layer analysis. For instance, in a boundary layer analysis of a flat plate aligned parallel to the free stream, the limiting value for the velocity far from the plate is simply a constant (equal to the free-stream velocity). Goldstein⁶ derived the first-order

boundary layer equations for flow along a curved surface, and Howarth^{7,8} analyzed the neighborhood of a stagnation point on a curved surface assuming a linear variation in the potential flow velocity with the distance along the wall. Murphy⁹ studied unseparated flow along a curved surface along which the free stream potential flow velocity was taken to be constant for the outer boundary condition in the boundary layer analysis. This, of course, dictated a specific surface shape. He later¹⁰ considered several cases, allowing the potential flow velocity to vary by powers of distance along the surface.

The curved surface continued to be investigated^{11,12} while a method of matching asymptotic expansions for the outer potential flow and of the inner boundary layer flow began to be developed¹⁴⁻¹⁷ to yield higher-order approximations to the solutions. Van Dyke^{18,19} has pioneered the application of the method as a systematic approach to solving many types of flow problems. Prandtl's original boundary layer assumption is considered to be the first term in an asymptotic expansion describing the boundary layer. The boundary layer expansion, valid only for small values of some perturbation

parameter (typically involving Reynolds number), is "matched" with an expansion for the potential flow, valid only for large values of the perturbation parameter, by summing the two expansions and deleting duplicate terms. However, this method does require that the external flow be computed first.¹⁵ The appropriate solution of the inviscid equations to describe limiting flow at large Reynolds number is not known for separated flows, and has a complicated mathematical description for unseparated flow past finite bodies.¹⁶ Also, serious difficulties arise for non-analytic shapes. With the exception of Kirchhoff's finite flat plate problem, all of the work cited above has been restricted to unseparated flows along infinite or semi-infinite surfaces. It is, therefore, aimed primarily at the analysis of laminar, incompressible flows. Schultz-Grunow and Breuer²⁰ derived the equations for self-similar solutions for a curved surface, leaving the potential flow velocity as an unknown function of flow path, and calculated the special case of constant potential flow velocity. Subsequently, several investigations^{21,22} were made into higher-order solutions to multi-dimensional, incompressible flows free of separation by retaining more terms in the asymptotic expansions.

It is known experimentally²⁵⁻³¹ that under some conditions stagnation flow can become unstable, causing stationary Görtler²⁴ vortices to arise in the boundary layer. Petitjeans³⁶ noted that Taylor-Görtler instability is a centrifugally driven instability in the boundary layer flow on a concave wall and appears in the form of counter-rotating longitudinal vortices. Theoretical treatments of the phenomenon are also numerous.³²⁻³⁶ Bogolepov and Lipatov³⁷ analyze asymptotically the uniform flow of a viscous liquid over a concave surface for large but subcritical Reynolds number. The wakes formed behind a curved plate subjected to an unsteady free stream have also been investigated experimentally.³⁸ Yet there remains to be found a general analytical solution of the potential flow equations for a finite plate with significant curvature and at large Reynolds numbers which accounts for flow separation. This information is needed in order to correctly specify the main flow for the boundary layer problem. It is known experimentally³⁹ that even at low to moderate Reynolds numbers, separation occurs downstream of a finite plate. The inherent three-dimensional, unsteady nature of the vortices formed at a stagnation point has motivated researchers to consider the concave surface in the context of turbulence. However, for the purposes of estimating the heat transfer at the surface of a

curved body, valuable insight might be gained by investigating the two-dimensional problem by the use of complex variable theory. Neglecting the vortices would provide a lower bound on the heat transfer estimate. This approach will now be discussed briefly.

Until Kirchhoff's treatment of free fluid jets, the conventional method of solving potential flow problems was to find a mapping which transformed some known, closed streamline contour into parallel flow, where the solution was known. This known solution was then mapped back to the closed streamline contour being studied, thereby obtaining the inviscid solution for the actual flow field. But with separation behind the plate, the location of the two streamlines along the discontinuity is unknown. It was Kirchhoff who first recognized that the free surface of separation (see Fig. 2) could be handled by considering the hodograph plane. In the hodograph plane, the two free streamlines form a curve of constant curvature equal to the magnitude of the free stream velocity (Fig. 3).

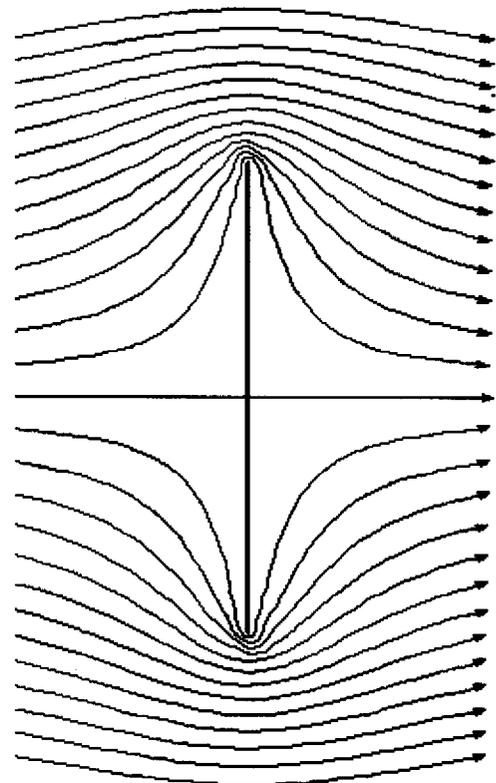


Figure 1 Finite plate without separation.

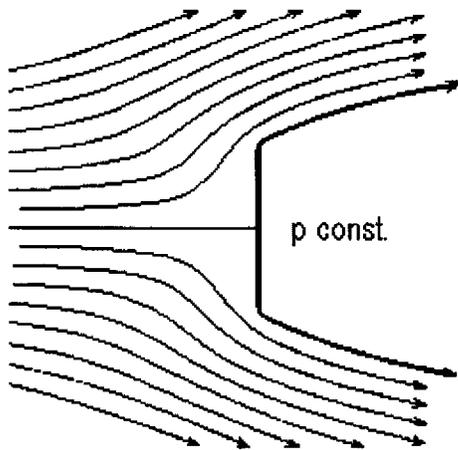


Figure 2 Finite plate with separation.⁴

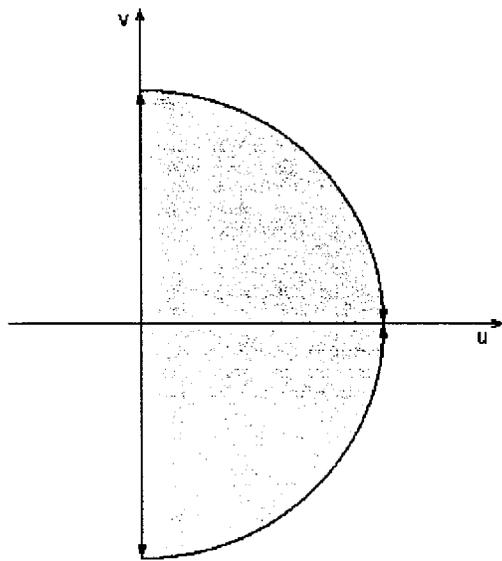


Figure 3. Hodograph for flat plate (z_2 -Plane).

The objective of the method is to find a mapping between the hodograph plane and simple parallel flow. For the flat plate case, the hodograph plane can be inverted using the transformation $z_3 = 1/z_2$. All velocities and lengths have been nondimensionalized so that the radius of the arc is unity. The graph of the z_3 -plane, shown in Figure 4, resembles the right half of a doublet.

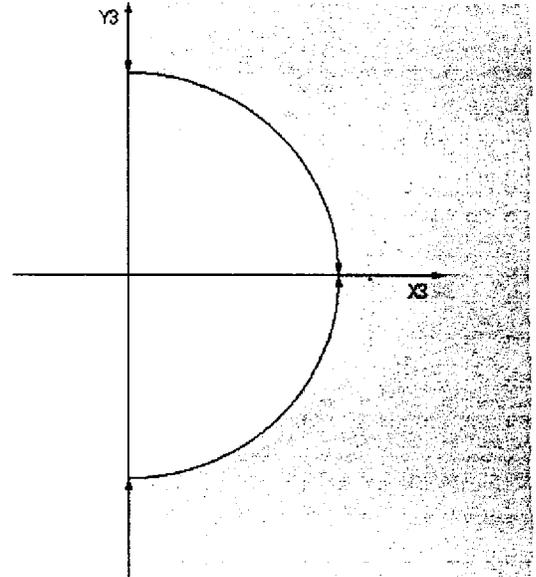


Figure 4. z_3 -Plane

Accordingly, a doublet is then subtracted out of the z_3 plane to give $z_4 = z_3 - a/z_3$ (where $a = 1$). The result is shown in Fig. 5.

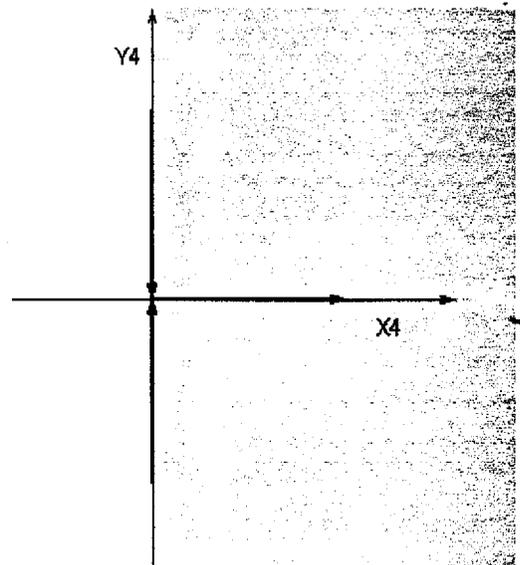


Figure 5. z_4 -Plane

The z_4 -plane can then be rotated by $z_5 = z_4 e^{i\pi/2}$, and transformed into parallel flow⁴⁰ by the transformation $z_6 = z_5^{-2}$. Since the complex potential for parallel flow is:

$$F(z_6) = Kz_6$$

the complex potential for the flat plate with separation can be found by back substituting the transformations just listed, to give:

$$F(z_2) = K \left[i \left(\frac{1}{z_2} - z_2 \right) \right]^{-2} \\ = \frac{-K}{\left(\frac{1}{W} - W \right)^2}$$

where $W = \frac{dF}{dz_1} = \sqrt{u^2 + v^2} e^{-i\theta} = Ve^{-i\theta}$.

Then $F = \phi + i\psi = \frac{-K}{\left(\frac{1}{Ve^{-i\theta}} - Ve^{-i\theta} \right)^2}$. Along the wall,

$\theta = \frac{\pi}{2}$ so that $\phi = \frac{K}{\left(v^2 + \frac{1}{v^2} + 2 \right)}$ and $\psi = 0$.

Then $v = \frac{\partial \phi}{\partial y} = \frac{-2K[v - v^{-3}]}{\left[v^2 + v^{-2} + 2 \right]^2} \frac{\partial v}{\partial y}$

can be rearranged and integrated as follows:

$$\int_0^y dy = \int_0^v \frac{-2K(v^2 - 1)}{(v^2 + 1)^3} dv.$$

Using the change of variables $v^2 + 1 = u$ gives:

$$y = \int_1^{v^2+1} \frac{-K}{u^2 \sqrt{u-1}} du + \int_1^{v^2+1} \frac{2K}{u^3 \sqrt{u-1}} du \\ = \frac{K}{2} \left[\frac{\sqrt{u-1}(u+2)}{u^2} + \tan^{-1} \sqrt{u-1} \right]_1^{v^2+1} \\ = \frac{K}{2} \left[\frac{v(v^2+3)}{(v^2+1)^2} + \tan^{-1}(v) \right]$$

The constant, K , can be found to be $\frac{8}{4+\pi}$ by setting $y = 1$ when $v = 1$, so that

$$y = \frac{4}{(4+\pi)} \left[\frac{v(v^2+3)}{(v^2+1)^2} + \tan^{-1}(v) \right].$$

This dimensionless velocity distribution is shown in Figure 6 plotted against that of the flat plate without separation, which approaches infinity due to the infinite curvature at the edge. As seen from the figure, both approaches predict a linear velocity profile in the neighborhood of the stagnation point. The dimensionless pressure is shown in Figure 7.

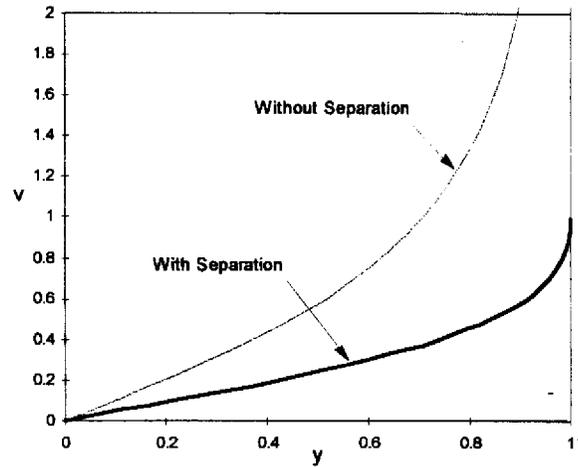


Figure 6. Flat Plate Velocity Profile.

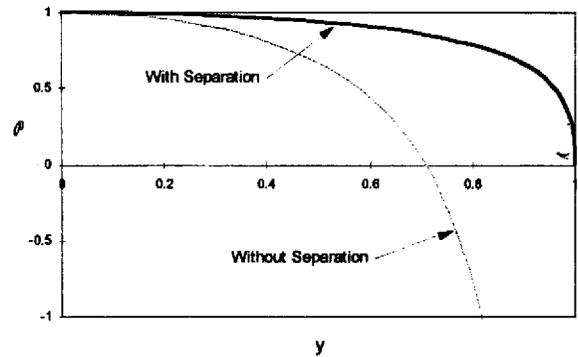


Figure 7. Flat Plate Pressure Distribution $\left(\frac{P - P_\infty}{\frac{1}{2} \rho U^2} \right)$.

The above example illustrates one possible procedure for predicting the potential flow past finite concave surfaces with flow separation. The method has traditionally been used only for flow along straight surfaces. The difficulty in applying the method to curved surfaces stems from the fact that the trajectory in the hodograph plane corresponding to the surface is

not known as it is in the case of the flat surface. The only piece of information initially available for the curved case is that

$$\frac{v}{u} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = f(x,y)$$

where f is known from the surface chosen. So at any point on the surface, the direction of flow (v/u) is known, but the magnitude is not. For the flat plate case, at any point in the hodograph plane corresponding to a point on the surface, the magnitude as well as the direction of flow are known simultaneously. The only unknown is the corresponding location on the actual surface, which is found by solving for the flow field. The straight line in the hodograph plane, together with the semicircle representing flow at the discontinuity downstream, form a closed contour of known shape. The complex potential of some simple flow field, such as parallel flow, is then mapped conformally into the shape of this closed contour just constructed in the hodograph plane. The composition of mappings form a function describing the streamlines in the new configuration. Although the direction and magnitude are both known at the tip of the curved surface, there are infinitely many possible curves connecting the stagnation point to the end point. This possibility of many different acceleration patterns is shown qualitatively in Fig. 8.

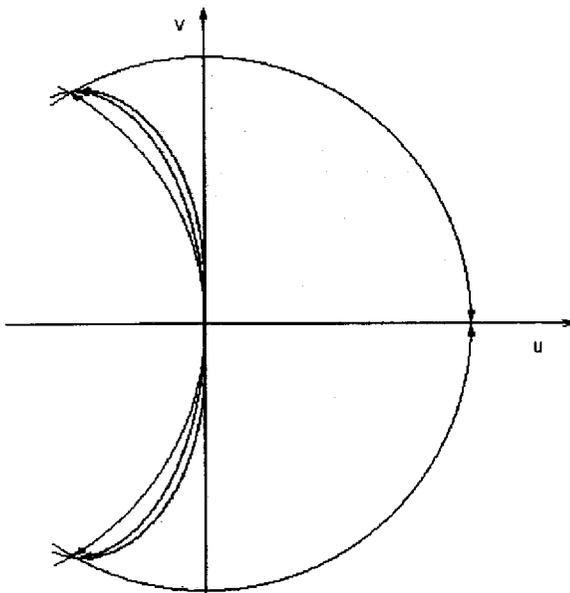


Figure 8. Hodograph of Finite, Curved Plate.

This can also be seen by considering the equations of the surface:

$$\begin{aligned} x &= \cos\theta & \text{and} & & y &= \sin\theta \\ x' &= -\sin\theta \theta' & & & y' &= \cos\theta \theta' \end{aligned}$$

The radius, R , of the curve traced by the velocity along the surface in the hodograph plane can be expressed as

$$\begin{aligned} R &= \sqrt{u'^2 + v'^2} = \sqrt{x'^2 + y'^2} \\ R &= \sqrt{(-\sin\theta \theta')^2 + (\cos\theta \theta')^2} \\ R &= |\theta'| \end{aligned}$$

where θ' is some unknown but increasing function of θ . Therefore the radius in the hodograph plane increases with increase angle, creating an outward spiral as depicted in Fig. 8.

One method of circumventing this problem might be to start first with a known hodograph plane rather than the actual plane. That is, solve the entire flow field for a family of hodograph curves and then determine what surface shape they represent. This reverse approach appears to be quite mathematically involved, even when simple circular arcs are used in the hodograph. The first choice would be to use a linear fractional transformation to map one of the two points of intersection to the origin and its conjugate to the ideal point. For the third point in the transformation, the origin could be mapped to some point on the real axis, such as $x = 1$. However, since the actual surface shape will not be known a priori, the simplification obtained in the flat plate case by setting one of the independent variables to zero is not likely to be possible. This may require numerical integration to obtain the velocity field.

Conclusion

Numerous researchers have investigated stagnation flow on curved surfaces. Due to the difficulty arising from the presence of the wake after separation, many investigations have been restricted to unseparated (laminar) flow or to the region immediately neighboring the stagnation point where the velocity profile can be assumed linear. It is thought that two-dimensional complex variable theory, while unable to predict the three-dimensional vortices which can occur at a stagnation point, could provide valuable

information as to the influence of concave curvature on the heat transfer at the surface.

Acknowledgments

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